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Positive Approximation Properties of Banach Lattices

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Abstract. In this paper, an equivalent formulation of extendable local reflexivity (ELR) introduced by Oikhberg and Rosenthal is given. We introduced the positive version (PELR) of the ELR in Banach lattices to solve the lifting problem for the bounded positive approximation property (BPAP). It is proved that a Banach lattice X has the BPAP and is PELR if and only if the dual space X^* of X has the BPAP. Finally, we give isometric factorizations of positive weakly compact operators and establish some new characterizations of positive approximation properties.

1. Introduction and main results

Recall that a Banach space X has the *approximation property* (AP) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator S on X such that $\|Sx - x\| < \varepsilon$ for all $x \in K$. If, in addition, there exists $\lambda \geq 1$ such that we can always choose a finite rank operator S on X with $\|S\| \leq \lambda$, then we say that X has the λ-*bounded approximation property* (λ-BAP). If a Banach space X has the λ-BAP for some λ, we say that X has the BAP.

A Banach lattice X is said to have the *positive approximation property* (PAP) if for every compact subset K of X and $\varepsilon > 0$, there exists a positive finite rank operator S on X such that $\|Sx - x\| < \varepsilon$ for all $x \in K$. If the positive finite rank operator S can be chosen with $\|S\| \leq \lambda$, then we say that X has the λ-*bounded positive approximation property* (λ-BPAP for short). If a Banach lattice X has the λ-BPAP for some λ, we say that X has the BPAP.

It is an open problem whether, in Banach lattices, the AP (BAP) implies the PAP (BPAP) (see [3, Problem 2.18]). There are many already classical results on AP. However, there has been little attention paid to PAP. Thus "to what extent things about AP (BAP) work for PAP (BPAP) in Banach lattices" becomes an interesting topic.

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How the geometric structure of Banach spaces permits to lift BAP from Banach spaces to their dual spaces has been studied in [8, 10, 11, 13, 20–22], etc. A new geometric property of Banach spaces was introduced by Oikhberg and Rosenthal [19], and Johnson and Oikhberg [11] proved that this property permits to lift BAP from a Banach space to its dual space. Let us recall the definition and result.

Definition 1.1. [19] A Banach space X is said to be *μ-extendably locally reflexive* (μ-ELR) ($\mu \geq 1$) if for every finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for every $\varepsilon > 0$, there exists an operator $T \colon X^{**} \to X^{**}$ such that

(i) $\|T\| \leq \mu + \varepsilon$;

(ii) $T(E) \subset i_X(X)$;

(iii) $(Tx^{**})(x^*) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$.

We say that a Banach space X is *extendably locally reflexive* (ELR) if X is μ-ELR for some μ.

A well-known result is:

Theorem 1.2. [11, Theorem 3.1] *Let X be a Banach space and let $\lambda, \mu \geq 1$.*

(a) *If X has the λ-BAP and is μ-ELR, then X^* has the $\lambda\mu$-BAP.*

(b) *If X^* has the λ-BAP, then X is λ-ELR.*

Oja [22] proved that if a Banach space X is ELR and has the AP, then X^* has the AP whenever X is complemented in X^{**}. Moreover, it is said in [22] that it is not known whether the ELR permits lifting the AP. Å. Lima [13] answered this question and proved that if a Banach space X is ELR and has the AP, then X^* has the AP. In [21], the lifting problem was considered for a more general situation of bounded approximation properties defined by operator ideals. In contrast, whether the ELR permits to lift BPAP seems to be unknown so far.

In this paper, an equivalent formulation of ELR is given as follows.

Lemma 1.3. *Let X be a Banach space and $\mu \geq 1$. The following statements are equivalent.*

(i) *X is μ-ELR;*

(ii) *for every finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for every $\varepsilon > 0$, there exists an operator $T \colon X^{**} \to X^{**}$ with $\|T\| \leq \mu + \varepsilon$ such that $T(E) \subset i_X(X)$ and $|(Tx^{**})(x^*) - x^{**}(x^*)| \leq \varepsilon \|x^{**}\| \|x^*\|$ for all $x^{**} \in E$, $x^* \in F$.*

In Section 2, we present an elementary and self-contained proof of Lemma 1.3. The proof is due to Professor W. B. Johnson. The author is grateful to him for providing it.

Based on Lemma 1.3, I introduce the positive version of ELR in Banach lattices as follows.

Definition 1.4. A Banach lattice X is *μ-positively extendably locally reflexive* (μ-PELR) ($\mu \geq 1$) if for every finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for every $\varepsilon > 0$, there exists a positive operator $T\colon X^{**} \to X^{**}$ such that

(i) $\|T\| \leq \mu + \varepsilon$;

(ii) $T(E) \subset i_X(X)$;

(iii) $|(Tx^{**})(x^*) - x^{**}(x^*)| \leq \varepsilon \|x^{**}\| \|x^*\|$ for all $x^{**} \in E$ and $x^* \in F$.

We say that a Banach lattice X is *positively extendably locally reflexive* (PELR) if X is μ-PELR for some $\mu \geq 1$.

Then we can use the PELR to lift the BPAP from Banach lattices to their dual spaces.

Theorem 1.5. *Let X be a Banach lattice and let $\lambda, \mu \geq 1$.*

(a) *If X has the λ-BPAP and is μ-PELR, then X^* has the $\lambda\mu$-BPAP.*

(b) *If X^* has the λ-BPAP, then X is λ-PELR and has the λ-BPAP.*

Consequently, a Banach lattice X has the BPAP and is PELR if and only if X^* has the BPAP.

It is proved in [16] that X has the 1-BPAP if X^* has the 1-BPAP. Actually, the proof is applicable to the λ-BPAP for every $\lambda \geq 1$. In the proof of Theorem 1.5(b), I provide a slightly direct proof of it. The proof of Theorem 1.5 will be presented in Section 2.

It is a classical result due to Grothendieck [9] that a Banach space X has the AP if and only if for every Banach space Y, $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$. But, in the case of PAP, there is a long-standing open problem:

Open Problem. [18, p. 102] *A Banach lattice X has the PAP if and only if for every Banach lattice Y, $\mathcal{K}(Y, X)_+ = \overline{\mathcal{F}(Y, X)_+}$.*

Kim [12] showed that a Banach space X has the AP if and only if for every Banach space Y, $\mathcal{K}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\tau_c}$. Subsequently, Choi, Kim and Lee [4] showed that a Banach space X has the AP if and only if for every Banach space Y, $\mathcal{K}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\tau_c}$.

In the final section, I shall be concerned with to what extent the above two characterizations of AP work for PAP. In Theorems 3.8 and 3.9, I characterize the PAP by positive weakly compact operators under the condition "KB-space". I do not know whether the positive weakly compact operators in Theorems 3.8 and 3.9 could be strengthened to be the positive compact operators. Furthermore, I still do not known whether the condition "KB-space" could be removed. However, it should be noted that, if we relax the Banach lattices Y in Theorems 3.8 and 3.9 to Banach spaces Y, then we can replace positive weakly compact operators by compact operators. Moreover, we can remove the condition "KB-space". Thus, in this sense, Theorems 3.8 and 3.9 are optimal so far. In Theorems 3.8 and 3.9, it is assumed that X^* or X is a KB-space. It is worth mentioning that KB-spaces do not necessarily enjoy the PAP. Actually, Szankowski [25] constructed a reflexive Banach lattice Z without the AP. So it follows from [17, Theorem 2.4.15] that Z and Z^* are KB-spaces, but Z fails the PAP.

Our notations and terminologies are standard as may be found in [15] and [17]. Throughout this paper, all Banach lattices are real lattices. By an operator, we always mean a bounded linear operator. For a Banach space X, $i_X\colon X \to X^{**}$ denotes the canonical embedding. Let X, Y be Banach spaces. We denote $\mathcal{L}(X,Y)$ (resp. $\mathcal{F}(X,Y)$) by the space of all operators (resp. finite rank operators) from X to Y. If $X = Y$, we denote $\mathcal{L}(X,Y)$ (resp. $\mathcal{F}(X,Y)$) by $\mathcal{L}(X)$ (resp. $\mathcal{F}(X)$) simply. If X and Y are Banach spaces, we let τ_c denote the locally convex topology on $\mathcal{L}(X,Y)$ of uniform convergence on compact sets in X. Let A be a subset of a Banach space X. We denote by $\overline{co}(A)$ the closed convex hull of a set A. For Banach lattices X and Y, we denote by $\mathcal{F}(X,Y)_+$, $\mathcal{K}(X,Y)_+$ and $\mathcal{W}(X,Y)_+$ the set of all positive finite rank operators, positive compact operators and positive weakly compact operators from X to Y. For a Banach lattice X, we denote by X_+ the positive cone of X, i.e., $X_+ := \{x \in X : x \geq 0\}$.

2. Proofs of Lemma 1.3 and Theorem 1.5

Proof of Lemma 1.3. Let E and F be finite-dimensional subspaces of X^{**} and X^*, respectively, and let $\varepsilon > 0$ be given. Choose a basis $\{x_1^*, x_2^*, \ldots, x_n^*\}$ for F. Take biorthogonal functionals $\{u_1, u_2, \ldots, u_n\}$ in X^{**} to $\{x_1^*, x_2^*, \ldots, x_n^*\}$. We may enlarge E such that E contains $\{u_1, u_2, \ldots, u_n\}$. Thus E is total over F. This means that the operator $R\colon E \to F^*$ defined by $R(x^{**}) = x^{**}|_F$ is surjective. Thus the dimension of the quotient $E/(E \cap F^\perp)$ is n. Note that

$$\operatorname{span}\{u_1, u_2, \ldots, u_n\} \cap (E \cap F^\perp) = \{0\}.$$

Therefore E can be written as the direct sum of $\operatorname{span}\{u_1, u_2, \ldots, u_n\}$ and $E \cap F^\perp$. Take a basis $\{u_{n+1}, \ldots, u_m\}$ for $E \cap F^\perp$ ($\dim E = m$). Then $\{u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_m\}$ forms a basis for E. Let

$$Z := (\operatorname{span}\{u_1, u_2, \ldots, u_n\})_\perp = \{x^* \in X^* : u_k(x^*) = 0, k = 1, 2, \ldots, n\}.$$

Then Z is total over $\operatorname{span}\{u_{n+1},\dots,u_m\}$. This also implies the operator

$$R_1\colon Z\to(\operatorname{span}\{u_{n+1},\dots,u_m\})^*,\quad R_1(x^*)=x^*|_{\operatorname{span}\{u_{n+1},\dots,u_m\}}\quad(x^*\in Z)$$

is surjective. Thus there exist $x^*_{n+1},\dots,x^*_m\in Z$ such that $\{R_1(x^*_{n+1}),\dots,R_1(x^*_m)\}$ is biorthogonal to $\{u_{n+1},\dots,u_m\}$. So $\{u_{n+1},\dots,u_m\}$ is biorthogonal to $\{x^*_{n+1},\dots,x^*_m\}$. Let $F_1=\operatorname{span}\{x^*_1,x^*_2,\dots,x^*_n,x^*_{n+1},\dots,x^*_m\}$. Then F_1 contains F. Choose $x_1,x_2,\dots,x_m\in X$ such that $\{x^*_1,x^*_2,\dots,x^*_m\}$ is biorthogonal to $\{x_1,x_2,\dots,x_m\}$. Let $C>0$ be such that

$$\sum_{k=1}^m|a_k|\le C\left\|\sum_{k=1}^m a_ku_k\right\|,\quad\forall\,a_1,\dots,a_m.$$

Let $\delta>0$ be such that

$$\delta+\delta C\max_{1\le k\le m}\|u_k\|\left(\sum_{k=1}^m\|x^*_k\|\|x_k\|\right)\sqrt{m}<\varepsilon.$$

Now applying the assumption to E, F_1 and δ, we obtain an operator $T\colon X^{**}\to X^{**}$ such that $\|T\|\le\mu+\delta$, $T(E)\subset i_X(X)$ and

$$|(Tx^{**})(x^*)-x^{**}(x^*)|\le\delta\|x^{**}\|\|x^*\|$$

for all $x^{**}\in E$ and $x^*\in F_1$. Define an operator $S\colon E\to X$ by

$$Su_k=(1-Tu_k(x^*_k))x_k-\sum_{j\ne k,j=1}^m Tu_k(x^*_j)x_j,\quad k=1,2,\dots,m.$$

By the choices of $\{u_1,u_2,\dots,u_m\}$, $\{x^*_1,\dots,x^*_m\}$ and $\{x_1,x_2,\dots,x_m\}$, we get

$$(T+S)(u_k)(x^*_j)=u_k(x^*_j),\quad k,j=1,2,\dots,m.$$

This implies that

$$(T+S)(x^{**})(x^*)=x^{**}(x^*),\quad\forall\,x^{**}\in E,\ x^*\in F_1.$$

Choose a projection P from X^{**} onto E with $\|P\|\le\sqrt{m}$. Then the operator $T+SP$ satisfies

$$(T+SP)(x^{**})(x^*)=x^{**}(x^*),\quad\forall\,x^{**}\in E,\ x^*\in F;$$

and

$$(T+SP)(E)\subset X.$$

By the choices of $\{u_1, u_2, \dots, u_m\}$, $\{x_1^*, \dots, x_m^*\}$ and the inequality condition for T, we get, for all $k = 1, 2, \dots, m$,

$$\begin{aligned}\|Su_k\| &\le \|(1 - Tu_k(x_k^*))x_k\| + \left\|\sum_{j\neq k, j=1}^{m} Tu_k(x_j^*)x_j\right\| \\ &\le \delta\|x_k^*\|\|u_k\|\|x_k\| + \sum_{j\neq k, j=1}^{m} \delta\|x_j^*\|\|u_k\|\|x_j\| \\ &= \delta\|u_k\|\sum_{j=1}^{m}\|x_j^*\|\|x_j\| \\ &\le \delta \max_{1\le k\le m}\|u_k\|\sum_{j=1}^{m}\|x_j^*\|\|x_j\|.\end{aligned}$$

Thus, for all scalars $a_1, \dots, a_m$, one has

$$\begin{aligned}\left\|\sum_{k=1}^{m} a_k S u_k\right\| &\le \left(\sum_{k=1}^{m}|a_k|\right)\delta \max_{1\le k\le m}\|u_k\|\sum_{j=1}^{m}\|x_j^*\|\|x_j\| \\ &\le C\delta \max_{1\le k\le m}\|u_k\|\left(\sum_{j=1}^{m}\|x_j^*\|\|x_j\|\right)\left\|\sum_{k=1}^{m} a_k u_k\right\|.\end{aligned}$$

It follows that

$$\|S\| \le C\delta \max_{1\le k\le m}\|u_k\|\sum_{j=1}^{m}\|x_j^*\|\|x_j\|.$$

Thus

$$\|T + SP\| \le \mu + \delta + C\delta \max_{1\le k\le m}\|u_k\|\left(\sum_{j=1}^{m}\|x_j^*\|\|x_j\|\right)\sqrt{m} \le \mu + \varepsilon,$$

which completes the proof. □

To prove Theorem 1.5, we need the principle of local reflexivity in Banach lattices due to Conroy, Moore [5] and Bernau [2].

Theorem 2.1. [2, Theorem 2] *Let X be a Banach lattice and let E be a finite-dimensional sublattice of X^{**}. Then for every finite-dimensional subspace F of X^* and every $\varepsilon > 0$, there exists a lattice isomorphism R from E into $i_X(X)$ such that*

(i) $\|R\|, \|R^{-1}\| \le 1 + \varepsilon$;

(ii) $|x^{**}(x^*) - Rx^{**}(x^*)| \le \varepsilon\|x^{**}\|\|x^*\|$ *for all* $x^{**} \in E$ *and* $x^* \in F$.

We also need a lemma due to Lissitsin and Oja [16] which demonstrates the relationship between finite-dimensional subspaces and finite-dimensional sublattices in Banach lattices.

Lemma 2.2. [16, Lemma 5.5] *Let E be a finite-dimensional subspace of an order complete Banach lattice X and let $\varepsilon > 0$. Then there exist a sublattice Z of X containing E, a finite-dimensional sublattice G of Z, and a positive projection P from Z onto G such that $\|Px - x\| \leq \varepsilon \|x\|$ for all $x \in E$.*

Proof of Theorem 1.5. (a) Let E and F be finite-dimensional subspaces of X^{**} and of X^*, respectively, and let $\varepsilon > 0$ be given. We shall find an operator $U \in \mathcal{F}(X^*)_+$ with $\|U\| \leq (1+\varepsilon)\lambda\mu$ such that

$$|x^{**}(Ux^*) - x^{**}(x^*)| \leq \varepsilon \|x^{**}\| \|x^*\|$$

for all $x^{**} \in E$ and $x^* \in F$.

Choose a $\delta > 0$ so that $(1+\delta)^3 \leq 1+\varepsilon$ and

$$\delta(1+\delta)^2\lambda\mu + \delta(1+\delta)\lambda\mu + \delta(1+\delta)\mu + \delta \leq \varepsilon.$$

Since X is μ-PELR, there exists a positive operator $T\colon X^{**} \to X^{**}$ with $T(E) \subset i_X(X)$ and $\|T\| \leq (1+\delta)\mu$ such that

$$|(Tx^{**})(x^*) - x^{**}(x^*)| \leq \delta \|x^{**}\| \|x^*\|$$

for all $x^{**} \in E$ and $x^* \in F$. Since X also has the λ-BPAP, there exists an operator $S \in \mathcal{F}(X)_+$ with $\|S\| \leq \lambda$ such that

$$\|Si_X^{-1}Tx^{**} - i_X^{-1}Tx^{**}\| \leq \delta \|Tx^{**}\|$$

for all $x^{**} \in E$.

Now, consider the finite-dimensional subspace $G := T^* i_{X^*} S^*(X^*)$ of X^{***}. Then it follows from Lemma 2.2 that there exist a sublattice Z of X^{***} containing G, a finite-dimensional sublattice $\widetilde{G}$ of Z, and a positive projection P from Z onto $\widetilde{G}$ such that

$$\|Px^{***} - x^{***}\| \leq \delta \|x^{***}\|$$

for all $x^{***} \in G$. By Theorem 2.1, there exists a lattice isomorphism $R\colon \widetilde{G} \to i_{X^*}(X^*)$ with $\|R\|, \|R^{-1}\| \leq 1+\delta$ such that

$$|x^{***}(x^{**}) - (Rx^{***})(x^{**})| \leq \delta \|x^{***}\| \|x^{**}\|$$

for all $x^{***} \in \widetilde{G}$ and $x^{**} \in E$.

We define the desired map $U \in \mathcal{F}(X^*)_+$ via the composition of the following operators

$$X^* \xrightarrow{S^*} X^* \xrightarrow{i_{X^*}} X^{***} \xrightarrow{T^*} G \subset Z \xrightarrow{P} \widetilde{G} \xrightarrow{R} i_{X^*}(X^*) \xrightarrow{i_{X^*}^{-1}} X^*.$$

Then for every $x^* \in X^*$, we have

$$\|Ux^*\| \leq (1+\delta)\|PT^*i_{X^*}S^*x^*\| \leq (1+\delta)^2\|T^*i_{X^*}S^*x^*\| \leq (1+\delta)^3\lambda\mu\|x^*\|,$$

hence $\|U\| \leq (1+\varepsilon)\lambda\mu$, and for every $x^{**} \in E$ and $x^* \in F$, we also have

$$\begin{aligned}
&|x^{**}(Ux^*) - x^{**}(x^*)| \\
&= |x^{**}(i_{X^*}^{-1}RPT^*i_{X^*}S^*x^*) - x^{**}(x^*)| \\
&\leq |x^{**}(i_{X^*}^{-1}RPT^*i_{X^*}S^*x^*) - (PT^*i_{X^*}S^*x^*)(x^{**})| \\
&\quad + |(PT^*i_{X^*}S^*x^*)(x^{**}) - x^{**}(x^*)| \\
&\leq \delta\|PT^*i_{X^*}S^*x^*\|\|x^{**}\| + |(PT^*i_{X^*}S^*x^*)(x^{**}) - x^{**}(x^*)| \\
&\leq \delta(1+\delta)^2\lambda\mu\|x^{**}\|\|x^*\| + |(PT^*i_{X^*}S^*x^*)(x^{**}) - (T^*i_{X^*}S^*x^*)(x^{**})| \\
&\quad + |(T^*i_{X^*}S^*x^*)(x^{**}) - (Tx^{**})(x^*)| + |(Tx^{**})(x^*) - x^{**}(x^*)| \\
&\leq \delta(1+\delta)^2\lambda\mu\|x^{**}\|\|x^*\| + \delta(1+\delta)\lambda\mu\|x^{**}\|\|x^*\| \\
&\quad + |(S^{**}Tx^{**})(x^*) - (Tx^{**})(x^*)| + \delta\|x^{**}\|\|x^*\| \\
&\leq (\delta(1+\delta)^2\lambda\mu + \delta(1+\delta)\lambda\mu + \delta(1+\delta)\mu + \delta)\|x^{**}\|\|x^*\| \\
&\leq \varepsilon\|x^{**}\|\|x^*\|.
\end{aligned}$$

Then the same argument as in the proof of [11, Theorem 3.1] shows that

$$\mathrm{id}_{X^*} \in \overline{\{S \in \mathcal{F}(X^*)_+ : \|S\| \leq \lambda\mu\}}^{\tau_{wo}},$$

where τ_{wo} is the *weak operator topology*. Put $\mathcal{A} := \{S \in \mathcal{F}(X^*)_+ : \|S\| \leq \lambda\mu\}$. We have

$$\overline{\mathcal{A}}^{\tau_{wo}} = \overline{\mathcal{A}}^{\tau_{sto}} = \overline{\mathcal{A}}^{\tau_c},$$

where τ_{sto} is the *strong operator topology*. The first equality follows from $(\mathcal{L}(X^*), \tau_{wo})^* = (\mathcal{L}(X^*), \tau_{sto})^*$ (cf. [7, p. 477, Theorem 4]) and the convexity of the set $\mathcal{A}$, and the second one follows from the uniform boundedness of the set $\mathcal{A}$. Hence we complete the proof of (a).

(b) Let E and F be finite-dimensional subspaces of X^{**} and X^*, respectively, and let $\varepsilon > 0$ be given. Choose a $\delta > 0$ so that $(1+\delta)^2 < 1+\varepsilon$ and

$$\delta(1+\delta)\lambda + \delta\lambda + \delta < \varepsilon.$$

Since X^* has the λ-BPAP, there exists $U \in \mathcal{F}(X^*)_+$ with $\|U\| \leq \lambda$ such that

$$\|Ux^* - x^*\| \leq \delta\|x^*\|$$

for all $x^* \in F$. By Lemma 2.2 there exist a sublattice Z of X^{**} containing $E_0 := U^*(X^{**})$, a finite-dimensional sublattice G of Z, and a positive projection P from Z onto G such that

$$\|Px^{**} - x^{**}\| \leq \delta\|x^{**}\|$$

for all $x^{**} \in E_0$. It follows from Theorem 2.1 that there exists a lattice isomorphism $R\colon G \to i_X(X)$ with $\|R\|, \|R^{-1}\| \leq 1+\delta$ such that

$$|x^{**}(x^*) - (Rx^{**})(x^*)| \leq \delta\|x^{**}\|\|x^*\|$$

for all $x^{**} \in G$ and $x^* \in F$. Then the map $T := RPU^*$ is a positive finite rank operator from X^{**} to $i_X(X)$.

Now, for every $x^{**} \in X^{**}$, we have

$$\|Tx^{**}\| \leq (1+\delta)\|PU^*x^{**}\| \leq (1+\delta)^2\|U^*x^{**}\| \leq (1+\varepsilon)\lambda\|x^{**}\|,$$

hence $\|T\| \leq (1+\varepsilon)\lambda$, and for every $x^{**} \in E$ and $x^* \in F$, we also have

$$\begin{aligned}
&|(Tx^{**})(x^*) - x^{**}(x^*)| \\
&\leq |(RPU^*x^{**})(x^*) - (PU^*x^{**})(x^*)| + |(PU^*x^{**})(x^*) - x^{**}(x^*)| \\
&\leq \delta\|PU^*x^{**}\|\|x^*\| + |(PU^*x^{**})(x^*) - (U^*x^{**})(x^*)| + |(U^*x^{**})(x^*) - x^{**}(x^*)| \\
&\leq \delta(1+\delta)\|U^*x^{**}\|\|x^*\| + \delta\|U^*x^{**}\|\|x^*\| + \delta\|x^{**}\|\|x^*\| \\
&\leq (\delta(1+\delta)\lambda + \delta\lambda + \delta)\|x^{**}\|\|x^*\| \\
&\leq \varepsilon\|x^{**}\|\|x^*\|.
\end{aligned}$$

Hence X is λ-PELR.

In order to prove the second part, let E and F be finite-dimensional subspaces of X and X^*, respectively, and let $\varepsilon > 0$ be given. Let U, P, and R be the operators in the proof of the first part. Then $S := i_X^{-1}RPU^*i_X \in \mathcal{F}(X)_+$ and as in the proof of the first part, we have $\|S\| \leq (1+\varepsilon)\lambda$ and

$$|x^*(Sx) - x^*(x)| \leq \varepsilon\|x^*\|\|x\|$$

for every $x \in E$ and $x^* \in F$. We can now use the argument in the proof of (a) to complete the proof of the second part. $\square$

3. Isometric factorization of positive weakly compact operators and positive approximation properties

We need an isometric lattice version of the Davis-Figiel-Johnson-Pelczynski factorization lemma (DFJP factorization lemma) [6] to characterize positive approximation properties. A subset A of a Banach lattice X is said to be *solid* whenever $|x| \leq |y|$ ($x \in X$, $y \in A$) implies that $x \in A$. The solid hull $sol(A)$ of a subset A of X is the smallest solid set that contains A. A solid linear subspace of X is called an *ideal* of X. Recall that an operator $T\colon Z \to X$ between two Banach lattices is said to be:

(a) *interval preserving* whenever $T[0,z]=[0,Tz]$ holds for all $z\in Z_+$;

(b) a *lattice homomorphism* whenever $T(z_1\vee z_2)=Tz_1\vee Tz_2$ holds for all $z_1,z_2\in Z$.

Aliprantis and Burkinshaw [1] established a lattice version of DFJP factorization lemma as follows.

Theorem 3.1. [1, Theorem 1.7] *Let K be a convex, solid and norm bounded subset of a Banach lattice X. For each n, put $U_n:=2^nK+2^{-n}B_X$, and denote by $\|\cdot\|_n$ the Minkowski functional of U_n, i.e.,*

$$\|x\|_n:=\inf\{\alpha>0:x\in\alpha U_n\},\quad x\in X.$$

Set $Z:=\left\{x\in X:|||x|||:=\left(\sum_{n=1}^{\infty}\|x\|_n^2\right)^{1/2}<\infty\right\}$, and let $J\colon Z\to X$ be the inclusion map. Then

(a) *$(Z,|||\cdot|||)$ is a Banach lattice and Z is an ideal of X;*

(b) *$K\subset B_Z$;*

(c) *Both J and J^* are interval preserving lattice homomorphisms;*

(d) *$(Z,|||\cdot|||)$ is reflexive if and only if K is relatively weakly compact.*

Just following the same arguments as in [14], we obtain the isometric version of Theorem 3.1 as follows.

Corollary 3.2. *Let X be a Banach lattice and let K be a closed, convex and solid subset of the unit ball B_X of X. Then there exists a Banach lattice Z such that*

(a) *$K\subset B_Z$;*

(b) *Z is an ideal of X;*

(c) *the inclusion map $J\colon Z\to X$ is positive and $\|J\|\le 1$;*

(d) *Z is reflexive if and only if K is weakly compact.*

Corollary 3.2 yields the following result.

Corollary 3.3. *Let T be an operator from a Banach lattice Y to a Banach lattice X such that $sol(T(B_Y)/\|T\|)$ is relatively weakly compact in X. Let $K:=\overline{co}(sol(T(B_Y)/\|T\|))$ and let Z be the Banach lattice associated with the set K in Corollary 3.2. Define the map $R\colon Y\to Z$ by $Ry=Ty$ $(y\in Y)$. Then we have that*

(a) *the maps R and J are weakly compact operators;*

(b) *R is positive if T is positive;*

(c) *$\|T\| = \|R\|$ and $\|J\| = 1$;*

(d) *$T = JR$.*

Lemma 3.4. *Let X be a Banach lattice and let K be a weakly compact, convex and solid subset of B_{X^*}. Let Z be the Banach lattice associated with the set K in Corollary* 3.2. *Then*

$$(Z^*)_+ = \overline{J^* i_X(X_+)}^{weak^*},$$

where $J\colon Z \to X^*$ is the inclusion map.

Proof. Suppose not. Then there exists a $z_0^* \in (Z^*)_+ \setminus \overline{J^* i_X(X_+)}^{weak^*}$. By the separation theorem there exist a $z_0 \in Z$ and a real number c such that

$$J(z_0)(x) = J^* i_X(x)(z_0) < c < z_0^*(z_0)$$

for all $x \in X_+$. Then we see that $c > 0$ and $J(z_0) \le 0$ in X^*. Since $J\colon Z \to X^*$ is one-to-one and lattice preserving, $z_0 \le 0$ in Z and so $0 < c < z_0^*(z_0) \le 0$. This is a contradiction. □

Recall that a Banach lattice X is called a *Kantorovič-Banach space* (KB-space) if every increasing norm bounded sequence in X_+ is norm convergent. It should be noted that the convex solid hull of a relatively weakly compact subset in a Banach lattice need not be relatively weakly compact [17, p. 108]. However, in KB-spaces, the convex solid hull of a relatively weakly compact subset remains relatively weakly compact [17, Proposition 2.5.12]. We refer to [17] for more about KB-spaces.

The next well-known result in Banach lattice theory, which gives a method to perturb positive finite rank operators on Banach lattices, preserving positivity, will be very useful for us in the sequel. This is quite different from Banach space case. Let X and Y be Banach spaces. For $T \in \mathcal{F}(X,Y)$, the nuclear norm of T is defined by

$$\nu_0(T) := \inf\left\{\sum_{k=1}^{m} \|x_k^*\|\|y_k\| : T = \sum_{k=1}^{m} x_k^* \otimes y_k, \{x_k^*\}_{k=1}^m \subset X^*, \{y_k\}_{k=1}^m \subset Y, m \in \mathbb{N}\right\}.$$

Lemma 3.5. [24, p. 148] *Let X and Y be Banach lattices. Then the set*

$$\left\{\sum_{k=1}^{m} x_k^* \otimes y_k : \{x_k^*\}_{k=1}^m \subset (X^*)_+, \{y_k\}_{k=1}^m \subset Y_+, m \in \mathbb{N}\right\}$$

is ν_0-dense in $\mathcal{F}(X,Y)_+$.

We also need two results, which are slightly stronger than the positive versions of [23, Corollaries 4.3 and 4.4]. They enable to represent positive finite rank operators $U\colon X \to Y$ in the special form JS or SJ, where $J\colon X \to Y$ comes from the isometric lattice version of the DFJP factorization lemma and S are positive finite rank operators on X or Y. They are of independent interest. Their proofs use basic Banach lattice theory.

Proposition 3.6. *Let X be a Banach lattice and let K be a weakly compact, convex and solid subset of B_{X^*}. Let Z be the Banach lattice associated with the set K in Corollary 3.2. Then we have*

$$\mathcal{F}(X, Z^*)_+ \subset \overline{\{J^* i_X S : S \in \mathcal{F}(X)_+\}}^{\nu_0},$$

where $J\colon Z \to X^*$ is the inclusion map.

Proof. By Lemma 3.4 and the reflexivity of Z, we get

$$(Z^*)_+ = \overline{J^* i_X(X_+)}^{weak^*} = \overline{J^* i_X(X_+)}^{weak} = \overline{J^* i_X(X_+)}.$$

Now, let $T = \sum_{k=1}^m x_k^* \otimes z_k^* \in \mathcal{F}(X, Z^*)_+$ and let $\varepsilon > 0$ be given. By Lemma 3.5, we may assume that $x_k^* \in (X^*)_+$ and $z_k^* \in (Z^*)_+$ for all $k = 1, \ldots, m$. Choose a $\delta > 0$ so that $\delta\big(\sum_{k=1}^m \|x_k^*\|\big) \leq \varepsilon$. Then for each $1 \leq k \leq m$, there exists an $x_k \in X_+$ such that

$$\|z_k^* - J^* i_X(x_k)\| \leq \delta.$$

Define $S \in \mathcal{F}(X)_+$ by

$$S = \sum_{k=1}^m x_k^* \otimes x_k.$$

Then we have

$$\nu_0(T - J^* i_X S) \leq \sum_{k=1}^m \|x_k^*\| \|z_k^* - J^* i_X(x_k)\| \leq \varepsilon.$$ □

Proposition 3.7. *Let X be a Banach lattice and let K be a weakly compact, convex and solid subset of B_X. Let Z be the Banach lattice associated with K in Corollary 3.2. Then for every Banach lattice Y, we have*

$$\mathcal{F}(Z, Y)_+ \subset \overline{\{SJ : S \in \mathcal{F}(X, Y)_+\}}^{\nu_0},$$

where $J\colon Z \to X$ is the inclusion map.

Proof. As in the proof of Proposition 3.6, we have $(Z^*)_+ = \overline{J^*((X^*)_+)}$. Now, let $T = \sum_{k=1}^m z_k^* \otimes y_k \in \mathcal{F}(Z, Y)_+$ and let $\varepsilon > 0$ be given. By Lemma 3.5, we may assume that $z_k^* \in (Z^*)_+$ and $y_k \in Y_+$ for all $k = 1, \ldots, m$. Choose a $\delta > 0$ so that $\delta\big(\sum_{k=1}^m \|y_k\|\big) \leq \varepsilon$. Then for each $k = 1, \ldots, m$, there exists an $x_k^* \in (X^*)_+$ such that

$$\|z_k^* - J^*(x_k^*)\| \leq \delta.$$

Define $S \in \mathcal{F}(X,Y)_+$ by

$$S = \sum_{k=1}^{m} x_k^* \otimes y_k.$$

Then we have

$$\nu_0(T - SJ) \le \sum_{k=1}^{m} \|z_k^* - J^*(x_k^*)\| \|y_k\| \le \varepsilon. \qquad \square$$

Theorem 3.8. *Suppose that X^* is a KB-space. The following statements are equivalent.*

(a) *X has the PAP.*

(b) *For every Banach lattice Y and every $T \in \mathcal{L}(X,Y)_+$, we have*

$$T \in \overline{\{TS : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

(c) *For every reflexive Banach lattice Y, $\mathcal{W}(X,Y)_+ \subset \overline{\mathcal{F}(X,Y)_+}^{\tau_c}$.*

(d) *For every Banach lattice Y and every $T \in \mathcal{W}(X,Y)_+$, we have*

$$T \in \overline{\{TS : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

Proof. Simple verifications show (a) $\Leftrightarrow$ (b). It remains to prove that (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a).

(c) $\Rightarrow$ (d). We follow the argument in the proof of [23, Proposition 4.6]. Let Y be a Banach lattice and let $T \in \mathcal{W}(X,Y)_+$ (in fact, an arbitrary weakly compact operator). Since, in KB-spaces, a relatively weakly compact set has a relatively weakly compact solid hull, we can apply Corollary 3.3 to the subset $K := \overline{co}(sol(T^*(B_{Y^*})/\|T\|))$ of X^*. Then there exists a reflexive Banach lattice Z, which is an ideal of X^*, and an operator $R\colon Y^* \to Z$ such that $T^* = JR$, where $J\colon Z \to X^*$ is the positive inclusion map. Since $J^* i_X \in \mathcal{W}(X, Z^*)_+$ and ν_0 is stronger than τ_c, it follows from (c) and Proposition 3.6 that

$$J^* i_X \in \overline{\{J^* i_X S : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

Thus there exists a net (S_α) in $\mathcal{F}(X)_+$ such that

$$J^* i_X S_\alpha \xrightarrow{\tau_c} J^* i_X.$$

Since $i_Y T S_\alpha = T^{**} S_\alpha^{**} i_X = R^* J^* i_X S_\alpha$ and $i_Y T = T^{**} i_X = R^* J^* i_X$, we have

$$i_Y T S_\alpha \xrightarrow{\tau_c} i_Y T.$$

Hence

$$T S_\alpha \xrightarrow{\tau_c} T.$$

(d) $\Rightarrow$ (a). Let $f \in (\mathcal{L}(X), \tau_c)^*$ be arbitrary. We prove that $f(\mathrm{id}_X) \leq \sup\{f(S) : S \in \mathcal{F}(X)_+\}$. Then by an application of the separation theorem we are done.

Now, by Grothendieck's description of $(\mathcal{L}(X), \tau_c)^*$ [9] (see, e.g., [15, Proposition 1.e.3]) there exist sequences (x_n) and (x_n^*) in X and X^*, respectively, with $\sum_n \|x_n\|\|x_n^*\| < \infty$ such that

$$f(U) = \sum_{n=1}^{\infty} x_n^*(Ux_n)$$

for every $U \in \mathcal{L}(X)$. We may assume that $\|x_n^*\| \leq 1$ for every n, $\|x_n^*\| \to 0$ ($n \to \infty$), and $\sum_n \|x_n\| < \infty$. Consider the weakly compact convex and solid subset $K := \overline{co}(sol(\{x_n^*\}_{n=1}^{\infty}))$ of B_{X^*}. Then by Corollary 3.2 there exists a reflexive Banach lattice Z, which is an ideal of X^*, with $K \subset B_Z$ such that the inclusion map $J\colon Z \to X^*$ is a positive weakly compact operator and $\|J\| = 1$. By (d) we have

$$J^* i_X \in \overline{\{J^* i_X S : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

Since $K \subset B_Z$, there exists a sequence $(z_n)_n$ in B_Z such that $J(z_n) = x_n^*$ for each n. Define $g := \sum_n z_n \otimes x_n \in (\mathcal{L}(X, Z^*), \tau_c)^*$. It is clear that $f(\mathrm{id}_X) = g(J^* i_X)$. Thus

$$\begin{aligned} f(\mathrm{id}_X) &\leq \sup\{g(J^* i_X S) : S \in \mathcal{F}(X)_+\} \\ &= \sup\left\{\sum_{n=1}^{\infty} (J^* i_X S x_n)(x_n^*) : S \in \mathcal{F}(X)_+\right\} \\ &= \sup\{f(S) : S \in \mathcal{F}(X)_+\}. \end{aligned}$$

□

Theorem 3.9. *Suppose that X is a KB-space. The following statements are equivalent.*

(a) *X has the PAP.*

(b) *For every Banach lattice Y and every $T \in \mathcal{L}(Y, X)_+$, we have*

$$T \in \overline{\{ST : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

(c) *For every (reflexive) Banach lattice Y, $\mathcal{W}(Y, X)_+ \subset \overline{\mathcal{F}(Y, X)_+}^{\tau_c}$.*

(d) *For every Banach lattice Y and every $T \in \mathcal{W}(Y, X)_+$, we have*

$$T \in \overline{\{ST : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

Proof. We only prove that (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a).

(c) $\Rightarrow$ (d). Let Y be a Banach lattice and let $T \in \mathcal{W}(Y, X)_+$ (in fact, an arbitrary weakly compact operator). We can apply Corollary 3.3 to the weakly compact subset $K := \overline{co}(sol(T(B_Y)/\|T\|))$ of X. Then there exists a reflexive Banach lattice Z, which is an ideal of X, and an operator $R\colon Y \to Z$ such that $T = JR$, where $J\colon Z \to X$ is the positive inclusion map. By (c) and Proposition 3.7 we have

$$J \in \overline{\{SJ : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

Thus there exists a net (S_α) in $\mathcal{F}(X)_+$ such that

$$S_\alpha J \xrightarrow{\tau_c} J.$$

Hence

$$S_\alpha T = S_\alpha JR \xrightarrow{\tau_c} JR = T.$$

(d) $\Rightarrow$ (a). We follow the argument in the proof of Theorem 3.8(d)$\Rightarrow$(a). Let $f = \sum_n x_n^*(\cdot\, x_n)$ be arbitrary in $(\mathcal{L}(X), \tau_c)^*$, where $\sum_n \|x_n\|\|x_n^*\| < \infty$. We may assume that $\|x_n\| \leq 1$ for every n, $\|x_n\| \to 0$ $(n \to \infty)$, and $\sum_n \|x_n^*\| < \infty$. Consider the weakly compact convex and solid subset $K := \overline{co}(sol(\{x_n\}_{n=1}^\infty))$ of B_X. Then by Corollary 3.2 there exists a reflexive Banach lattice Z, which is an ideal of X, with $K \subset B_Z$ such that the inclusion map $J\colon Z \to X$ is a positive weakly compact operator and $\|J\| = 1$. By (d) we have

$$J \in \overline{\{SJ : S \in \mathcal{F}(X)_+\}}^{\tau_c}.$$

Since $K \subset B_Z$, I find a sequence $(z_n)_n$ in B_Z such that $J(z_n) = x_n$ for each n. Define $g := \sum_n x_n^* \otimes z_n \in (\mathcal{L}(Z, X), \tau_c)^*$ and it is obvious that $f(\mathrm{id}_X) = g(J)$.

Thus

$$\begin{aligned} f(\mathrm{id}_X) &\leq \sup\{g(SJ) : S \in \mathcal{F}(X)_+\} \\ &= \sup\{f(S) : S \in \mathcal{F}(X)_+\}. \end{aligned}$$ □

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