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Blow-up Solution for 4-dimensional Generalized Emden-Fowler Equation with Exponential Nonlinearity

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Abstract. Using some nonlinear domain decomposition method, we prove the existence of singular limits for solution of generalized Emden-Fowler equation with exponential nonlinearity in fourth-dimensional given by

$$\begin{cases} \Delta(a(x)\Delta u) - V(x)\operatorname{div}(a(x)\nabla u) = \rho^4 a(x) e^u & \text{in } \Omega \subset \mathbb{R}^4, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

The leading part Δ is, usually, called Laplacian operator. The potential $V(x)$ belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^4)$ it is smooth and bounded and $a = a(x)$ is a given smooth function over $\overline{\Omega}$, called the Schrödinger wave function. Namely, we are still looking for solutions which concentrate at the points $x^j \in \Omega$, $j = 1, \ldots, m$ as the parameter ρ tends to 0. We find sufficient conditions under which, as ρ tend to 0, there exists an explicit class of solutions which admit a concentration behavior with a prescribed bubble profile around some given m-points in Ω, for any given integer m. These are the so-called singular limits. The candidate m-points of concentration must be *nondegenerate* (in essential way) critical points of a suitable finite dimensional functional explicitly and the higher order Green's function with respect to the imposed boundary conditions.

1. Introduction and statement of the results

Let Ω be a regular bounded open domain in $\mathbb{R}^4$. We are interested in positive solutions of the generalized Emden-Fowler equation with exponential nonlinearity:

$$\begin{cases} \Delta(a(x)\Delta u) - V(x)\operatorname{div}(a(x)\nabla u) = \rho^4 a(x) e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

when the parameter ρ tends to 0 verifying $\rho^4 = 384\varepsilon^4/(1+\varepsilon^2)^4 \sim \varepsilon^4$ as ε tends to 0, the potential V belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^4)$ and it is smooth and bounded, the function $a = a(z)$ is

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a given smooth function over $\overline{\Omega}$, called the Schrödinger wave function, solution of the so called linear form of stationary smooth nonhomogenous Schrödinger problem

$$\text{(1.2)} \qquad \begin{cases} -\Delta a(z) + V(z)a(z) = \lambda f(z,a) & \text{in } \overline{\Omega}, \\ \|\nabla a\|_\infty \le \gamma \end{cases}$$

satisfying

(H) $0 < c_1 \le a(z) \le c_2 < \infty$,

λ and γ are small parameters and f is a smooth bounded function over $\overline{\Omega}$.

The existence and multiplicity of nontrivial solutions $a(z)$ of problem (1.2) have been extensively investigated in the literature, for various conditions of the potential $V(x)$ and the nonlinearity f. Precisely, in [18], the author presents general results on exponential decay of finite energy solutions to stationary nonlinear Schrödinger equations (1.2). Under certain natural assumptions, he shows that any such solution is continuous and vanishes at infinity and the solution is interpret as a finite multiplicity eigenfunction of a certain linear Schrödinger operator and, hence, apply well-known results on the decay of eigenfunctions. An interesting case is when the nonlinearity of f is supposed to satisfy the following assumption: f is a Carathéodory function, a.e. it is Lebesgue measurable with respect to $z \in \mathbb{R}^4$ for all $a \in \mathbb{R}$ and continuous with respect to $a \in \mathbb{R}$ for almost all $z \in \mathbb{R}^4$. Furthermore,

$$|f(z,a(z))| \le c(1 + |a(z)|^3), \quad z \in \mathbb{R}^4,\ a \in \mathbb{R}$$

with $c > 0$ and the potential V belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^4)$. (For more details about asymptotical behaviors (exponential decay) of the solution $a(z)$ of problem (1.2), see [18, Lemma 1, Theorems 2, 5, 6] and some references therein. See also [2,3]).

To describe our result, let us denote by

$$\Delta_a^2 - \Delta_a = \frac{1}{a}\Delta(a\Delta u) - \frac{V(x)}{a}\operatorname{div}(a\nabla u) = \Delta^2 u + \Sigma_a^1 u + \Sigma_a^2 u$$

where

$$\Sigma_a^1 u = 2\frac{\nabla a}{a}\cdot\nabla(\Delta u) - V(x)\nabla\log a\cdot\nabla u \quad \text{and} \quad \Sigma_a^2 u = \left(\frac{\Delta a}{a} - V(x)\right)\Delta u.$$

Then to solve (1.1) is equivalent to solve the following

$$\begin{cases} \Delta_a^2 u - \Delta_a u = \rho^4 e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

when the parameter ρ tends to 0. Denote by $G_a(x,\cdot)$ the solution of

$$\begin{cases} \Delta^2 G_a(x,\cdot) = 64\pi^2\delta_x & \text{in } \Omega, \\ G_a(x,\cdot) = \Delta G_a(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to check that the function

$$R_a(x,y) := G_a(x,y) + 8\log|x-y|$$

is a smooth function. We define the set of blow-up as

$$S := \{x \in \Omega \mid \exists\, x_n \to x \text{ such that } u_n(x_n) \to +\infty\}$$

and

$$W(x^1,\dots,x^m) := \sum_{j=1}^{m} R(x^j,x^j) + \sum_{j\neq\ell} G(x^j,x^\ell) \tag{1.3}$$

for $x^1, x^2, \dots, x^m$ m-points in Ω.

Obviously, the application of the implicit function theorem yields the existence of a smooth one parameter family of solutions $(u_{\rho,\lambda,\gamma})$ which converges uniformly to 0 as ρ tends to 0, taking into account that λ and γ are small enough. This branch of solutions is usually referred to as the branch of *minimal solutions* and there is by now quite an important literature which is concerned with the understanding this particular branch of solutions [13]. The question we would like to study is concerned with the existence of other branches of solutions of (1.1) as ρ tends to 0 with λ and γ are small enough.

Problem $(1.1)_{|V=0}$ has been treated in [15] in radial case. Problem (1.1) without the terms $\Delta(a(x)\Delta u)$ has been studied by many authors, see [6, 21]. Problem (1.1) with $a = cte$ becomes:

$$\begin{cases} \Delta^2 u - V(x)\Delta u = \rho^4 e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Problem $(1.4)_{|V=0}$ has treated by [1] as ρ tends to 0. Recently in [16] the authors have treated problem $(1.4)_{V=\gamma}$ with nonlinear dissipative q-gradient terms $\lambda|\nabla u|^q$, for $q \in [1,4]$ as ρ, λ and γ tend to 0 under some hypothesis for $q = 4$. In [7] Chen and McKenna have suggested to investigate the following equation

$$u_{xxxx} + \widetilde{c}u_{xx} = e^u, \tag{1.5}$$

where they give some existence and nonexistence results. In a note on an exponential semilinear equation of the fourth order, D. Mugnai, in [14], considers the related problem to (1.5). More precisely he takes the problem:

$$\begin{cases} \Delta^2 u + \widetilde{c}\Delta u = b(e^u - 1) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^n$, $\widetilde{c} \in \mathbb{R}$ and $b \in \mathbb{R}$. The author proves some existence and nonexistence results for (1.6) via variational techniques. Such equations may occur while studying traveling waves in suspension bridges. For more general problem, see [19], for the following Navier boundary value problem:

$$\begin{cases} \Delta^2 u + \widetilde{c}\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

in $\mathbb{R}^n$, $n \geq 4$ and f is nonlinear growth function. In conformal dimensional i.e., $n = 4$ and f has the subcritical (exponential) growth on Ω, i.e.,

$$\lim_{t\to+\infty} \frac{|f(x,t)|}{\exp(\alpha t)} = 0$$

uniformly on $x \in \Omega$ for all $\alpha > 0$ and in some cases and hypothesis and using Adams inequality, (see [10]), for the fourth-order derivative, namely,

$$\sup_{\{u\in H^2(\Omega)\cap H_0^1(\Omega), \|u\|\leq 1\}} \int_\Omega e^{32\pi^2 u^2}\, dx \leq C|\Omega|,$$

the authors show that the problem (1.7) has at least two nontrivial solutions (for more details see Theorem 1.3 in [10]) or infinitely many nontrivial solutions (for more details see Theorem 1.4 in [10]). Many papers have been devoted to the case $(a, V) = (cte, 0)$, where the problem (1.1) becomes

$$\begin{cases} \Delta^2 u = \rho^4 e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

when the parameter ρ tends to 0 (see for example [1]). Semilinear equations involving fourth order elliptic operator and exponential nonlinearity appear naturally in conformal geometry and in particular in the prescription of the so called Q-curvature on 4-dimensional Riemannian manifolds [4, 5]

$$Q_g = \frac{1}{12}\left(-\Delta_g S_g + S_g^2 - 3|\operatorname{Ric}_g|^2\right)$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g. Recall that the Q-curvature changes under a conformal change of metric

$$g_w = e^{2w} g,$$

according to

$$P_g w + 2Q_g = 2\widetilde{Q}_{g_w} e^{4w} \tag{1.8}$$

where

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}S_g I - 2\,\mathrm{Ric}_g\right) d$$

is the Panietz operator, which is an elliptic 4-th order partial differential operator [5] and which transforms according to

$$e^{4w} P_{e^{2w}g} = P_g,$$

under a conformal change of metric $g_w := e^{2w}g$. In the special case where the manifold is the Euclidean space, the Panietz operator is simply given by

$$P_{g_{\mathrm{eucl}}} = \Delta^2$$

in which case (1.8) reduces to

$$\Delta^2 w = \widetilde{Q} e^{4w}$$

the solutions of which give rise to conformal metric $g_w = e^{2w} g_{\mathrm{eucl}}$ whose Q-curvature is given by $\widetilde{Q}$. There is by now an extensive literature about this problem and we refer to [5, 8] for references and recent developments. In dimension 4, Wei in [20], have studied the behavior of solutions to the following nonlinear eigenvalue problem for the biharmonic operator Δ^2 in $\mathbb{R}^4$. More precisely, consider the following problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

and u^* the solution of

$$\begin{cases} \Delta^2 u^* = 64\pi^2 \sum_{i=1}^m \delta_{x^i} & \text{in } \Omega, \\ u^* = \Delta u^* = 0 & \text{on } \partial\Omega. \end{cases}$$

The author proved the following result.

Theorem 1.1. [20] *Let Ω be a smooth bounded domain in $\mathbb{R}^4$ and f a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \text{ and } e^{-u}\int_0^u f(s)\,ds \quad \text{tend to } 1 \text{ as } u \to +\infty.$$

For u_λ solution of (1.9), denote by $\Sigma_\lambda = \lambda\int_\Omega f(u_\lambda)\,dx$. Then many cases occur:

(i) $\Sigma_\lambda \to 0$ *therefore,* $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ *as* $\lambda \to 0$.

(ii) $\Sigma_\lambda \to +\infty$ *then* $u_\lambda \to +\infty$ *as* $\lambda \to 0$.

(iii) $\Sigma_\lambda \to 64\pi^2 m$ *for some positive integer m. Then the limiting function $u^* = \lim_{\lambda\to 0} u_\lambda$ has m blow-up points, $\{x^1, \ldots, x^m\}$, where $u_\lambda(x^i) \to +\infty$ as $\lambda \to 0$. Moreover, $(x^1, \ldots, x^m)$ is a critical point of W.*

Our main result reads:

Theorem 1.2. *Let $\alpha \in (0,1)$ and Ω be an open smooth bounded domain in $\mathbb{R}^4$ and $S = \{x^1, \ldots, x^m\} \subset \Omega$ be a nonempty set. Assume that $V(x)$ is smooth bounded potential and $a(x)$ is a smooth function, over $\overline{\Omega}$, solution of the linear form of stationary smooth nonhomogenous Schrödinger problem*

$$\begin{cases} -\Delta a(x) + V(x)a(x) = \lambda f(x,a) & \text{in } \overline{\Omega}, \\ \|\nabla a\|_\infty \leq \gamma \end{cases}$$

and satisfying (H) *and $(x^1, \ldots, x^m)$ is a nondegenerate critical point of W, then there exist $\rho_0 > 0$, $\lambda_0 > 0$, $\gamma_0 > 0$ and $\{u_{\rho,\lambda,\gamma}\}_{0<\rho<\rho_0, 0<\lambda<\lambda_0, 0<\gamma<\gamma_0}$ a family of solutions of*

$$\begin{cases} \Delta(a(x)\Delta u) - V(x)\operatorname{div}(a(x)\nabla u) = \rho^4 a(x) e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$\lim_{\rho,\lambda,\gamma \to 0} u_{\rho,\lambda,\gamma} = \sum_{j=1}^{m} G_a(x^j, \cdot\,)$$

in $\mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega - \{x^1, \ldots, x^m\})$.

Our result reduces the study of nontrivial branches of solutions of (1.1) to the search for critical points of the function W defined in (1.3). Observe that the assumption on the nondegeneracy of the critical point is a rather mild assumption since it is certainly fulfilled for generic choice of the open domain Ω.

We briefly describe the plan of the paper: In Section 2 we discuss rotationally symmetric solutions of (1.1) and we recall some studies about the linearized operator around the radially symmetric solution. In Section 3, we recall some Known results about the analysis of the bi-Laplace operator in weighted spaces. Both section strongly use the b-operator which has been developed by Melrose [12] in the context of weighted Sobolev spaces and by Mazzeo [11] in the context of weighted Hölder spaces (see also [17]). A first nonlinear problem is studied in Section 4 where the existence of an infinite dimensional family of solutions of (1.1) which are defined on a large ball and which are close to the rotationally symmetric solution is proven. In Section 5, we prove the existence of an infinite dimensional family of solutions of (1.1) which are defined on Ω with small ball removed. Finally, in Section 6, we show how elements of these infinite dimensional families can be connected to produce solutions of (1.1) described in Theorem 1.2. This last section borrows ideas from applied mathematics were domain decomposition methods are of common use. Throughout the paper, the symbol $c_\kappa > 0$ (which can depend only on κ) denotes always a positive constant independent of ε, λ and γ which might change from one line to another.

2. Rotationally symmetric solutions and linear fourth order elliptic operator on $\mathbb{R}^4$

We first describe the rotationally symmetric solutions of

$$\Delta_a^2 u - \Delta_a u - \rho^4 e^u = 0.$$

Recall that, given $\varepsilon > 0$, then

$$u_\varepsilon(x) := 4\log(1+\varepsilon^2) - 4\log(\varepsilon^2 + |x|^2)$$

is a solution of

$$\Delta^2 u = \rho^4 e^u \tag{2.1}$$

when

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Let us notice that equation (2.1) is invariant under some dilation in the following sense: If u is a solution of (2.1) and if $\tau > 0$, then $u(\tau\,\cdot\,) + 4\log\tau$ is also a solution of (2.1). With this observation in mind, we define, for all $\tau > 0$

$$u_{\varepsilon,\tau}(x) := 4\log(1+\varepsilon^2) + 4\log\tau - 4\log(\varepsilon^2 + \tau^2|x|^2).$$

With these notations we define the linear fourth order elliptic operator

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+|x|^2)^4}$$

which corresponds to the linearization of (2.1) about the solution u_1 ($= u_{\varepsilon=1}$).

We are interested in the classification of bounded solutions of $\mathbb{L}w = 0$ in $\mathbb{R}^4$. Some solutions are easy to find. For example, we can define

$$\phi_0(x) := r\partial_r u_1(x) + 4 = 4\frac{1-r^2}{1+r^2},$$

where $r = |x|$. Clearly $\mathbb{L}\phi_0 = 0$ and this reflects the fact that (2.1) is invariant under the group of dilations $\tau \to u(\tau\,\cdot\,) + 4\log\tau$. We also define, for $i = 1, \ldots, 4$,

$$\phi_i(x) := -\partial_{x_i} u_1(x) = \frac{8x_i}{1+|x|^2},$$

which are also solutions of $\mathbb{L}\phi_j = 0$ since these solutions correspond to the invariance of the equation under the group of translations $a \to u(\,\cdot + a)$.

The following result classifies all bounded solutions of $\mathbb{L}w = 0$ which are defined in $\mathbb{R}^4$.

Lemma 2.1. [1] *Any bounded solution of $\mathbb{L}w = 0$ defined in $\mathbb{R}^4$ is a linear combination of ϕ_i for $i = 0, 1, \ldots, 4$.*

Let B_r denote the ball of radius r centered at the origin in $\mathbb{R}^4$.

Definition 2.2. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^4)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1)} + \sup_{r\geq 1}\Big((1+r^2)^{-\mu/2}\|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1-B_{1/2})}\Big)$$

is finite.

More details about these spaces and their use in nonlinear problems can be found in [17]. Roughly speaking, functions in $\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\mu/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{\mu/2}$ for $\ell = 1,\ldots,k+\alpha$. We also define

$$\mathcal{C}^{k,\alpha}_{\mathrm{rad},\mu}(\mathbb{R}^4) = \{f \in \mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^4) \mid f(x) = f(|x|), \forall\, x \in \mathbb{R}^4\}.$$

As a consequence of the result of Lemma 2.1, we have

Proposition 2.3. [1] (i) *Assume that $\mu > 1$ and $\mu \notin \mathbb{N}$, then*

$$\begin{array}{rccc} L_\mu\colon & \mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\mu-4}(\mathbb{R}^4) \\ & w & \longmapsto & \mathbb{L}w \end{array}$$

is surjective.

(ii) *Assume that $\delta > 0$ and $\delta \notin \mathbb{N}$ then*

$$\begin{array}{rccc} L_{\mathrm{rad},\delta}\colon & \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\mathrm{rad},\delta-4}(\mathbb{R}^4) \\ & w & \longmapsto & \mathbb{L}w \end{array}$$

is surjective.

We set $\overline{B}^*_1 = \overline{B}_1 - \{0\}$.

Definition 2.4. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_{\mu}(\overline{B}^*_1)$ as the space of functions in $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{B}^*_1)$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_{\mu}(\overline{B}^*_1)} = \sup_{r\leq 1/2}\Big(r^{-\mu}\|u(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2-B_1)}\Big)$$

is finite.

Then we define the subspace of radial functions in $\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\overline{B}^*_1)$ by

$$\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\overline{B}^*_1) = \{f \in \mathcal{C}^{k,\alpha}_{\delta}(\mathbb{R}^4) \mid f(x) = f(|x|), \forall\, x \in \overline{B}^*_1\}.$$

For all $\varepsilon, \tau, \lambda, \gamma > 0$, we define $R_{\varepsilon,\lambda,\gamma} := \tau r_{\varepsilon,\lambda,\gamma}/\varepsilon$ where

$$r_{\varepsilon,\lambda,\gamma} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}, \sqrt{\gamma}).$$

We would like to find a solution u of

$$\Delta_a^2 u - \Delta_a u - \rho^4 e^u = 0 \tag{2.2}$$

in $\overline{B}_{r_{\varepsilon,\lambda,\gamma}}$. Using the transformation

$$v(x) = u\left(\frac{\varepsilon}{\tau}x\right) + 8\log\varepsilon - 4\log\left(\frac{\tau(1+\varepsilon^2)}{2}\right),$$

then equation (2.2) is equivalent to

$$\Delta^2 v + \Sigma^1_{\widetilde{a},\varepsilon} v + \Sigma^2_{\widetilde{a},\varepsilon} v - 24 e^v = 0 \tag{2.3}$$

in $\overline{B}_{R_{\varepsilon,\lambda}}$, where

$$\Sigma^1_{\widetilde{a},\varepsilon} v = 2\frac{\nabla\widetilde{a}}{\widetilde{a}} \cdot \nabla(\Delta v) - \widetilde{V}(x)\left(\frac{\varepsilon}{\tau}\right)^2 \nabla \log \widetilde{a} \cdot \nabla v$$

and

$$\Sigma^2_{\widetilde{a},\varepsilon} v = \left(\frac{\varepsilon}{\tau}\right)^2 \frac{1}{\widetilde{a}} \left(\left(\frac{\varepsilon}{\tau}\right)^{-2} \Delta\widetilde{a} - \widetilde{V}(x)\widetilde{a}\right) \Delta v,$$

where $\widetilde{a}(x) = a\left(\frac{\varepsilon}{\tau}x\right)$ and $\widetilde{V}(x) = V\left(\frac{\varepsilon}{\tau}x\right)$. Now we look for a solution of (2.3) of the form

$$v(x) = u_1(x) + h(x),$$

this amounts to solve

$$\mathbb{L}h = \frac{384}{(1+|x|^2)^4}(e^h - h - 1) - \Sigma^1_{\widetilde{a},\varepsilon}(u_1 + h) - \Sigma^2_{\widetilde{a},\varepsilon}(u_1 + h) \tag{2.4}$$

in $\overline{B}_{R_{\varepsilon,\lambda}}$.

We will need the following definition.

Definition 2.5. Given $\overline{r} \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_{\mu}(B_{\overline{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\overline{r}})$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\mu}(\overline{B}_{\overline{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1\leq r\leq \overline{r}} \left(r^{-\mu}\|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1 - B_{1/2})}\right).$$

For all $\sigma \geq 1$, we denote by

$$\mathcal{E}_\sigma \colon \mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma) \to \mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)$$

the extension operator defined by

$$\mathcal{E}_\sigma(f)(x) = \begin{cases} f(x) & \text{for } |x| \leq \sigma, \\ \chi\left(\frac{|x|}{\sigma}\right) f\left(\sigma \frac{x}{|x|}\right) & \text{for } |x| \geq \sigma, \end{cases}$$

where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \leq 1$ and identically equal to 0 for $t \geq 2$. It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\mathcal{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)} \leq c\|w\|_{\mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma)}. \tag{2.5}$$

We fix

$$\delta \in (0,1).$$

Denote by $\mathcal{G}_\delta$ to be a right inverse of $\mathbb{L}_{\mathrm{rad},\delta}$ provided by Proposition 2.3. To find a solution of (2.4) it is enough to find a fixed point h, in a small ball of $\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$, solution of

$$h = \aleph(h) \tag{2.6}$$

where $\aleph(h) := \mathcal{G}_\delta \circ \mathcal{E}_\delta \circ \mathfrak{R}(h)$ with

$$\mathfrak{R}(h) = \frac{384}{(1+|x|^2)^4}(e^h - h - 1) - \Sigma^1_{\widetilde{a},\varepsilon}(u_1 + h) - \Sigma^2_{\widetilde{a},\varepsilon}(u_1 + h).$$

For $|x| = r$, we have $\mathfrak{R}(0) = -\Sigma^1_{\widetilde{a},\varepsilon}(u_1) - \Sigma^2_{\widetilde{a},\varepsilon}(u_1)$, where

$$\Sigma^1_{\widetilde{a},\varepsilon}(u_1) = 2\frac{\nabla\widetilde{a}}{\widetilde{a}} \cdot \nabla(\Delta u_1) - \widetilde{V}(x)\left(\frac{\varepsilon}{\tau}\right)^2 \nabla \log \widetilde{a} \cdot \nabla u_1$$

and

$$\Sigma^2_{\widetilde{a},\varepsilon}(u_1) = \left(\frac{\varepsilon}{\tau}\right)^2 \frac{1}{\widetilde{a}} \left(\left(\frac{\varepsilon}{\tau}\right)^{-2} \Delta\widetilde{a} - \widetilde{V}(x)\widetilde{a}\right) \Delta u_1.$$

Given $\kappa > 0$, there exist $c_\kappa > 0$ (which can depend only on κ), such that for $\delta \in (0,1)$, we have

$$\begin{aligned} \sup_{r \leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(0)| &\leq c_\kappa \sup_{r \leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\Sigma^1_{\widetilde{a},\varepsilon}(u_1) + c_\kappa \sup_{r \leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\Sigma^2_{\widetilde{a},\varepsilon}(u_1) \\ &\leq c_\kappa \sup_{r \leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\left(2\left|\frac{\nabla\widetilde{a}}{\widetilde{a}}\right| |\nabla(\Delta u_1)| + \widetilde{V}(x)\left(\frac{\varepsilon}{\tau}\right)^2 |\nabla \log \widetilde{a}|\, |\nabla u_1|\right) \\ &\quad + c_\kappa \left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r \leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\left|\frac{1}{\widetilde{a}}\left(\left(\frac{\varepsilon}{\tau}\right)^{-2}\Delta\widetilde{a} - V(x)\widetilde{a}\right)\right| |\Delta u_1|\,. \end{aligned}$$

Using the fact that $a(x)$ is solution of (1.2) satisfying (H) and $V(x)$ is smooth bounded potential, we deduce

$$\begin{aligned}\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(0)| \le{}& c_\kappa\|\nabla a\|_\infty\left(\frac{\varepsilon}{\tau}\right)\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\frac{r(3+r^2)}{(1+r^2)^3}\\ &+c_\kappa\|\nabla a\|_\infty\left(\frac{\varepsilon}{\tau}\right)^3\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\frac{r}{1+r^2}\\ &+c_\kappa\left(\frac{\varepsilon}{\tau}\right)^2\lambda\|f\|_\infty\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\frac{2+r^2}{(1+r^2)^2}.\end{aligned}$$

Taking into account that for r very large we have $(1+r^2)^{-\beta}\sim r^{-2\beta}$, the fact that $\|\nabla a\|_\infty<\gamma$, recall that $r_{\varepsilon,\lambda,\gamma}:=\max(\sqrt{\varepsilon},\sqrt{\lambda},\sqrt{\gamma})$ and $\delta\in(0,1)$, then we obtain

$$\begin{aligned}\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(0)| &\le c_\kappa\varepsilon\gamma+c_\kappa\varepsilon R^{1-\delta}_{\varepsilon,\lambda,\gamma}+c_\kappa\gamma\varepsilon^3R^{3-\delta}_{\varepsilon,\lambda,\gamma}+c_\kappa\lambda\varepsilon^2+c_\kappa\lambda\varepsilon^2R^{2-\delta}_{\varepsilon,\lambda,\gamma}\\ &\le c_\kappa\varepsilon^\delta\gamma\varepsilon^{1-\delta}+c_\kappa\gamma\varepsilon^\delta r^{1-\delta}_{\varepsilon,\lambda,\gamma}+c_\kappa\gamma\varepsilon^\delta r^{3-\delta}_{\varepsilon,\lambda,\gamma}+c_\kappa\lambda\varepsilon^2+c_\kappa\lambda\varepsilon^\delta r^{2-\delta}_{\varepsilon,\lambda,\gamma}\\ &\le c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}.\end{aligned}$$

Recall that $\aleph(h):=\mathcal{G}_\delta\circ\mathcal{E}_\delta\circ\mathfrak{R}(h)$, then there exist $c_\kappa>0$ (which can depend only on κ), such that

$$\|\aleph(0)\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\le c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}.$$

Making use of Proposition 2.3 together with (2.5), hence,

$$\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\le 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}. \tag{2.7}$$

Now, let h_1, h_2 in $B(0,2c_\kappa r^2_{\varepsilon,\lambda,\gamma})$ of $\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$, satisfying for each $x\in\overline{B}_{R_{\varepsilon,\lambda,\gamma}}$

$$|h_i(x)|\le c_\kappa r^{2+\delta}_{\varepsilon,\lambda,\gamma},$$

we prove that $|h_i(x)|\to 0$ as ε, λ and γ tend to 0. Then we have

$$\begin{aligned}&\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(h_2)-\mathfrak{R}(h_1)|\\ &\le c_\kappa\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}(1+|x|^2)^{-4}\left|e^{h_2}-e^{h_1}+h_1-h_2\right|\\ &\quad+c_\kappa\lambda\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\left|\Sigma^1_{\tilde{a},\varepsilon}(h_2-h_1)\right|+c_\kappa\lambda\sup_{r\le R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\left|\Sigma^2_{\tilde{a},\varepsilon}(h_2-h_1)\right|.\end{aligned}$$

Recall that a functions w in $\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\delta/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\delta-\ell)/2}$, for $\ell=1,\dots,k+\alpha$ (a.e. $|\nabla^\ell w|\le c_\kappa r^{\delta-\ell}\|w\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}$, $(1+r^2)^{(\delta-\ell)/2}\sim r^{\delta-\ell}$ for r very large), then there exist

$c_\kappa > 0$ (only depend on κ), using the fact that $a(x)$ is solution of (1.2) satisfying (H) and $V(x)$ is smooth bounded potential, we deduce

$$\begin{aligned}
&\sup_{r\leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(h_2)-\mathfrak{R}(h_1)| \\
&\leq c_\kappa \sup_{r\leq R_{\varepsilon,\lambda,\gamma}} r^{-4-\delta}|h_2-h_1||h_2+h_1| \\
&\quad + c_\kappa\gamma \sup_{r\leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\left(2\left|\frac{\nabla\widetilde{a}}{\widetilde{a}}\right||\nabla(\Delta(h_2-h_1))| + \widetilde{V}(x)\left(\frac{\varepsilon}{\tau}\right)^2|\nabla\log\widetilde{a}|\,|\nabla(h_2-h_1)|\right) \\
&\quad + c_\kappa\left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}\frac{1}{\widetilde{a}}\left(\left(\frac{\varepsilon}{\tau}\right)^{-2}\Delta\widetilde{a} - \widetilde{V}(x)\widetilde{a}\right)|\Delta(h_2-h_1)| \\
&\leq c_\kappa\sum_{i=1}^{2}\|h_i\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + c_\kappa\gamma\varepsilon R_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \\
&\quad + c_\kappa\gamma\varepsilon^3 R^3_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + c_\kappa\lambda\varepsilon^2 R^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.
\end{aligned}$$

Provided $h_i \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ satisfies $\|h_i\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \leq 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$, then the last estimate, is given by

$$\begin{aligned}
&\sup_{r\leq R_{\varepsilon,\lambda,\gamma}} r^{4-\delta}|\mathfrak{R}(h_2)-\mathfrak{R}(h_1)| \\
&\leq c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + c_\kappa\gamma r_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \\
&\quad + c_\kappa\gamma r^3_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} + c_\kappa\lambda r^2_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.
\end{aligned}$$

Similarly, making use Proposition 2.3 together with (2.5), we conclude that given $\kappa > 0$, there exist ε_κ, λ_κ, γ_κ and $\bar{c}_\kappa > 0$ (only depend on κ) such that

$$\|\aleph(h_2)-\aleph(h_1)\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_{\varepsilon,\lambda,\gamma}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}. \tag{2.8}$$

Reducing ε_κ, λ_κ and γ_κ if necessary, we can assume that

$$\bar{c}_\kappa r_{\varepsilon,\lambda,\gamma} \leq \frac{1}{2}$$

for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\gamma \in (0,\gamma_\kappa)$. Then, (2.7) and (2.8) are enough to show that

$$h \mapsto \aleph(h)$$

is a contraction from the ball

$$\left\{h \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4) \mid \|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \leq 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}\right\}$$

into itself and hence has a unique fixed point h in this set. This fixed point is a solution of (2.6) in $\overline{B}_{R_{\varepsilon,\lambda,\gamma}}$.

We summarize this in the following proposition.

Proposition 2.6. *Let $\delta \in (0,1)$. Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$ (which can depend only on κ) and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$, there exists a unique solution $h \in \mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ of (2.6) such that*

$$v(x) = u_1(x) + h(x)$$

solves (2.3) in $\overline{B}_{R_{\varepsilon,\lambda,\gamma}}$. In addition

$$\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}.$$

3. Known results [1]

3.1. Analysis of the bi-Laplace operator in weighted spaces

Given $x^1, \ldots, x^m \in \Omega$ we define $X := (x^1, \ldots, x^m)$ and

$$\overline{\Omega}^*(X) := \overline{\Omega} - \{x^1, \ldots, x^m\},$$

and we choose $r_0 > 0$ so that the balls $B_{r_0}(x^i)$ of center x^i and radius r_0 are mutually disjoint and included in Ω. For all $r \in (0, r_0)$ we define

$$\overline{\Omega}_r(X) := \overline{\Omega} - \bigcup_{j=1}^m B_r(x^j).$$

With these notations, we have

Definition 3.1. Given $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(X))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{\Omega}^*(X))$ which is endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(X))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(X))} + \sum_{j=1}^m \sup_{r\in(0,r_0/2)} \left(r^{-\nu} \|w(x^j + r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2 - B_1)} \right)$$

is finite.

When $k \geq 2$, we denote by $[\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(X))]_0$ the subspace of functions $w \in \mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(X))$ satisfying $w = \Delta w = 0$.

We will use the following

Proposition 3.2. [1] *Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\begin{array}{rccc} \mathcal{L}_\nu\colon & [\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(X))]_0 & \longrightarrow & \mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(X)) \\ & w & \longmapsto & \Delta^2 w \end{array}$$

is surjective.

3.2. Bi-harmonic extensions

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ we define H^i $(= H^i(\varphi, \psi; \cdot))$ to be the solution of

$$\begin{cases} \Delta^2 H^i = 0 & \text{in } B_1, \\ H^i = \varphi & \text{on } \partial B_1, \\ \Delta H^i = \psi & \text{on } \partial B_1, \end{cases}$$

where, as already mentioned, B_1 denotes the unit ball in $\mathbb{R}^4$.

We set $B_1^* = B_1 - \{0\}$. As in the previous section, we define

Definition 3.3. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_{\mu}(\overline{B}_1^*)$ as the space of function in $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{B}_1^*)$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_{\mu}(\overline{B}_1^*)} = \sup_{r \leq 1/2} \left(r^{-\mu} \|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2 - B_1)} \right)$$

is finite.

This corresponds to the space and norm already defined in the previous section when $\Omega = B_1$, $m = 1$ and $x^1 = 0$.

Let $e_1, \ldots, e_4$ be the coordinate functions on S^3. We prove

Lemma 3.4. [1] *Assume that*

$$\text{(3.1)} \qquad \int_{S^3} (8\varphi - \psi)\, dv_{S^3} = 0 \quad \textit{and also that} \quad \int_{S^3} (12\varphi - \psi) e_\ell \, dv_{S^3} = 0$$

for $\ell = 1, \ldots, 4$*. Then there exists* $c > 0$ *such that*

$$\|H^i(\varphi, \psi; \cdot)\|_{\mathcal{C}^{4,\alpha}_{2}(\overline{B}_1^*)} \leq c \left(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \right).$$

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ we define (when it exists!) H^e $(= H^e(\varphi, \psi; \cdot))$ to be the solution of

$$\begin{cases} \Delta^2 H^e = 0 & \text{in } \mathbb{R}^4 - B_1, \\ H^e = \varphi & \text{on } \partial B_1, \\ \Delta H^e = \psi & \text{on } \partial B_1, \end{cases}$$

which decays at infinity.

Definition 3.5. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_{\nu}(\mathbb{R}^4 - B_1)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4 - B_1)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\nu}(\mathbb{R}^4 - B_1)} = \sup_{r \geq 1} \left(r^{-\nu} \|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}_{\nu}(\overline{B}_2 - B_1)} \right)$$

is finite.

We recall

Lemma 3.6. [1] *Assume that*

$$\int_{S^3} \psi \, dv_{S^3} = 0. \tag{3.2}$$

Then there exists $c > 0$ such that

$$\|H^e(\varphi, \psi; \cdot\,)\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1)} \leq c(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}).$$

We will need

Lemma 3.7. [1] *The mapping*

$$\begin{array}{rccc} \mathcal{P}\colon & \mathcal{C}^{4,\alpha}(S^3)^\perp \times \mathcal{C}^{2,\alpha}(S^3)^\perp & \longrightarrow & \mathcal{C}^{3,\alpha}(S^3)^\perp \times \mathcal{C}^{1,\alpha}(S^3)^\perp \\ & (\varphi, \psi) & \longmapsto & (\partial_r H^i - \partial_r H^e, \partial_r \Delta H^i - \partial_r \Delta H^e) \end{array}$$

where $H^i = H^i(\varphi, \psi; \cdot\,)$ and $H^e = H^e(\varphi, \psi; \cdot\,)$, is an isomorphism.

4. The first nonlinear Dirichlet problem

Recall that for all $\varepsilon, \tau, \lambda, \gamma > 0$, we define

$$R_{\varepsilon,\lambda,\gamma} := \tau r_{\varepsilon,\lambda,\gamma}/\varepsilon$$

where

$$r_{\varepsilon,\lambda,\gamma} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}, \sqrt{\gamma}).$$

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfying (3.1), we define

$$\boldsymbol{u} := u_1 + h + H^i(\varphi, \psi; (\,\cdot\,/R_{\varepsilon,\lambda,\gamma})).$$

We would like to find a solution u of

$$\Delta^2 u + \Sigma^1_{\widetilde{a},\varepsilon} u + \Sigma^2_{\widetilde{a},\varepsilon} u - 24 e^u = 0 \tag{4.1}$$

which is defined in $B_{R_{\varepsilon,\lambda,\gamma}}$ and which is a perturbation of $\boldsymbol{u}$. Writing $u = \boldsymbol{u} + v$, this amounts to solve the equation

$$\begin{aligned} \mathbb{L} v = {} & \frac{384}{(1+r^2)^4} e^h \left(e^{H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))+v} - 1 - v \right) + \frac{384}{(1+r^2)^4}(e^h - 1) v \\ & - \Sigma^1_{\widetilde{a},\varepsilon} \left(u_1 + h + H^i(\varphi, \psi; (\,\cdot\,/R_{\varepsilon,\lambda,\gamma})) + v \right) + \Sigma^1_{\widetilde{a},\varepsilon}(u_1 + h) \\ & - \Sigma^2_{\widetilde{a},\varepsilon} \left(u_1 + h + H^i(\varphi, \psi; (\,\cdot\,/R_{\varepsilon,\lambda,\gamma})) + v \right) + \Sigma^2_{\widetilde{a},\varepsilon}(u_1 + h), \end{aligned}$$

since H^i is bi-harmonic. Then taking in to account that $\left(\Sigma^i_{\tilde{a},\varepsilon}\right)_i$, $i = 1,2$ are linear operators, then

$$\begin{aligned}\mathbb{L}v = {} & \frac{384}{(1+r^2)^4} e^h \left(e^{H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))+v} - 1 - v \right) + \frac{384}{(1+r^2)^4}(e^h - 1)v \\ & - \Sigma^1_{\tilde{a},\varepsilon}\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma})) + v\right) - \Sigma^2_{\tilde{a},\varepsilon}\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma})) + v\right). \end{aligned} \tag{4.2}$$

In the following, we will denote by $\mathcal{K}(v)$ the right-hand side of (4.2). We will need the following definition.

Definition 4.1. Given $\overline{r} \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_\mu(B_{\overline{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\overline{r}})$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\overline{B}_{\overline{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1\leq r\leq \overline{r}} \left(r^{-\mu} \|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1 - B_{1/2})} \right).$$

For all $\sigma \geq 1$, we denote by

$$\mathcal{E}_\sigma \colon \mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma) \to \mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)$$

the extension operator defined by

$$\mathcal{E}_\sigma(f)(x) = \chi\left(\frac{|x|}{\sigma}\right) f\left(\sigma \frac{x}{|x|}\right)$$

where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \geq 2$ and identically equal to 0 for $t \leq 1$. It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\mathcal{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)} \leq c\|w\|_{\mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma)}. \tag{4.3}$$

We fix

$$\mu \in (1,2)$$

and denote by $\mathcal{G}_\mu$ a right inverse provided by Proposition 2.3. To find *a* solution of (4.2), it is enough to find $v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solution of

$$v = N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v) \tag{4.4}$$

where we have defined

$$N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v) := \mathcal{G}_\mu \circ \mathcal{E}_{R_{\varepsilon,\lambda,\gamma}} \mathcal{K}(v)).$$

Given $\kappa > 1$ (whose value will be fixed later on), we now further assume that the functions $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ and the constant $\tau > 0$ satisfy

$$\frac{1}{\log(1/r^2_{\varepsilon,\lambda,\gamma})} |\log(\tau/\tau_*)| \leq \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} \leq \kappa r^2_{\varepsilon,\lambda,\gamma} \quad \text{and} \quad \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \leq \kappa r^2_{\varepsilon,\lambda,\gamma}, \tag{4.5}$$

where $\tau_* > 0$ is fixed later.

We have the following technical:

Lemma 4.2. *Given $\kappa > 0$, $\mu \in (1,2)$, $\delta \in (0,1)$, $V(z)$ is smooth bounded potential and $a(z)$ satisfies* (H) *and solution of*

$$\begin{cases} -\Delta a(z) + V(z)a(z) = \lambda f(z,a) & \text{in } \overline{\Omega}, \\ \|\nabla a\|_\infty \leq \gamma \end{cases}$$

then there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$,

$$\|N(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; 0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}. \tag{4.6}$$

Moreover,

$$\begin{aligned} &\|N(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; v_2) - N(\varepsilon, \lambda, \gamma, \tau, \varphi, \psi; v_1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\ &\leq \bar{c}_\kappa r_{\varepsilon,\lambda,\gamma} \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \end{aligned} \tag{4.7}$$

provided $\widetilde{v} = v_1, v_2 \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$, $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfy

$$\|\widetilde{v}\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}, \quad \|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} \leq \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \leq \kappa r^2_{\varepsilon,\lambda,\gamma},$$

and $|\log(\tau/\tau_*)| \leq \kappa r^2_{\varepsilon,\lambda,\gamma} \log(1/r^2_{\varepsilon,\lambda,\gamma})$.

Proof. The proof of these estimates follows from the result of Lemma 3.4 together with the assumption on the norms of φ and ψ and recall that a functions in $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\mu/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\mu-\ell)/2}$, for $\ell = 1, \ldots, k+\alpha$. Indeed, let c_κ denote constants which only depend on κ (provided ε, λ and γ are chosen small enough), it follows from Lemma 3.4 and the estimates given by (4.5) and under the hypothesis (3.1), the coefficients of r^0 and r^1 vanish and hence, at least formally, the expansion of H^i only involves powers of r that are greater than or equal to 2, then

$$\|H^i(\varphi, \psi; (\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})} \leq c_\kappa R^{-2}_{\varepsilon,\lambda,\gamma} (\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}) \leq c_\kappa \varepsilon^2. \tag{4.8}$$

Therefore, using the fact that for each $x \in \overline{B}_{R_{\varepsilon,\lambda,\gamma}}$, we have $|h(x)| \leq c_\kappa r^{2+\delta}_{\varepsilon,\lambda,\gamma}$, which tends to 0 as ε, λ and γ tend to 0, we get

$$\left\| (1+|\cdot|^2)^{-4} e^h \left(e^{H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))} - 1 \right) \right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})} \leq c_\kappa \varepsilon^2.$$

Using the fact that $\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)} \leq 2c\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$, then for all $x \in \overline{B}_{R_{\varepsilon,\lambda,\gamma}}$, $|h(x)| \leq c r^{2+\delta}_{\varepsilon,\lambda,\gamma}$ tends to 0 as ε, λ and γ tend to 0 and from the asymptotic behavior of H^i given by the estimate (4.8) and since $a(x)$ is solution of (1.2) satisfying (H) and $V(x)$ is smooth bounded potential, we deduce

$$\begin{aligned}
&\left\|\Sigma^1_{\widetilde{a},\varepsilon}\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\
&\leq \Bigg\|2\frac{\nabla\widetilde{a}}{\widetilde{a}}\cdot\nabla\left(\Delta\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\right)\\
&\qquad -\widetilde{V}(x)\left(\frac{\varepsilon}{\tau}\right)^2\nabla\log\widetilde{a}\cdot\nabla\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\Bigg\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\
&\leq c_\kappa\gamma r^{5-\mu}_{\varepsilon,\lambda,\gamma}\varepsilon^{\mu-2}\|H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\Sigma^2_{\widetilde{a},\varepsilon}\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\
&\leq \left\|\left(\frac{\varepsilon}{\tau}\right)^2\frac{1}{\widetilde{a}}\left(\left(\frac{\varepsilon}{\tau}\right)^{-2}\Delta\widetilde{a}-\widetilde{V}(x)\widetilde{a}\right)\Delta\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\
&\leq c_\kappa\lambda r^{4-\mu}_{\varepsilon,\lambda,\gamma}\varepsilon^{\mu-2}\|H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}.
\end{aligned}$$

Provided H^i satisfies (4.8), we deduce that

$$\left\|\Sigma^1_{\widetilde{a},\varepsilon}\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\leq c_\kappa\varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}$$

and

$$\left\|\Sigma^2_{\widetilde{a},\varepsilon}\left(H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\leq c_\kappa\varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}.$$

Using Proposition 2.3 and (4.3), we conclude that

$$\|N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\leq c_\kappa\varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}$$

which derive the desired estimates given by (4.6).

To derive the second estimate, we recall that a functions w in $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\mu/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\mu-\ell)/2}$, for $\ell=1,\ldots,k+\alpha$ (a.e. $|\nabla^\ell w|\leq c_\kappa r^{\mu-\ell}\|w\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}$, $(1+r^2)^{(\mu-\ell)/2}\sim r^{\mu-\ell}$ for r very large), then there exist $c_\kappa>0$ (only depend on κ) such that we use the fact that for each $x\in\overline{B}_{R_{\varepsilon,\lambda,\gamma}}$, we have $|h(x)|\leq c_\kappa r^{2+\delta}_{\varepsilon,\lambda,\gamma}$, and $|v_i(x)|\leq c_\kappa r^{2+\mu}_{\varepsilon,\lambda,\gamma}$, which they tend to 0 as ε, λ and γ tend to 0, then given $\kappa>0$, there exists $c_\kappa>0$ such that

$$\begin{aligned}
&\left\|(1+|\cdot|^2)^{-4}e^{H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma}))+h}(e^{v_2}-e^{v_1}-v_2+v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\
&\leq c_\kappa\varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}
\end{aligned}$$

and

$$\left\|(1+|\cdot|^2)^{-4}e^h\left(e^{H^i(\varphi,\psi;(\cdot/R_{\varepsilon,\lambda,\gamma}))}-1\right)(v_2-v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\le c_\kappa\varepsilon^2\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.$$

Provided $h\in\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ satisfies $\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\le 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}$, we deduce that

$$\begin{aligned}\left\|(1+|\cdot|^2)^{-4}(e^h-1)(v_2-v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}&\le c_\kappa\|h\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\\&\le c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda,\gamma}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.\end{aligned}$$

Since $a(x)$ satisfying (H) and $V(x)$ is smooth bounded potential, then

$$\begin{aligned}&\left\|\Sigma^1_{\widetilde{a},\varepsilon}(v_2-v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\&\le\left\|2\frac{\nabla\widetilde{a}}{\widetilde{a}}\cdot\nabla(\Delta(v_2-v_1))-\widetilde{V}(x)\left(\frac{\varepsilon}{\tau}\right)^2\nabla\log\widetilde{a}\cdot\nabla(v_2-v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\&\le c_\kappa\gamma r_{\varepsilon,\lambda,\gamma}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+c_\kappa\gamma r^3_{\varepsilon,\lambda,\gamma}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.\end{aligned}$$

Since $a(x)$ is solution of (1.2) satisfying (H), and $V(x)$ is smooth bounded potential, we deduce

$$\begin{aligned}\left\|\Sigma^2_{\widetilde{a},\varepsilon}(v_2-v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}&\le\left\|\left(\frac{\varepsilon}{\tau}\right)^2\frac{1}{\widetilde{a}}\left(\left(\frac{\varepsilon}{\tau}\right)^{-2}\Delta\widetilde{a}-\widetilde{V}(x)\widetilde{a}\right)\Delta(v_2-v_1)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\overline{B}_{R_{\varepsilon,\lambda,\gamma}})}\\&\le c_\kappa\lambda r^2_{\varepsilon,\lambda,\gamma}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.\end{aligned}$$

Using Proposition 2.3 and (4.3), we derive the desired estimates given by (4.7). □

Reducing ε_κ, λ_κ and γ_κ if necessary, we can assume that

$$\bar{c}_\kappa r^2_{\varepsilon,\lambda,\gamma}\le\frac{1}{2},$$

there exist $\varepsilon_\kappa>0$, $\lambda_\kappa>0$, $\gamma_\kappa>0$, $c_\kappa>0$ and $\bar{c}_\kappa>0$ such that, for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\lambda\in(0,\lambda_\kappa)$ and $\gamma\in(0,\gamma_\kappa)$. Then, (4.6) and (4.7) in Lemma 4.2 are enough to show that

$$v\mapsto N(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;v)$$

is a contraction from

$$\left\{v\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\mid\|v\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\le 2c_\kappa\varepsilon^\mu r^2_{\varepsilon,\lambda,\gamma}\right\}$$

into itself and hence has a unique fixed point $v(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;\cdot)$ in this set. This fixed point is a solution of (4.4) in $B_{R_{\varepsilon,\lambda,\gamma}}$.

We summarize this in

Proposition 4.3. *Given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$ and $c_\kappa > 0$ (only depending on κ) such that given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfying* (3.1) *and $\tau > 0$ satisfying*

$$|\log(\tau/\tau_*)| \le \kappa r_{\varepsilon,\lambda,\gamma}^2 \log(1/r_{\varepsilon,\lambda,\gamma}^2), \quad \|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa r_{\varepsilon,\lambda,\gamma}^2 \quad \text{and} \quad \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa r_{\varepsilon,\lambda,\gamma}^2,$$

the function

$$u(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;\cdot\,) := u_1 + h + H^i(\varphi,\psi;(\,\cdot\,/R_{\varepsilon,\lambda,\gamma})) + v(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;\cdot\,)$$

solves (4.1) *in $B_{R_{\varepsilon,\lambda,\gamma}}$. In addition,*

$$\|v(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;\cdot\,)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa \varepsilon^\mu r_{\varepsilon,\lambda,\gamma}^2.$$

Observe that the function $v(\varepsilon,\lambda,\gamma,\tau,\varphi,\psi;\cdot\,)$ being obtained as a fixed point for contraction mapping, it depends continuously on the parameter τ.

5. The second nonlinear Dirichlet problem

For all $(\varepsilon,\lambda,\gamma) \in (0,r_0^2)^3$, we recall that

$$r_{\varepsilon,\lambda,\gamma} := \max(\sqrt{\varepsilon},\sqrt{\lambda},\sqrt{\gamma}).$$

Recall that $G_a(x,\cdot\,)$ denotes the unique solution of

$$\Delta^2 G_a(x,\cdot\,) = 64\pi^2\delta_x$$

in Ω, with $G_a(x,\cdot\,) = \Delta G_a(x,\cdot\,) = 0$ on $\partial\Omega$. In addition, the following decomposition holds

$$G_a(x,y) = -8\log|x-y| + R_a(x,y)$$

where $y \mapsto R_a(x,y)$ is a smooth function.

We recall in this section a result which concerns the properties of the Green's function in the following lemma.

Lemma 5.1. *There exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have that*

$$|\nabla^i G_a(x,y)| \le C|x-y|^{-i}, \quad i \ge 1.$$

Proof. This estimate is originally due to Krasovskiĭ [9] and some reference therein. □

Given $x^1,\dots,x^m \in \Omega$. The data we will need are the following:

(i) Points $Y = (y^1,\dots,y^m) \in \Omega^m$ close enough to $X = (x^1,\dots,x^m)$.

(ii) Parameters $\widetilde{\eta} = (\widetilde{\eta}^1, \ldots, \widetilde{\eta}^m) \in \mathbb{R}^m$ close to 0.

(iii) Boundary data $\Phi = (\varphi^1, \ldots, \varphi^m) \in (\mathcal{C}^{4,\alpha}(S^3))^m$ and $\Psi = (\psi^1, \ldots, \psi^m) \in (\mathcal{C}^{2,\alpha}(S^3))^m$ each of which satisfies (3.2).

With all these data, we define

$$\widetilde{\boldsymbol{u}} := \sum_{j=1}^{m} (1 + \widetilde{\eta}^j) G_a(y^j, \cdot\,) + \sum_{j=1}^{m} \chi_{r_0}(\,\cdot - y^j) H^e(\varphi^j, \psi^j; (\,\cdot - y^j)/r_{\varepsilon,\lambda,\gamma})$$

where χ_{r_0} is a cutoff function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}.

We recall that $\rho > 0$ is defined by

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

We would like to find a solution of the equation

$$\Delta^2 u + \Sigma_a^1 u + \Sigma_a^2 u - \rho^4 e^u = 0, \tag{5.1}$$

which is defined in $\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)$ and which is a perturbation of $\widetilde{\boldsymbol{u}}$. Writing $u = \widetilde{\boldsymbol{u}} + \widetilde{v}$, this amounts to solve

$$\Delta^2 \widetilde{v} = \rho^4 e^{\widetilde{\boldsymbol{u}} + \widetilde{v}} - \Delta^2 \widetilde{\boldsymbol{u}} - \Sigma_a^1(\widetilde{\boldsymbol{u}} + \widetilde{v}) - \Sigma_a^2(\widetilde{\boldsymbol{u}} + \widetilde{v}). \tag{5.2}$$

We need to define an auxiliary weighed space:

Definition 5.2. Given $\overline{r} \in (0, r_0/2)$, $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the Hölder weighted space $\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}_{\overline{r}}(X))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}(\overline{\Omega}_{\overline{r}}(X))$ which is endowed with the norm

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}_{\overline{r}}(X))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(X))} + \sum_{j=1}^{m} \sup_{r \in [\overline{r}, r_0/2)} \left(r^{-\nu} \|w(x^j + r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2 - B_1)} \right).$$

For all $\sigma \in (0, r_0/2)$ and all $Y \in \Omega^m$ such that $\|X - Y\| \leq r_0/2$, we denote by

$$\widetilde{\mathcal{E}}_{\sigma,Y} \colon \mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}_\sigma(Y)) \to \mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}^*(Y)),$$

the extension operator defined by $\widetilde{\mathcal{E}}_{\sigma,Y}(f) = f$ in $\overline{\Omega}_\sigma(Y)$

$$\widetilde{\mathcal{E}}_{\sigma,Y}(f)(y^i + x) = \widetilde{\chi}\left(\frac{|x|}{\sigma}\right) f\left(y^i + \sigma \frac{x}{|x|}\right)$$

for each $j = 1, \ldots, m$ and $\widetilde{\mathcal{E}}_{\sigma,Y}(f) = 0$ in each $B_{\sigma/2}(y^j)$, where $t \mapsto \widetilde{\chi}(t)$ is a cutoff function identically equal to 1 for $t \geq 1$ and identically equal to 0 for $t \leq 1/2$. It is easy to check that there exists a constant $c = c(\nu) > 0$ only depending on ν such that

$$\|\widetilde{\mathcal{E}}_{\sigma,Y}(w)\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}^*(X))} \leq c\|w\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}_\sigma(X))}. \tag{5.3}$$

We fix

$$\nu \in (-1, 0),$$

and denote by $\widetilde{\mathcal{G}}_{\nu,Y}$ the right inverse provided by Proposition 3.2. Clearly, it is enough to find $\widetilde{v} \in \mathcal{C}^{4,\alpha}_{\nu}(\Omega^*(Y))$ solution of

$$\widetilde{v} = \widetilde{N}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \widetilde{v})$$

where we have defined

$$\begin{aligned}
\widetilde{N}(\widetilde{v}) &:= \widetilde{N}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \widetilde{v}) \\
&:= \widetilde{\mathcal{G}} \circ \widetilde{\mathcal{E}}_{r_{\varepsilon,\lambda,\gamma},Y} \left(\rho^4 e^{\widetilde{\boldsymbol{u}}+\widetilde{v}} - \Delta^2 \widetilde{\boldsymbol{u}} - \Sigma^1_a(\widetilde{\boldsymbol{u}} + \widetilde{v}) - \Sigma^2_a(\widetilde{\boldsymbol{u}} + \widetilde{v})\right) \\
&:= \widetilde{\mathcal{G}}_{\nu,Y} \circ \widetilde{\mathcal{E}}_{r_{\varepsilon,\lambda,\gamma},Y}(\widetilde{S}(v)).
\end{aligned}$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that Φ and Ψ satisfy

$$\|\Phi\|_{(\mathcal{C}^{4,\alpha}(S^3))^m} \leq \kappa r^2_{\varepsilon,\lambda,\gamma} \quad \text{and} \quad \|\Psi\|_{(\mathcal{C}^{2,\alpha}(S^3))^m} \leq \kappa r^2_{\varepsilon,\lambda,\gamma}. \tag{5.4}$$

Moreover, we assume that the parameters $\widetilde{\eta}$ and the points Y are chosen to satisfy

$$|\widetilde{\eta}| \leq \kappa r^2_{\varepsilon,\lambda,\gamma} \quad \text{and} \quad \|Y - X\| \leq \kappa r_{\varepsilon,\lambda,\gamma}. \tag{5.5}$$

Then, the following result holds.

Lemma 5.3. *Given $\kappa > 1$. There exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$, we have*

$$\|\widetilde{N}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; 0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(Y))} \leq c_\kappa r^2_{\varepsilon,\lambda,\gamma}. \tag{5.6}$$

Moreover,

$$\begin{aligned}
&\|\widetilde{N}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \widetilde{v}_2) - \widetilde{N}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \widetilde{v}_1)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(Y))} \\
&\leq \bar{c}_\kappa r^2_{\varepsilon,\lambda,\gamma} \|\widetilde{v}_2 - \widetilde{v}_1\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(Y))}
\end{aligned} \tag{5.7}$$

provided $\widetilde{v} = v_1, v_2 \in \mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(Y))$, $\widetilde{\Phi} = \Phi_1, \Phi_2 \in (\mathcal{C}^{4,\alpha}(S^3))^m$, $\widetilde{\Psi} = \Psi_1, \Psi_2 \in (\mathcal{C}^{2,\alpha}(S^3))^m$ satisfy

$$\|\widetilde{v}\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(Y))} \leq 2c_\kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\widetilde{\Phi}\|_{(\mathcal{C}^{4,\alpha}(S^3))^m} \leq \kappa r^2_{\varepsilon,\lambda,\gamma}, \quad \|\widetilde{\Psi}\|_{(\mathcal{C}^{2,\alpha}(S^3))^m} \leq \kappa r^2_{\varepsilon,\lambda,\gamma},$$

and $|\widetilde{\eta}| \leq \kappa r^2_{\varepsilon,\lambda,\gamma}$, $\|Y - X\| \leq \kappa r_{\varepsilon,\lambda,\gamma}$.

Proof. The proof of the first estimate follows from the asymptotic behavior of H^e together with the assumption on the norm of boundary data $\widetilde{\varphi}^i$ given by (5.4). Recall that functions in $\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(X))$ are bounded by a constant times the distance to X to the power ν and have their ℓ-th partial derivatives that are bounded by a constant times the distance to X to the power $\nu-\ell$, for $\ell=1,\dots,k+\alpha$. Indeed, let c_κ be a constant depending only on κ (provided ε, λ and γ are chosen small enough) it follows from the estimate of $H^e := H^e_{\widetilde{\varphi}^j,\widetilde{\psi}^j}$ (Observe that (3.2) implies that the expansion of H^e only involves powers of r that are lower than or equal to -1), given by Lemma 3.6, then

$$\left|H^e_{\widetilde{\varphi}^j,\widetilde{\psi}^j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})\right| \le c_\kappa r^3_{\varepsilon,\lambda,\gamma} r^{-1}.$$

Recall that $\widetilde{N}(\widetilde{v}) = \widetilde{\mathcal{G}}_\nu \circ \widetilde{\xi}_{r_{\varepsilon,\lambda,\gamma}} \circ \widetilde{S}(\widetilde{v})$, we will estimate $\widetilde{N}(0)$ in different subregions of $\overline{\Omega}^*$.

• In $B_{r_0/2}(y^j)$ for $1\le j\le m$, we have $\chi_{r_0}(x-y^j)=1$ and $\Delta^2\widetilde{\boldsymbol{u}}=0$ so that

$$\begin{aligned}
|\widetilde{S}(0)| &\le c_\kappa \varepsilon^4 \prod_{j=1}^m \left[e^{(1+\widetilde{\eta}^j)G_a(y^j,\cdot)+H^e_{\widetilde{\varphi}^j,\widetilde{\psi}^j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})} + \left|\Sigma_a^1(\widetilde{\boldsymbol{u}})\right| + \left|\Sigma_a^2(\widetilde{\boldsymbol{u}})\right|\right] \\
&\le c_\kappa\varepsilon^4 \prod_{j=1}^m |x-y^j|^{-8(1+\widetilde{\eta}^j)} \prod_{\ell=1,\ell\neq j}^m |x-y^\ell|^{-8(1+\widetilde{\eta}^j)} + \left|\Sigma_a^1(\widetilde{\boldsymbol{u}})\right| + \left|\Sigma_a^2(\widetilde{\boldsymbol{u}})\right| \\
&\le c_\kappa\varepsilon^4 |x-y^j|^{-8(1+\widetilde{\eta}^j)} \\
&\quad + \left|\sum_{j=1}^m (1+\widetilde{\eta}^j)\Sigma_a^1(G_a(y^j,\cdot))\right| + \left|\sum_{j=1}^m \Sigma_a^1\left(\chi_{r_0}(\cdot-y^j)(H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma}))\right)\right| \\
&\quad + \left|\sum_{j=1}^m (1+\widetilde{\eta}^j)\Sigma_a^2(G_a(y^j,\cdot))\right| + \left|\sum_{j=1}^m \Sigma_a^2\left(\chi_{r_0}(\cdot-y^j)(H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma}))\right)\right|,
\end{aligned}$$

where we recall

$$\Sigma_a^1 w = 2\frac{\nabla a}{a}\cdot\nabla(\Delta w) - V(x)\nabla\log a\cdot\nabla w \quad \text{and} \quad \Sigma_a^2 w = \left(\frac{\Delta a}{a} - V(x)\right)\Delta w.$$

Since $a(x)$ is solution of (1.2) satisfying (H), and $V(x)$ is smooth bounded potential we deduce,

$$\begin{aligned}
|\widetilde{S}(0)| &\le c_\kappa\varepsilon^4|x-y^j|^{-8(1+\widetilde{\eta}^j)} \\
&\quad + \left|\sum_{j=1}^m (1+\widetilde{\eta}^j)\left[2\frac{\nabla a}{a}\cdot\nabla(\Delta G_a(y^j,\cdot)) - V(x)\nabla\log a\cdot\nabla(G_a(y^j,\cdot))\right]\right| \\
&\quad + \left|\sum_{j=1}^m 2V(x)\frac{\nabla a}{a}\cdot\nabla\left(\Delta(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma}))\right)\right| \\
&\quad + \left|\sum_{j=1}^m \nabla\log a\cdot\nabla\left(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma})\right)\right|
\end{aligned}$$

$$+\left|\sum_{j=1}^{m}(1+\widetilde{\eta}^j)\left(\frac{\Delta a}{a}-V(x)\right)\Delta(G_a(y^j,\cdot))\right|$$
$$+\left|\sum_{j=1}^{m}\left(\frac{\Delta a}{a}-c\right)\Delta\left(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma})\right)\right|$$
$$\le c_\kappa\varepsilon^4|x-y^j|^{-8(1+\widetilde{\eta}^j)}+c_\kappa\gamma(1+\widetilde{\eta}^j)|x-y^j|^{-3}+c_\kappa(1+\widetilde{\eta}^j)\gamma|x-y^j|^{-1}$$
$$+c_\kappa(1+\widetilde{\eta}^j)\gamma r_{\varepsilon,\lambda,\gamma}^3|x-y^j|^{-4}+c_\kappa\gamma r_{\varepsilon,\lambda,\gamma}^3|x-y^j|^{-2}+c_\kappa(1+\widetilde{\eta}^j)\lambda|x-y^j|^{-2}$$
$$+c_\kappa\lambda r_{\varepsilon,\lambda,\gamma}^3|x-y^j|^{-3}.$$

Hence, for $\nu\in(-1,0)$ and $\widetilde{\eta}^j$ small enough, we get

$$\begin{aligned}\|\widetilde{N}(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\bigcup_{j=1}^m B(y^j,r_0/2))}&\le\sup_{r_{\varepsilon,\lambda,\gamma}\le r\le r_0/2}r^{4-\nu}|\widetilde{N}(0)|\\&\le c_\kappa\varepsilon^4 r_{\varepsilon,\lambda,\gamma}^{-4}+2c_\kappa\gamma+2c_\kappa\gamma r_{\varepsilon,\lambda,\gamma}^3+c_\kappa\lambda+c_\kappa\lambda r_{\varepsilon,\lambda,\gamma}^3\\&\le c_\kappa r_{\varepsilon,\lambda,\gamma}^2.\end{aligned}$$

• In Ω_{r_0,y^j} (recall that $\Omega_{r_0,y^j}=\Omega\setminus\bigcup_j B_{r_0}(y^j)$), we have $\chi_{r_0}(x-y^j)=0$ and $\Delta^2\widetilde{\boldsymbol{u}}=0$, then

$$\begin{aligned}|\widetilde{S}(0)|&\le c_\kappa\varepsilon^4|x-y^j|^{-8(1+\widetilde{\eta}^j)}\prod_{\ell=1,\ell\ne j}^{m}|x-y^\ell|^{-8(1+\widetilde{\eta}^j)}+\left|\Sigma_a^1G_a(x,y^j)\right|+\left|\Sigma_a^1G_a(x,y^j)\right|\\&\le c_\kappa\varepsilon^4|x-y^j|^{-8(1+\widetilde{\eta}^j)}+c_\kappa\gamma(1+\widetilde{\eta}^j)|x-y^j|^{-3}\\&\quad+c_\kappa(1+\widetilde{\eta}^j)\gamma|x-y^j|^{-1}+c_\kappa(1+\widetilde{\eta}^j)\lambda|x-y^j|^{-2}.\end{aligned}$$

Thus

$$\|\widetilde{N}(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega_{r_0,\widetilde{x}})}\le c_\kappa\sup_{r\ge r_0}r^{4-\nu}|\widetilde{S}(0)|\le c_\kappa\varepsilon^4+2c_\kappa\gamma+c_\kappa\lambda\le c_\kappa r_{\varepsilon,\lambda,\gamma}^2.$$

• In $B_{r_0}(y^j)-B_{r_0/2}(y^j)$, for $j=1,\dots,m$, taking into account that $\Delta^2G_a(x,y^j)=0$, using the fact that $a(x)$ is solution of (1.2) satisfying (H), and $V(x)$ is smooth bounded potential, we obtain

$$\begin{aligned}|\widetilde{S}(0)|&\le c_\kappa\varepsilon^4\prod_{j=1}^{m}e^{(1+\widetilde{\eta}^j)G_a(y^j,\cdot)}e^{\chi_{r_0}(x-y^j)H^e_{\widetilde{\varphi}^j,\widetilde{\psi}^j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})}+\left|\Delta^2\widetilde{\boldsymbol{u}}\right|+\left|\Sigma_a^1\widetilde{\boldsymbol{u}}\right|+\left|\Sigma_a^1\widetilde{\boldsymbol{u}}\right|\\&\le c_\kappa\varepsilon^4|x-y^j|^{-8(1+\widetilde{\eta}^j)}+\left|\sum_{j=1}^{m}(1+\widetilde{\eta}^j)\Delta^2(G_a(y^j,\cdot))\right|\\&\quad+\left|\sum_{j=1}^{m}\Delta^2\left(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma})\right)\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\sum_{j=1}^{m}(1+\widetilde{\eta}^j)\left[2\frac{\nabla a}{a}\cdot\nabla(\Delta G_a(y^j,\cdot))-V(x)\nabla\log a\cdot\nabla(G_a(y^j,\cdot))\right]\right|\\
&+\left|\sum_{j=1}^{m}2\frac{\nabla a}{a}\cdot\nabla\left(\Delta(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma}))\right)\right|\\
&+\left|\sum_{j=1}^{m}\nabla\log a\cdot\nabla\left(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma})\right)\right|\\
&+\left|\sum_{j=1}^{m}(1+\widetilde{\eta}^j)\left(\frac{\Delta a}{a}-V(x)\right)\Delta(G_a(y^j,\cdot))\right|\\
&+\left|\sum_{j=1}^{m}\left(\frac{\Delta a}{a}-V(x)\right)\Delta\left(\chi_{r_0}(\cdot-y^j)H^e(\varphi^j,\psi^j;(\cdot-y^j)/r_{\varepsilon,\lambda,\gamma})\right)\right|\\
\le\;& c_\kappa\varepsilon^4|x-y^j|^{-8(1+\widetilde{\eta}^j)}+c_\kappa\varepsilon^2+c_\kappa\gamma(1+\widetilde{\eta}^j)|x-y^j|^{-3}+c_\kappa(1+\widetilde{\eta}^j)\gamma|x-y^j|^{-1}\\
&+c_\kappa(1+\widetilde{\eta}^j)\gamma r^3_{\varepsilon,\lambda,\gamma}|x-y^j|^{-4}+c_\kappa\gamma r^3_{\varepsilon,\lambda,\gamma}|x-y^j|^{-2}+c_\kappa(1+\widetilde{\eta}^j)\lambda|x-y^j|^{-2}\\
&+c_\kappa\lambda r^3_{\varepsilon,\lambda,\gamma}|x-y^j|^{-3}.
\end{aligned}
$$

So, for $|x-y^j|=r$, we have $r_0/2\le r\le r_0$ then all quantity of type $|x-y^j|^\ell$, which appear to estimate $|\widetilde{S}(0)|$ are bounded, then using (5.3) and Proposition 3.2, we derive

$$
\begin{aligned}
\|\widetilde{N}(0)\|_{\mathcal{C}^{4,\alpha}_\nu(B(y^j,r_0)-B(y^j,r_0/2))}&\le c_\kappa\sup_{r_0/2\le r\le r_0} r^{4-\nu}|\widetilde{N}(0)|\\
&\le c_\kappa\varepsilon^4+c_\kappa\varepsilon^2+2c_\kappa\gamma+2c_\kappa\gamma r^3_{\varepsilon,\lambda,\gamma}+c_\kappa\lambda+c_\kappa\lambda r^3_{\varepsilon,\lambda,\gamma}\\
&\le c_\kappa r^2_{\varepsilon,\lambda,\gamma}.
\end{aligned}
$$

Finally in each subregions of $\overline{\Omega}^*$, we conclude that

$$
\|\widetilde{N}(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\Omega-\bigcup_{j=1}^m B(y^j,r_{\varepsilon,\lambda,\gamma}))}\le c_\kappa r^2_{\varepsilon,\lambda,\gamma}.
$$

To derive the second estimate, we use the fact that for $\|\widetilde{v}_i\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\le c_\kappa r^2_{\varepsilon,\lambda,\gamma}$ for $i=1,2$ in different subregions of $\overline{\Omega}^*$, we derive the following estimates

$$
\begin{aligned}
&\left\|\rho^4(e^{\widetilde{\boldsymbol{u}}+v_2}-e^{\widetilde{\boldsymbol{u}}+v_1})\right\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y))}\\
&=\rho^4\sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^{4-\nu}(e^{\widetilde{\boldsymbol{u}}+\widetilde{v}_2}-e^{\widetilde{\boldsymbol{u}}+\widetilde{v}_1})\\
&\le c_\kappa\varepsilon^4\prod_{j=1}^{m}e^{(1+\widetilde{\eta}^j)G_{y^j}}e^{\chi_{r_0}(x-y^j)H^e_{\widetilde{\varphi}^j,\widetilde{\psi}^j}((x-y^j)/r_{\varepsilon,\lambda,\gamma})}\left|e^{\widetilde{v}_2}-e^{\widetilde{v}_1}\right|\\
&\le c_\kappa\max(\varepsilon^4 r^{-4}_{\varepsilon,\lambda,\gamma},\varepsilon^4)\|\widetilde{v}_2-\widetilde{v}_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))}\\
&\le c_\kappa\varepsilon^2\|\widetilde{v}_2-\widetilde{v}_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))}
\end{aligned}
$$

and the fact that for all $w \in \mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(Y))$, there exist $c > 0$ such that $|\nabla^\ell w| \le c r^{\nu-\ell}$ $\|w\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(Y))}$, the fact that the function $a(x)$ satisfying (H) and $V(x)$ is smooth bounded potential and $\|\nabla a\|_\infty < \gamma$, then

$$\begin{aligned}
&\left\|\Sigma_a^1(\widetilde{\boldsymbol{u}}+\widetilde{v}_2) - \Sigma_a^1(\widetilde{\boldsymbol{u}}+\widetilde{v}_1)\right\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(Y))} \\
&\le c_\kappa \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^{4-\nu}\left|\Sigma_a^1(\widetilde{v}_2-\widetilde{v}_1)\right| \\
&\le c_\kappa \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^{4-\nu}\left|2\frac{\nabla a}{a}\cdot\nabla(\Delta(\widetilde{v}_2-\widetilde{v}_1)) - V(x)\nabla\log a\cdot\nabla(\widetilde{v}_2-\widetilde{v}_1)\right| \\
&\le c_\kappa\gamma \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r\|\widetilde{v}_2-\widetilde{v}_1\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)} + c_\kappa\gamma \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^3\|\widetilde{v}_2-\widetilde{v}_1\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)} \\
&\le 2c_\kappa\gamma\|\widetilde{v}_1-\widetilde{v}_2\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)}
\end{aligned}$$

and

$$\begin{aligned}
\left\|\Sigma_a^2(\widetilde{\boldsymbol{u}}+\widetilde{v}_2) - \Sigma_a^2(\widetilde{\boldsymbol{u}}+\widetilde{v}_1))\right\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(Y))} &\le c_\kappa \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^{4-\nu}\left|\Sigma_a^2(\widetilde{v}_2-\widetilde{v}_1)\right| \\
&\le c_\kappa \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^{4-\nu}\left|\left(\frac{\Delta a}{a} - V(x)\right)\Delta(\widetilde{v}_2-\widetilde{v}_1)\right| \\
&\le c_\kappa\|f\|_\infty\lambda \sup_{r\in\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)} r^2\|\widetilde{v}_2-\widetilde{v}_1\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)} \\
&\le c_\kappa\lambda\|\widetilde{v}_1-\widetilde{v}_2\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)}.
\end{aligned}$$

Using (5.3) and Proposition 3.2, we conclude that

$$\|\widetilde{N}(\widetilde{v}_1) - \widetilde{N}(\widetilde{v}_2)\|_{\mathcal{C}_\nu^{4,\alpha}(\Omega_{r_{\varepsilon,\lambda,\gamma},y^j})} \le c_\kappa r_{\varepsilon,\lambda,\gamma}^2\|\widetilde{v}_1-\widetilde{v}_2\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)}. \qquad \square$$

Reducing ε_κ, λ_κ and γ_κ if necessary, we can assume that

$$\bar{c}_\kappa r_{\varepsilon,\lambda,\gamma}^2 \le \frac{1}{2}$$

for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\gamma \in (0,\gamma_\kappa)$. Then, (5.6) and (5.7) are enough to show that

$$\widetilde{v} \mapsto \widetilde{N}(\varepsilon,\lambda,\gamma,\widetilde{\eta},Y,\Phi,\Psi;\widetilde{v})$$

is a contraction from

$$\left\{\widetilde{v}\in\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(Y)) \;\middle|\; \|\widetilde{v}\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(Y))} \le 2c_\kappa r_{\varepsilon,\lambda,\gamma}^2\right\}$$

into itself and hence has a unique fixed point $\widetilde{v}(\varepsilon,\widetilde{\eta},Y,\Phi,\Psi;\cdot)$ in this set. This fixed point is *a* solution of (5.2).

We summarize this in

Proposition 5.4. *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\gamma_\kappa > 0$, and $c_\kappa > 0$ (only depending on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\gamma \in (0, \gamma_\kappa)$ and for all set of parameters $\widetilde{\eta}$, points Y satisfying*

$$|\widetilde{\eta}| \leq \kappa r_{\varepsilon,\lambda,\gamma}^2 \quad \text{and} \quad \|Y - X\| \leq \kappa r_{\varepsilon,\lambda,\gamma}$$

and boundary functions Φ and Ψ satisfying (3.2) *and*

$$\|\Phi\|_{(\mathcal{C}^{4,\alpha}(S^3))^m} \leq \kappa r_{\varepsilon,\lambda,\gamma}^2 \quad \text{and} \quad \|\Psi\|_{(\mathcal{C}^{2,\alpha}(S^3))^m} \leq \kappa r_{\varepsilon,\lambda,\gamma}^2.$$

The function

$$\begin{aligned}\widetilde{u}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \cdot\,) := &\sum_{j=1}^{m}(1 + \widetilde{\eta}^j)G_a(y^j, \cdot\,) + \sum_{j=1}^{m} \chi_{r_0}(\,\cdot - y^j)H^e(\varphi^j, \psi^j; (\,\cdot - y^j)/r_{\varepsilon,\lambda,\gamma}) \\ &+ \widetilde{v}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \cdot\,)\end{aligned}$$

solves (5.1) *in $\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)$. In addition,*

$$\|\widetilde{v}(\varepsilon, \lambda, \gamma, \widetilde{\eta}, Y, \Phi, \Psi; \cdot\,)\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)} \leq 2c_\kappa r_{\varepsilon,\lambda,\gamma}^2.$$

Observe that the function $\widetilde{v}_{\varepsilon,\lambda,\gamma,\widetilde{\eta},Y,\Phi,\Psi}$ being obtained as a fixed point for contraction mapping, it depends continuously on the parameters $\widetilde{\eta}$ and the points Y.

6. The nonlinear Cauchy-data matching

Keeping the notations of the previous sections, we gather the results of Propositions 4.3 and 5.4. From now let $\kappa > 1$ is fixed large enough (we will shortly see how) and assume that $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\gamma \in (0, \gamma_\kappa)$.

Assume that $X = (x^1, \ldots, x^m) \in \Omega^m$ is a nondegenerate critical point of the function W defined in the introduction. For all $j = 1, \ldots, m$, we define $\tau_*^j > 0$ by

$$-4 \log \tau_*^j = R(x^j, x^j) + \sum_{\ell \neq j} G_a(x^\ell, x^j).$$

We assume that we are given:

(i) points $Y := (y^1, \ldots, y^m) \in \Omega^m$ close to $X := (x^1, \ldots, x^m)$ satisfying (5.5).

(ii) parameters $\widetilde{\eta} := (\widetilde{\eta}^1, \ldots, \widetilde{\eta}^m) \in \mathbb{R}^m$ satisfying (5.5).

(iii) parameters $T := (\tau^1, \ldots, \tau^m) \in (0, \infty)^m$ satisfying (4.5) (where, for each $j = 1, \ldots, m$, τ_* is replaced by τ_*^j).

We set

$$R^j_{\varepsilon,\lambda,\gamma} := \tau^j / r_{\varepsilon,\lambda,\gamma}.$$

First, we consider some set of boundary data

$$\Phi := (\varphi^1, \dots, \varphi^m) \in (\mathcal{C}^{4,\alpha}(S^3))^m \quad \text{and} \quad \Psi := (\psi^1, \dots, \psi^m) \in (\mathcal{C}^{2,\alpha}(S^3))^m$$

satisfying (3.1) and (4.5).

Thanks to the result of Proposition 4.3, we can find u_{int} a solution of

$$\Delta^2 u + \Sigma^1_a u + \Sigma^2_a u - \rho^4 e^u = 0$$

in each $B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$, which can be decomposed as

$$\begin{aligned} u_{\text{int}}(\varepsilon,\lambda,\gamma,T,Y,\Phi,\Psi;x) := {} & u_{\varepsilon,\tau^j}(x-y^j) + h(R^j_{\varepsilon,\lambda,\gamma}(x-y^j)/r_{\varepsilon,\lambda,\gamma}) \\ & + H^i(\varphi^j,\psi^j;(x-y^j)/r_{\varepsilon,\lambda,\gamma}) \\ & + v(\varepsilon,\lambda,\gamma,\tau^j,\varphi^j,\psi^j;R^j_{\varepsilon,\lambda,\gamma}(x-y^j)/r_{\varepsilon,\lambda,\gamma}) \end{aligned}$$

in $B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$.

Similarly, given some boundary data

$$\widetilde{\Phi} := (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m) \in (\mathcal{C}^{4,\alpha}(S^3))^m \quad \text{and} \quad \widetilde{\Psi} := (\widetilde{\psi}^1, \dots, \widetilde{\psi}^m) \in (\mathcal{C}^{2,\alpha}(S^3))^m$$

satisfying (3.2) and (5.4), we use the result of Proposition 5.4, to find u_{ext} a solution of

$$\Delta^2 u + \Sigma^1_a u + \Sigma^2_a u - \rho^4 e^u = 0$$

in $\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)$, which can be decomposed as

$$\begin{aligned} u_{\text{ext}}(\varepsilon,\lambda,\gamma,\widetilde{\eta},\widetilde{\Phi},\widetilde{\Psi};x) = {} & \sum_{j=1}^m (1+\widetilde{\eta}^j) G_a(y^j,x) + \sum_{j=1}^m \chi_{r_0}(x-y^j) H^e(\widetilde{\varphi}^j,\widetilde{\psi}^j;(x-y^j)/r_{\varepsilon,\lambda,\gamma}) \\ & + \widetilde{v}(\varepsilon,\lambda,\gamma,\widetilde{\eta},Y,\widetilde{\Phi},\widetilde{\Psi};x). \end{aligned}$$

It remains to determine the parameters and the boundary functions in such a way that the function which is equal to u_{int} in $\bigcup_j B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$ and which is equal to u_{ext} in $\overline{\Omega}_{r_{\varepsilon,\lambda,\gamma}}(Y)$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1, \dots, m$

$$u_{\text{int}} = u_{\text{ext}}, \quad \partial_r u_{\text{int}} = \partial_r u_{\text{ext}}, \quad \Delta u_{\text{int}} = \Delta u_{\text{ext}}, \quad \partial_r \Delta u_{\text{int}} = \partial_r \Delta u_{\text{ext}},$$

on $\partial B_{r_{\varepsilon,\lambda,\gamma}}(y^j)$. Assuming we have already done so, this provides for each ε, λ and γ are small enough a function $w_{\varepsilon,\lambda,\gamma} \in \mathcal{C}^{4,\alpha}(\overline{\Omega})$ (which is obtained by patching together the function u_{int} and the function u_{ext}) solution of $\Delta(a(x)\Delta u) - V(x)\,\mathrm{div}(a(x)\nabla u) - \rho^4 a(x) e^u =$

0 and elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result by applying Schauder's fixed point theorem in the ball of radius $\kappa r^2_{\varepsilon,\lambda,\gamma}$ in the product space, as ε, λ and γ tend to 0. The sequence of solutions we have obtained satisfies the required properties, namely, away from the points x^j the sequence $w_{\varepsilon,\lambda,\gamma}$ converges to $\sum_j G_a(x^j,\cdot)$. For reader's convenience see [1] about the rest of the proof.

Remark 6.1. To see how the condition $(x^1,\dots,x^m)$ is a *nondegenerate* critical point of the function W, defined in (1.3), enters in our analysis and in order to make the exposition of this "nonlinear domain decomposition technique" as clear as possible, the reader is referred to a comments in [1, pp. 18–19].

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