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Blow-up Phenomena for a Porous Medium Equation with Time-dependent Coefficients and Inner Absorption Term Under Nonlinear Boundary Flux

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Abstract. This paper deals with blow-up phenomena for an initial boundary value problem of a porous medium equation with time-dependent coefficients and inner absorption term in a bounded star-shaped region under nonlinear boundary flux. Using the auxiliary function method and modified differential inequality technique, we establish some conditions on time-dependent coefficient and nonlinear functions to guarantee that the solution $u(x,t)$ exists globally or blows up at some finite time t^*. Moreover, the upper and lower bounds for t^* are derived in the higher dimensional space. Finally, some examples are presented to illustrate applications of our results.

1. Introduction

Our main interest lies in the following porous medium equation with time-dependent coefficient and inner absorption term

$$u_t = \Delta u^m - k(t)f(u), \quad (x,t) \in \Omega \times (0,t^*), \tag{1.1}$$

subject to the nonlinear Neumann boundary and initial conditions

$$\frac{\partial u^m}{\partial \nu} = g(u), \quad (x,t) \in \partial\Omega \times (0,t^*), \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$ and ν is the unit outward normal vector on $\partial\Omega$, $m \geq 1$. The coefficient $k(t)$ is a nonnegative differentiable function, t^* is a possible blow-up time when blow-up occurs, otherwise $t^* = +\infty$. The nonlinear functions $f(u)$ and $g(u)$ are nonnegative continuous functions which satisfy some appropriate conditions, and the initial data $u_0(x)$ is a nonnegative C^1-function which satisfies a compatibility condition. By the degenerate parabolic theory, one can deduce that the local weak solution of (1.1)–(1.3) exists uniquely and is nonnegative,

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see [1, 6]. Moreover, for convenience, we may assume that the appropriate weak solution is smooth, and no longer consider approximation problem.

Equation (1.1) describes the diffusion of concentration of some Newtonian fluids via porous medium or the density of some biological species in many physical phenomena and biological species theories. It has been known that the nonlocal source term presents a more realistic model for population dynamics, see [2, 13]. In the nonlinear diffusion theory, there exist obvious differences among the situations of slow ($m > 1$), fast ($0 < m < 1$), and linear ($m = 1$) diffusions. For example, there is a finite speed propagation in the slow and linear diffusion situations, whereas an infinite speed propagation exists in the fast diffusion situation. The nonlinear boundary flux (1.2) can be physically interpreted as the nonlinear radial law, see [11].

In the past decades, there have been many works dealing with existence and nonexistence of global solutions, blow-up of solutions, bounds for blow-up time, blow-up rates, blow-up sets, and asymptotic behavior of the solutions to nonlinear parabolic equations, refer to [1, 3, 6, 12, 21]. Roughly, it has been seen that existence of global and nonglobal solutions and behavior of the solutions to parabolic equations depend on nonlinearity, dimension, initial data, and nonlinear boundary flux. In this paper, we are particularly interested in the topic about the upper and lower bounds for the blow-up time of the blow-up solution of parabolic equation. A variety of methods have been used to determine the blow-up of solutions and to indicate an upper bound for the blow-up time (see [15] and the references therein). However, lower bounds for blow-up time may be harder to be determined. Recently, the study of the lower bound estimate for the blow-up time of the blow-up solutions also makes some new progress. For the studies on initial boundary problems of semilinear parabolic equation with constant-dependent coefficients under nonlinear boundary flux, Payne and Schaefer [20] studied initial boundary value problems of linear heat equation

$$u_t = \Delta u, \quad (x, t) \in \Omega \times (0, t^*).$$

Under the suitable conditions on the nonlinearities, they determined a lower bound on the blow-up time in $\mathbb{R}^3$ when blow-up occurs. In addition, a sufficient condition which implies that blow-up does occur was determined and an upper bound for t^* was derived. Payne et al. [19] studied the semilinear parabolic equation with inner absorption term

$$u_t = \Delta u - f(u), \quad (x, t) \in \Omega \times (0, t^*),$$

under the nonlinear Neumann boundary condition, where Ω is a bounded star-shaped region in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$. By virtue of a differential inequality technique, they introduced some appropriate conditions on nonlinearities sufficient to guarantee $u(x, t)$ exists for all time $t > 0$ or blows up at some finite time t^*. Moreover, an upper bound for t^* was derived. Under somewhat more restrictive conditions, a lower bound for t^* was derived in $\mathbb{R}^3$ when blow-up occurs. Recently, Baghaei et al. [4] investigated the following nonlinear divergence form of semilinear parabolic equation with absorption term and nonlinear boundary source

$$u_t = \sum_{i,j=1}^{N} \left(a^{ij}(x)u_{x_i}\right)_{x_j} - f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

and obtained a lower bound for the blow-up time in the high-dimensional space. Hu et al. [14] studied the slow diffusion equation with inner absorption terms

$$u_t = \Delta u^m - f(u), \quad (x,t) \in \Omega \times (0,T)$$

under the nonlinear Neumann boundary condition. Under the suitable conditions, they established a lower bound of blow-up time in three-dimensional space. In [7], Enache considered the quasilinear parabolic equation with source term

$$u_t = (g(u)_{,i})_{,i} + f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, and nonlinear terms $f(u)$ and $g(u)$ satisfy some appropriate conditions. Using a first-order differential inequality technique, they introduced some sufficient conditions to guarantee that the solution exists globally or blows up and estimated a lower bound for the blow-up time in $\mathbb{R}^3$ under Robin boundary condition. To see some studies on porous medium equations with nonlocal source under null Dirichlet and Neumann boundary conditions in three-dimensional space and on p-Laplacian parabolic equations under nonlinear boundary flux, refer to [8, 9, 17].

For the studies on initial boundary problems of parabolic equation with time-dependent coefficients under nonlinear boundary flux, Fang and Wang [10] investigated the divergence form of a parabolic equation with time-dependent coefficient and inner absorption term

$$u_t = \sum_{i,j=1}^{N} \left(a^{ij}(x)u_{x_i}\right)_{x_j} - k(t)f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

under nonlinear boundary flux, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$. They established some conditions on nonlinear terms $f(u)$, $g(u)$ and initial data which guarantee blow-up or global existence of the solutions and derived an upper bound of the blow-up time. They also obtained a lower bound of blow-up time under more restrictive conditions. Recently, Liu and Fang [16] promoted this conclusion to quasilinear case. For the studies on quasilinear problems under homogeneous Dirichlet boundary condition, refer to [18].

In view of the works mentioned above, one can find that research on the blow-up phenomena of the solutions to porous medium equations with absorption terms having time-dependent coefficients under nonlinear boundary flux has not been started yet. A difficulty lies in finding an influence of $k(t)$ and a competitive relationship between diffusion, inner absorption and boundary source in determining blow-up of the solutions. Particularly, in contrast to other quasilinear problems, dealing with quasilinear diffusion terms Δu^m in a porous media model (1.1) has some difficulties. By virtue of the modified differential inequality technique, we establish some conditions on time-dependent coefficient and nonlinear functions to guarantee that the solution $u(x,t)$ exists globally or blows up at some finite time t^*, and we also derive the lower and upper bounds for t^* in the higher dimensional space.

The rest of our paper is organized as follows. In Sections 2 and 3, we establish some conditions on $k(t)$, $f(u)$, and $g(u)$ to guarantee that the solution $u(x,t)$ exists globally or blows up in finite time t^*, and then obtain an upper bound for t^*. A lower bound of t^* is derived in Section 4. Finally, some examples are presented to illustrate applications of our results.

2. The global existence

In this section, we establish some conditions on $k(t)$ and the nonlinear functions f and g to guarantee the existence of global solution. In order to prove our result, we first recall a lemma in [10] and state it as follows:

Lemma 2.1. [10] *Suppose that Ω is a bounded star-shaped region in $\mathbb{R}^N$ and $N \geq 2$. Then for any nonnegative C^1-function u and constant $n > 0$, we have the inequality*

$$\int_{\partial\Omega} u^n \, dS \leq \frac{N}{\rho_0} \int_{\Omega} u^n \, dx + \frac{nd}{\rho_0} \int_{\Omega} u^{n-1} |\nabla u| \, dx,$$

where

$$\rho_0 = \min_{x \in \partial\Omega} (x \cdot \nu), \quad d = \max_{x \in \overline{\Omega}} |x|.$$

Theorem 2.2. *Suppose that the nonnegative functions $f(u)$ and $g(u)$ satisfy*

$$f(\xi) \geq k_1 \xi^p, \quad \xi \geq 0, \tag{2.1}$$

$$g(\xi) \leq k_2 \xi^q, \quad \xi \geq 0, \tag{2.2}$$

where $k_1 > 0$, $k_2 \geq 0$, $q > m$, $p + m > 2q$ and

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \leq -r_1, \quad t > 0 \tag{2.3}$$

for a positive constant r_1. Then the (nonnegative) solution $u(x,t)$ of problem (1.1)–(1.3) *does not blow up; that is, $u(x,t)$ exists for all $t > 0$.*

Remark 2.3. From the conditions $q > m$ and $p + m > 2q$ in Theorem 2.2, one can easily obtain that $p > q$.

Proof. Set

$$\phi(t) := k(t) \int_\Omega u^2\,dx.$$

We first show that ϕ is not an increasing function. To this end, we compute the derivative

$$\begin{aligned}\phi'(t) &= k'(t) \int_\Omega u^2\,dx + 2k(t) \int_\Omega u u_t\,dx \\ &= k'(t) \int_\Omega u^2\,dx + 2k(t) \int_\Omega u\Delta u^m\,dx - 2k^2(t) \int_\Omega u f(u)\,dx.\end{aligned}$$

Making use of (2.1)–(2.3) and the divergence theorem, we have the inequality

$$\phi'(t) \le -r_1\phi + 2k(t)k_2 \int_{\partial\Omega} u^{q+1}\,dS - 2mk(t) \int_\Omega u^{m-1}|\nabla u|^2\,dx - 2k^2(t)k_1 \int_\Omega u^{p+1}\,dx. \tag{2.4}$$

By Lemma 2.1, one can have the inequality

$$\int_{\partial\Omega} u^{q+1}\,dS \le \frac{N}{\rho_0} \int_\Omega u^{q+1}\,dx + \frac{(q+1)d}{\rho_0} \int_\Omega u^q|\nabla u|\,dx, \tag{2.5}$$

and if Ω is with respect to x_0, $x_0 \ne 0$, by using the technique of translation in Lemma 2.1, and setting

$$\rho_0 = \min_{x\in\partial\Omega} \left((x - x_0)\cdot\nu\right), \quad d = \max_{x\in\overline{\Omega}} |x - x_0|,$$

we can get (2.5) easily. It follows from Schwarz's and Young's inequalities that

$$\begin{aligned}\int_\Omega u^q|\nabla u|\,dx &\le \left(\int_\Omega u^{m-1}|\nabla u|^2\,dx\right)^{1/2} \left(\int_\Omega u^{2q-m+1}\,dx\right)^{1/2} \\ &\le \frac{1}{2\varepsilon_1} \int_\Omega u^{m-1}|\nabla u|^2\,dx + \frac{\varepsilon_1}{2} \int_\Omega u^{2q-m+1}\,dx,\end{aligned} \tag{2.6}$$

where ε_1 is a positive constant to be determined later. Substituting (2.5) and (2.6) into (2.4), we get the inequality

$$\begin{aligned}\phi'(t) \le -r_1\phi &+ \frac{2Nk_2}{\rho_0}k(t) \int_\Omega u^{q+1}\,dx + \frac{k_2 d(q+1)\varepsilon_1}{\rho_0}k(t) \int_\Omega u^{2q-m+1}\,dx \\ &+ \left(\frac{k_2 d(q+1)}{\rho_0\varepsilon_1} - 2m\right) k(t) \int_\Omega u^{m-1}|\nabla u|^2\,dx - 2k_1k^2(t) \int_\Omega u^{p+1}\,dx.\end{aligned}$$

Selecting

$$\frac{k_2 d(q+1)}{\rho_0\varepsilon_1} - 2m = 0,$$

then we have $\varepsilon_1 = k_2 d(q+1)/(2m\rho_0)$ and obtain the inequality

$$\begin{aligned}\phi'(t) \le -r_1\phi + \frac{2Nk_2}{\rho_0}k(t)\int_\Omega u^{q+1}\,dx + \frac{k_2 d(q+1)\varepsilon_1}{\rho_0}k(t)\int_\Omega u^{2q-m+1}\,dx \\ - 2k_1k^2(t)\int_\Omega u^{p+1}\,dx.\end{aligned} \tag{2.7}$$

We now focus our attention on $k(t)\int_\Omega u^{2q-m+1}\,dx$. From Hölder's inequality, we can have the inequality

$$k(t)\int_\Omega u^{2q-m+1}\,dx \le \left(k(t)\int_\Omega u^{p+1}\,dx\right)^{1-\alpha_1}\left(k(t)\int_\Omega u^{q+1}\,dx\right)^{\alpha_1},$$

where $\alpha_1 = (p+m-2q)/(p-q) \in (0,1)$. Furthermore, we can obtain the inequalities

$$\begin{aligned}k(t)\int_\Omega u^{2q-m+1}\,dx &\le \left[\varepsilon_2 k(t)\int_\Omega u^{p+1}\,dx\right]^{1-\alpha_1}\left[\varepsilon_2^{(\alpha_1-1)/\alpha_1}k(t)\int_\Omega u^{q+1}\,dx\right]^{\alpha_1} \\ &\le (1-\alpha_1)\varepsilon_2 k(t)\int_\Omega u^{p+1}\,dx + \alpha_1\varepsilon_2^{(\alpha_1-1)/\alpha_1}k(t)\int_\Omega u^{q+1}\,dx\end{aligned} \tag{2.8}$$

for arbitrary $\varepsilon_2 > 0$ by the arithmetic and geometric inequality

$$a^s b^{1-s} \le as + b(1-s) \quad \text{for } a, b > 0,\ 0 < s < 1.$$

Substituting (2.8) into (2.7) yields the inequalities

$$\begin{aligned}\phi'(t) &\le -r_1\phi + \frac{2k_2N}{\rho_0}k(t)\int_\Omega u^{q+1}\,dx - 2k_1k^2(t)\int_\Omega u^{p+1}\,dx \\ &\quad + \frac{k_2(q+1)d\varepsilon_1}{\rho_0}\left[(1-\alpha_1)\varepsilon_2 k(t)\int_\Omega u^{p+1}\,dx + \alpha_1\varepsilon_2^{(\alpha_1-1)/\alpha_1}k(t)\int_\Omega u^{q+1}\,dx\right] \\ &\le M_1 k(t)\int_\Omega u^{q+1}\,dx - M_2 k(t)\int_\Omega u^{p+1}\,dx,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}M_1 &= \frac{2k_2N}{\rho_0} + \frac{k_2(q+1)d\varepsilon_1}{\rho_0}\alpha_1\varepsilon_2^{(\alpha_1-1)/\alpha_1} > 0, \\ M_2 &= 2k_1k(t) - \frac{k_2(q+1)d\varepsilon_1}{\rho_0}(1-\alpha_1)\varepsilon_2,\end{aligned}$$

and $\varepsilon_2 > 0$ is a sufficiently small constant so that $M_2 > 0$. Applying Hölder's inequality to the second term on the right-hand side of (2.9), we get

$$\int_\Omega u^{q+1}\,dx \le \left(\int_\Omega u^{p+1}\,dx\right)^{(q+1)/(p+1)}|\Omega|^{(p-q)/(p+1)}, \tag{2.10}$$

where $|\Omega| = \int_\Omega dx$ is the N-volume of Ω. Inserting (2.10) into (2.9), we have

$$\begin{aligned} \phi'(t) &\le M_1 k(t)|\Omega|^{(p-q)/(p+1)} \left(\int_\Omega u^{p+1}\,dx\right)^{(q+1)/(p+1)} - M_2 k(t)\int_\Omega u^{p+1}\,dx \\ &= M_1|\Omega|^{(p-q)/(p+1)}k(t)\left(\int_\Omega u^{p+1}\,dx\right)^{(q+1)/(p+1)} \\ &\quad\times\left[1 - |\Omega|^{(q-p)/(p+1)}\frac{M_2}{M_1}\left(\int_\Omega u^{p+1}\,dx\right)^{(p-q)/(p+1)}\right]. \end{aligned} \tag{2.11}$$

By using Hölder's inequality, we can have the inequality

$$\phi(t) = k(t)\int_\Omega u^2\,dx \le k(t)\left(\int_\Omega u^{p+1}\,dx\right)^{2/(p+1)}|\Omega|^{(p-1)/(p+1)},$$

i.e.,

$$\int_\Omega u^{p+1}\,dx \ge \left[|\Omega|^{(1-p)/(p+1)}k^{-1}(t)\phi(t)\right]^{(p+1)/2}. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\begin{aligned} \phi'(t) &\le M_1|\Omega|^{(p-q)/(p+1)}k(t)\left(\int_\Omega u^{p+1}\,dx\right)^{(q+1)/(p+1)} \\ &\quad\times\left[1 - |\Omega|^{(q-p)/2}\frac{M_2}{M_1}k^{(q-p)/2}(t)\phi^{(p-q)/2}\right] \end{aligned} \tag{2.13}$$

with $(p-q)/2 > 0$. Since $k'(t)/k(t) \le -r_1$ and the positive coefficient $k(t)$ is a nonincreasing function, one can conclude from (2.13) that $\phi(t)$ is bounded for all $t > 0$ under the conditions in Theorem 2.2. In fact, if $u(x,t)$ blows up at finite time t^*, then $\phi(t)$ is unbounded near t^*, which forces $\phi'(t) \le 0$ in some interval $[t_0, t^*)$, and hence, we have $\phi(t) \le \phi(t_0)$ in $[t_0, t^*)$, which implies that $\phi(t)$ is bounded in $[t_0, t^*)$. This is a contradiction. Therefore, $u(x,t)$ exists for all $t > 0$, which completes the proof. □

Remark 2.4. The proof is suitable for the process of $m > 0$.

For the special case $q = 1$, $0 < m < 1$, one can obtain the same result under slightly different conditions.

Theorem 2.5. *Suppose that the nonlinear functions $f(u)$ and $g(u)$ satisfy* (2.1) *and* (2.2) *in Theorem* 2.2 *with constants $k_1 > 0$, $k_2 \ge 0$, $q = 1$, $p + m > 2$ and*

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \le r_2, \quad t > 0,$$

where r_2 is a nonnegative constant. Then the (nonnegative) solution $u(x,t)$ of problem (1.1)–(1.3) *does not blow up; that is, $u(x,t)$ exists for all $t > 0$.*

Proof. Set

$$\phi_1(t) := (k(t))^\mu \int_\Omega u^\sigma \, dx,$$

where $\mu > 0$, $\sigma > 3 - m$. By using similar arguments as used in the proof of Theorem 2.2, we can have the inequality

$$\begin{aligned}\phi_1'(t) &= \mu \frac{k'(t)}{k(t)} \phi_1 + \sigma (k(t))^\mu \int_\Omega u^{\sigma-1} u_t \, dx \\ &= \mu \frac{k'(t)}{k(t)} \phi_1 + \sigma (k(t))^\mu \int_\Omega u^{\sigma-1} \left[\Delta u^m - k(t) f(u) \right] dx \\ &\le \mu r_2 \phi_1 + \sigma k_2 (k(t))^\mu \int_{\partial\Omega} u^\sigma \, dS - k_1 \sigma (k(t))^{\mu+1} \int_\Omega u^{\sigma+p-1} \, dx \\ &\quad - m\sigma(\sigma-1)(k(t))^\mu \int_\Omega u^{m+\sigma-3} |\nabla u|^2 \, dx. \end{aligned} \tag{2.14}$$

By Lemma 2.1, one can have the inequality

$$\int_{\partial\Omega} u^\sigma \, dS \le \frac{N}{\rho_0} \int_\Omega u^\sigma \, dx + \frac{\sigma d}{\rho_0} \int_\Omega u^{\sigma-1} |\nabla u| \, dx, \tag{2.15}$$

and if Ω is with respect to x_0, $x_0 \neq 0$, by using the technique of translation in Lemma 2.1, and setting

$$\rho_0 = \min_{x \in \partial\Omega} \left((x - x_0) \cdot \nu \right), \quad d = \max_{x \in \overline{\Omega}} |x - x_0|,$$

we can get (2.15) easily. Let us consider the second term on the right-hand side of (2.15). By Hölder's inequality and Young's inequality, we can obtain the inequalities

$$\begin{aligned}\int_\Omega u^{\sigma-1} |\nabla u| \, dx &\le \left(\int_\Omega u^{\sigma+m-3} |\nabla u|^2 \, dx \right)^{1/2} \left(\int_\Omega u^{\sigma-m+1} \, dx \right)^{1/2} \\ &\le \frac{1}{2\varepsilon_3} \int_\Omega u^{\sigma+m-3} |\nabla u|^2 \, dx + \frac{\varepsilon_3}{2} \int_\Omega u^{\sigma-m+1} \, dx, \end{aligned} \tag{2.16}$$

where ε_3 is a positive constant to be determined later. Combining (2.14), (2.15) with (2.16) and taking $\varepsilon_3 = k_2 \sigma d / [2m\rho_0(\sigma-1)]$, we get the inequality

$$\begin{aligned}\phi_1'(t) &\le \left(\mu r_2 + \frac{\sigma N k_2}{\rho_0} \right) \phi_1 + \frac{k_2 \sigma^2 d \varepsilon_3}{2\rho_0} (k(t))^\mu \int_\Omega u^{\sigma-m+1} \, dx \\ &\quad - k_1 \sigma (k(t))^{\mu+1} \int_\Omega u^{\sigma+p-1} \, dx. \end{aligned} \tag{2.17}$$

We now focus our attention on $(k(t))^\mu \int_\Omega u^{\sigma-m+1} \, dx$. From Hölder's inequality, we can have the inequality

$$(k(t))^\mu \int_\Omega u^{\sigma-m+1} \, dx \le \left[(k(t))^\mu \int_\Omega u^\sigma \, dx \right]^{\alpha_2} \left[(k(t))^\mu \int_\Omega u^{\sigma+p-1} \, dx \right]^{1-\alpha_2},$$

where $\alpha_2 = (m+p-2)/(p-1) \in (0,1)$. We can use Young's inequalities and get

$$(k(t))^\mu \int_\Omega u^{\sigma-m+1}\,dx \le \alpha_2\varepsilon_4\phi_1 + (1-\alpha_2)\varepsilon_4^{\alpha_2/(\alpha_2-1)}(k(t))^\mu \int_\Omega u^{\sigma+p-1}\,dx \tag{2.18}$$

for arbitrary $\varepsilon_4 > 0$. Substituting (2.18) into (2.17), we have

$$\begin{aligned} \phi_1'(t) &\le \left(\mu r_2 + \frac{\sigma N k_2}{\rho_0}\right)\phi_1 - k_1\sigma(k(t))^{\mu+1}\int_\Omega u^{\sigma+p-1}\,dx \\ &\quad + \frac{k_2\sigma^2 d\varepsilon_3}{2\rho_0}\left[\alpha_2\varepsilon_4\phi_1 + (1-\alpha_2)\varepsilon_4^{\alpha_2/(\alpha_2-1)}(k(t))^\mu\int_\Omega u^{\sigma+p-1}\,dx\right] \\ &= C_1\phi_1 - C_2(k(t))^\mu\int_\Omega u^{\sigma+p-1}\,dx, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} C_1 &= \mu r_2 + \frac{\sigma N k_2}{\rho_0} + \frac{k_2 d\sigma^2\alpha_2\varepsilon_3\varepsilon_4}{2\rho_0} > 0, \\ C_2 &= k_1\sigma k(t) - \frac{k_2\sigma^2 d\varepsilon_3}{2\rho_0}(1-\alpha_2)\varepsilon_4^{\alpha_2/(\alpha_2-1)}, \end{aligned}$$

and $\varepsilon_4 > 0$ is a sufficiently small constant so that $C_2 > 0$. Applying Hölder's inequality to the second term on the right-hand side of (2.19), we get

$$(k(t))^\mu \int_\Omega u^{\sigma+p-1}\,dx \ge \phi_1^{(\sigma+p-1)/\sigma}\left(|\Omega|(k(t))^\mu\right)^{(1-p)/\sigma}, \tag{2.20}$$

where $|\Omega| = \int_\Omega dx$ is the N-volume of Ω. Inserting (2.20) into (2.19), we have

$$\phi_1'(t) \le C_1\phi_1\left[1 - \frac{C_2}{C_1}|\Omega|^{(1-p)/\sigma}(k(t))^{\mu(1-p)/\sigma}\phi_1^{(p-1)/\sigma}\right].$$

By an analogous analysis as in the proof of Theorem 2.2, one can easily conclude that the solution $u(x,t)$ exists for all $t > 0$, which completes the proof. □

3. Upper bound of blow-up time t^*

In this section, Ω needs not to be star-sharped. We assume some conditions to assure that the solution $u(x,t)$ of (1.1)–(1.3) blows up at finite time t^* and derive an upper bound for T. The result can be summarized as follows:

Theorem 3.1. *Suppose that Ω is a bounded region in $\mathbb{R}^N$ $(N \ge 2)$ with smooth boundary $\partial\Omega$ and $u(x,t)$ is a nonnegative solution of problem (1.1)–(1.3), and assume that the nonnegative integrable functions $f(u)$ and $g(u)$ satisfy the conditions*

$$\xi f(\xi) \le 2(1+\alpha)F(\xi), \quad \xi \ge 0, \tag{3.1}$$

$$\xi g(\xi) \ge 2(1+\beta)G(\xi), \quad \xi \ge 0, \tag{3.2}$$

where

$$F(\xi) := \int_0^\xi f(\eta)\, d\eta, \quad G(\xi) := \int_0^\xi g(\eta)\, d\eta, \tag{3.3}$$

$$0 \le \alpha \le \beta. \tag{3.4}$$

Let

$$\begin{aligned} &\psi(t) := 2\int_{\partial\Omega} G(u)\, dS - \frac{4m}{(m+1)^2}\int_\Omega \left|\nabla u^{(m+1)/2}\right|^2 dx - 2k(t)\int_\Omega F(u)\, dx, \\ &\psi(0) > 0, \end{aligned} \tag{3.5}$$

and

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \le -r_3, \quad t > 0 \tag{3.6}$$

for a positive constant r_3.

Then the solution $u(x,t)$ of problem (1.1)–(1.3) *blows up at some finite time $t^* \le T$ with*

$$T = \frac{\chi(0)}{2\beta(1+\beta)\psi(0)}, \quad \beta > 0,$$

where $\chi(t) := \int_\Omega u^2\, dx$ and $\chi(0) > 0$. If $\beta = 0$, then $u(x,t)$ blows up at infinite time.

Remark 3.2. If we take the local term $f(\xi) \le k_1\xi^p$ with $p \le 2\alpha + 1$ and $g(\xi) \ge k_2\xi^q$ with $q \ge 2\beta + 1$, then the functions f and g satisfy the conditions (3.1) and (3.2).

Proof. In order to prove that the solution blows up in finite time under the assumption of Theorem 3.1. When $\beta > 0$, we first assume the solution $u(x,t)$ is global to get a contradiction. In this way, the auxiliary function $\chi(t)$ is bounded for all $t > 0$. We compute the derivative

$$\begin{aligned} \chi'(t) &= 2\int_\Omega u u_t\, dx = 2\int_\Omega u\left[\Delta u^m - k(t) f(u)\right] dx \\ &= 2\int_\Omega u\Delta u^m\, dx - 2k(t)\int_\Omega u f(u)\, dx. \end{aligned}$$

By using hypotheses (3.1)–(3.5) given in Theorem 3.1, one can see that

$$\begin{aligned} \chi'(t) &= 2\int_{\partial\Omega} u\frac{\partial u^m}{\partial\nu}\, dS - 2\int_\Omega \nabla u \nabla u^m\, dx - 2k(t)\int_\Omega u f(u)\, dx \\ &\ge 4(1+\beta)\int_{\partial\Omega} G(u)\, dS - 2(1+\beta)\int_\Omega \frac{4m}{(m+1)^2}\left|\nabla u^{(m+1)/2}\right|^2 dx \\ &\quad - 4(1+\alpha)k(t)\int_\Omega F(u)\, dx \\ &\ge 2(1+\beta)\left[2\int_{\partial\Omega} G(u)\, dS - \frac{4m}{(m+1)^2}\int_\Omega \left|\nabla u^{(m+1)/2}\right|^2 dx - 2k(t)\int_\Omega F(u)\, dx\right] \\ &= 2(1+\beta)\psi(t). \end{aligned} \tag{3.7}$$

Computing the derivative of $\psi(t)$, it can be seen that

$$\begin{aligned}\psi'(t) &= 2\int_{\partial\Omega} g(u)u_t\,dS - \frac{4m}{(m+1)^2}\int_\Omega \left(\left|\nabla u^{(m+1)/2}\right|^2\right)_t dx \\ &\quad - 2k(t)\int_\Omega f(u)u_t\,dx - 2k'(t)\int_\Omega F(u)\,dx \\ &= 2\int_{\partial\Omega} g(u)u_t\,dS - 2\int_\Omega \nabla u^m \nabla u_t\,dx - 2k(t)\int_\Omega f(u)u_t\,dx - 2k'(t)\int_\Omega F(u)\,dx \\ &= 2\int_\Omega u_t\left[\Delta u^m - k(t)f(u)\right]dx - 2k'(t)\int_\Omega F(u)\,dx \\ &\geq 2\int_\Omega u_t^2\,dx + 2r_3k(t)\int_\Omega F(u)\,dx \geq 2\int_\Omega u_t^2\,dx \geq 0,\end{aligned} \tag{3.8}$$

which implies $\psi(t) > 0$ for all $t \in (0, t^*)$, since $\psi(0) > 0$.

Making use of the Schwarz's inequality and (3.8), we can have the inequalities

$$(\chi'(t))^2 = 4\left(\int_\Omega uu_t\,dx\right)^2 \leq 4\int_\Omega u^2\,dx \int_\Omega u_t^2\,dx \leq 2\chi(t)\psi'(t). \tag{3.9}$$

By (3.7) and (3.9), we can deduce

$$\chi(t)\psi'(t) \geq \frac{1}{2}(\chi'(t))^2 \geq (1+\beta)\chi'(t)\psi(t),$$

i.e.,

$$\left(\psi\chi^{-(1+\beta)}\right)' \geq 0. \tag{3.10}$$

By $\psi(0) > 0$, $\psi(t) > 0$, $\chi(0) > 0$ and (3.7), we get

$$\chi(t) > 0, \quad t \geq 0.$$

Integrating (3.10) over $[0, t]$, one can see that

$$\psi(t)\chi^{-(1+\beta)}(t) \geq \psi(0)\chi^{-(1+\beta)}(0) =: M > 0. \tag{3.11}$$

It follows from (3.7) and (3.11) that

$$\chi'(t) \geq 2(1+\beta)\psi(t) \geq 2M(1+\beta)\chi^{1+\beta}(t). \tag{3.12}$$

Now, integrating (3.12) we have the following inequality

$$\chi^{-\beta}(t) \leq \chi^{-\beta}(0) - 2\beta(1+\beta)Mt. \tag{3.13}$$

Obviously, (3.13) cannot hold for all time t, which is a contradiction. Hence the solution $u(x,t)$ blows up in finite time. Therefore, (3.13) leads to

$$t^* \leq T = \frac{\chi(0)}{2\beta(1+\beta)\psi(0)}, \quad \forall\, \beta > 0.$$

If $\beta = 0$, we have the inequalities

$$(\psi\chi^{-1})' \geq 0, \quad \chi'(t) \geq 2M\chi(t), \quad \chi(t) \geq e^{2Mt}\chi(0),$$

which is valid for all $t > 0$, implying that the solution $u(x,t)$ blows up at infinite time. This completes the proof. □

Remark 3.3. If the condition (3.6) in Theorem 3.1 is replaced by the following condition:

$$k(t) < 0, \quad 0 < \frac{k'(t)}{k(t)} \leq r_4, \quad t > 0, \quad r_4 > 0, \tag{3.14}$$

then we can easily obtain similar results as the ones in Theorem 3.1. In fact, (3.14) implies $k'(t) < 0$ and (3.8) becomes

$$\psi'(t) \geq 2\int_\Omega u_t^2\,dx - 2k'(t)\int_\Omega F(u)\,dx \geq 2\int_\Omega u_t^2\,dx \geq 0.$$

4. Lower bounds for t^*

In this section, the domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ needs to be a convex bounded domain with smooth boundary. Moreover, we make some appropriate assumptions on nonlinear functions $f(u)$, $g(u)$ and $k(t)$ to seek a lower bound for blow-up time t^*. We state our result below.

Theorem 4.1. *Suppose that $u(x,t)$ is the nonnegative solution of problem* (1.1)–(1.3), *$u(x,t)$ blows up at t^*, and assume that the nonnegative functions $f(u)$ and $g(u)$ satisfy* (2.1) *and* (2.2), *where $k_1 > 0$, $k_2 > 0$, $p > m$ and*

$$k(t) \geq r_5, \quad t > 0, \quad r_5 > 0. \tag{4.1}$$

Define a function

$$\varphi(t) := \int_\Omega u^\theta\,dx,$$

where θ is a parameter such that

$$\theta > \max\left\{\frac{4(2q-m-1)(N-2)-(m-1)N}{2}, m\right\}$$

If $2q > m+p$, $q > p$, then the blow-up time t^* is bounded below, i.e.,

$$t^* \geq T_0 = \int_{\varphi(0)}^\infty \frac{d\varphi}{A_1 + A_2\varphi^{3(N-2)/(3N-8)}}.$$

If $2q = m+p$, then the blow-up time t^* is bounded below, i.e.,

$$t^* \geq T_1 = \int_{\varphi(0)}^\infty \frac{d\varphi}{B_1 + B_2\varphi + B_3\varphi^{3(N-2)/(3N-8)}},$$

where $\varphi(0) = \int_\Omega u_0^\theta\,dx$, and A_1, A_2, B_1, B_2 and B_3 are positive constants to be determined later.

Remark 4.2. From the conditions $p > m \geq 1$ and $2q \geq m + p$ in Theorem 4.1, one can easily obtain that $2q > m + 1$ and $q > m$.

Proof. Let $v = u^m$, so

$$(v^{1/m})_t = \Delta v - k(t) f(v^{1/m}), \quad (x,t) \in \Omega \times (0,t^*), \tag{4.2}$$

$$\frac{\partial v}{\partial \nu} = g(v^{1/m}), \quad (x,t) \in \partial\Omega \times (0,t^*), \tag{4.3}$$

$$v^{1/m}(x,0) = u_0^{1/m}(x) \geq 0, \quad x \in \Omega,$$

and $\varphi(t) = \int_\Omega (v^{1/m})^\theta \, dx$. Computing the derivative of $\varphi(t)$ and using (4.2), (4.3), (2.1), (2.2) and (4.1), we get

$$\begin{aligned}
\varphi'(t) &= \theta \int_\Omega (v^{1/m})^{\theta-1} (v^{1/m})_t \, dx = \theta \int_\Omega (v^{1/m})^{\theta-1} \left[\Delta v - k(t) f(v^{1/m}) \right] dx \\
&\leq \theta k_2 \int_{\partial\Omega} v^{(\theta+q-1)/m} \, dS - \theta \frac{\theta-1}{m} \int_\Omega v^{(\theta-1)/m-1} |\nabla v|^2 \, dx \\
&\quad - \theta k_1 r_5 \int_\Omega v^{(\theta-1)/m} v^{p/m} \, dx \\
&= \theta k_2 \int_{\partial\Omega} v^{(\theta+q-1)/m} \, dS - \frac{4m\theta(\theta-1)}{(m+\theta-1)^2} \int_\Omega \left| \nabla v^{(m+\theta-1)/(2m)} \right|^2 dx \\
&\quad - \theta k_1 r_5 \int_\Omega v^{(\theta+p-1)/m} \, dx.
\end{aligned} \tag{4.4}$$

By Lemma 2.1, we have

$$\int_{\partial\Omega} v^{(\theta+q-1)/m} \, dS \leq \frac{N}{\rho_0} \int_\Omega v^{(\theta+q-1)/m} \, dx + \frac{(\theta+q-1)d}{m\rho_0} \int_\Omega v^{(\theta+q-1)/m-1} |\nabla v| \, dx, \tag{4.5}$$

and if Ω is with respect to x_0, $x_0 \neq 0$, by using the technique of translation in Lemma 2.1 and setting

$$\rho_0 = \min_{x \in \partial\Omega} \left((x - x_0) \cdot \nu \right), \quad d = \max_{x \in \overline{\Omega}} |x - x_0|,$$

we can get (4.5) easily. Applying Hölder's and Young's inequalities to the second term on the right-hand side of (4.5), we get

$$\begin{aligned}
&\frac{(\theta+q-1)d}{m\rho_0} \int_\Omega v^{(\theta+q-1)/m-1} |\nabla v| \, dx \\
&\leq \left[\frac{(\theta+q-1)^2 d^2}{m^2 \rho_0^2} \int_\Omega v^{(\theta+2q-m-1)/m} \, dx \right]^{1/2} \left[\int_\Omega v^{(\theta-m-1)/m} |\nabla v|^2 \, dx \right]^{1/2} \\
&\leq \frac{2m^2 \varepsilon_5}{(\theta+m-1)^2} \int_\Omega \left| \nabla v^{(m+\theta-1)/(2m)} \right|^2 dx + \frac{(\theta+q-1)^2 d^2}{2\varepsilon_5 m^2 \rho_0^2} \int_\Omega v^{(\theta+2q-m-1)/m} \, dx,
\end{aligned} \tag{4.6}$$

where ε_5 is a positive constant to be determined later. For the first term on the right-hand side of (4.5), we make two situation to discuss.

(1) If $2q > p+m$, making use of Hölder's and Young's inequalities, we have

$$\begin{aligned}\int_\Omega v^{(\theta+q-1)/m}\,dx &\le \left(\int_\Omega v^{(\theta+2q-m-1)/m}\,dx\right)^{1-\alpha_3}\left(\int_\Omega v^{(\theta+p-1)/m}\,dx\right)^{\alpha_3}\\ &\le \alpha_3\varepsilon_6\int_\Omega v^{(\theta+p-1)/m}\,dx\\ &\quad+(1-\alpha_3)\varepsilon_6^{\alpha_3/(\alpha_3-1)}\int_\Omega v^{(\theta+2q-m-1)/m}\,dx,\end{aligned}\tag{4.7}$$

where $\alpha_3=(q-m)/(2q-m-p)\in(0,1)$, and ε_6 is a positive constant to be determined later. From (4.4)–(4.7), we get the inequality

$$\begin{aligned}\varphi'(t) &\le \left(\frac{N\theta k_2}{\rho_0}\alpha_3\varepsilon_6-\theta k_1 r_5\right)\int_\Omega v^{(\theta+p-1)/m}\,dx\\ &\quad+\theta k_2\left[\frac{N(1-\alpha_3)}{\rho_0}\varepsilon_6^{\alpha_3/(\alpha_3-1)}+\frac{(\theta+q-1)^2d^2}{2\varepsilon_5 m^2\rho_0^2}\right]\int_\Omega v^{(\theta+2q-m-1)/m}\,dx\\ &\quad+\frac{2m^2\varepsilon_5\theta k_2-4m\theta(\theta-1)}{(m+\theta-1)^2}\int_\Omega\left|\nabla v^{(m+\theta-1)/(2m)}\right|^2\,dx.\end{aligned}\tag{4.8}$$

By Young's inequality, we get

$$\int_\Omega v^{(\theta+2q-m-1)/m}\,dx\le\alpha_4\int_\Omega v^{\frac{(4\theta+m-1)N-6\theta}{4m(N-2)}}\,dx+(1-\alpha_4)|\Omega|,\tag{4.9}$$

where $\alpha_4=\frac{4(\theta+2q-m-1)(N-2)}{(4\theta+m-1)N-6\theta}\in(0,1)$. Applying Schwarz's inequality to the first term on the right-hand side of (4.9), we have

$$\begin{aligned}\int_\Omega v^{\frac{(4\theta+m-1)N-6\theta}{4m(N-2)}}\,dx &\le \left(\int_\Omega v^{\theta/m}\,dx\right)^{1/2}\left(\int_\Omega v^{\frac{(2\theta+m-1)N-2\theta}{2m(N-2)}}\,dx\right)^{1/2}\\ &\le \left(\int_\Omega v^{\theta/m}\,dx\right)^{3/4}\left(\int_\Omega v^{\frac{\theta+m-1}{2m}\frac{2N}{N-2}}\,dx\right)^{1/4}.\end{aligned}\tag{4.10}$$

To estimate the bound of $\int_\Omega v^{\frac{\theta+m-1}{2m}\frac{2N}{N-2}}\,dx$, we use the following Sobolev inequality ($N\ge 3$) given in [5]:

$$\left\|u^\theta\right\|_{L^{2N/(N-2)}(\Omega)}\le C_s\left\|u^\theta\right\|_{W^{1,2}(\Omega)},$$

where C_s is a constant depending on Ω and N, i.e.,

$$\begin{aligned}&\left\|v^{(m+\theta-1)/(2m)}\right\|_{L^{2N/(N-2)}(\Omega)}^{N/[2(N-2)]}\\ &\le (C_s)^{N/[2(N-2)]}\left\|v^{(m+\theta-1)/(2m)}\right\|_{W^{1,2}(\Omega)}^{N/[2(N-2)]}\\ &\le C\left(\left\|\nabla v^{(m+\theta-1)/(2m)}\right\|_{L^2(\Omega)}^{N/[2(N-2)]}+\left\|v^{(m+\theta-1)/(2m)}\right\|_{L^2(\Omega)}^{N/[2(N-2)]}\right),\end{aligned}\tag{4.11}$$

where

$$C = \begin{cases} 2^{1/2}(C_s)^{3/2} & \text{if } N = 3, \\ (C_s)^{N/[2(N-2)]} & \text{if } N > 3. \end{cases}$$

Inserting (4.11) into (4.10), we can obtain the inequality

$$\begin{aligned} &\int_\Omega v^{\frac{(4\theta+m-1)N-6\theta}{4m(N-2)}}\,dx \\ &\le C\left(\int_\Omega v^{\theta/m}\,dx\right)^{3/4}\left(\int_\Omega v^{(\theta+m-1)/m}\,dx\right)^{N/[4(N-2)]} \\ &\quad + C\left(\int_\Omega v^{\theta/m}\,dx\right)^{3/4}\left(\int_\Omega \left|\nabla v^{(\theta+m-1)/(2m)}\right|^2\,dx\right)^{N/[4(N-2)]}. \end{aligned} \tag{4.12}$$

Now, we use Young's inequality on the right-hand side of (4.12) and get the inequalities

$$\begin{aligned} &C\left(\int_\Omega v^{\theta/m}\,dx\right)^{3/4}\left(\int_\Omega \left|\nabla v^{(m+\theta-1)/(2m)}\right|^2\,dx\right)^{N/[4(N-2)]} \\ &\le \frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_8^{N/(3N-8)}}\left(\int_\Omega v^{\theta/m}\,dx\right)^{3(N-2)/(3N-8)} + \frac{N\varepsilon_8}{4(N-2)}\int_\Omega \left|\nabla v^{(m+\theta-1)/(2m)}\right|^2\,dx, \\ &C\left(\int_\Omega v^{\theta/m}\,dx\right)^{3/4}\left(\int_\Omega v^{(\theta+m-1)/m}\,dx\right)^{N/[4(N-2)]} \\ &\le \frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_9^{N/(3N-8)}}\left(\int_\Omega v^{\theta/m}\,dx\right)^{3(N-2)/(3N-8)} + \frac{N\varepsilon_9}{4(N-2)}\int_\Omega v^{(\theta+m-1)/m}\,dx, \end{aligned}$$

i.e.,

$$\begin{aligned} &\int_\Omega v^{\frac{(4\theta+m-1)N-6\theta}{4m(N-2)}}\,dx \\ &\le \left(\frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_8^{N/(3N-8)}} + \frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_9^{N/(3N-8)}}\right)\left(\int_\Omega v^{\theta/m}\,dx\right)^{3(N-2)/(3N-8)} \\ &\quad + \frac{N\varepsilon_8}{4(N-2)}\int_\Omega \left|\nabla v^{(m+\theta-1)/(2m)}\right|^2\,dx + \frac{N\varepsilon_9}{4(N-2)}\int_\Omega v^{(m+\theta-1)/m}\,dx, \end{aligned} \tag{4.13}$$

where $\varepsilon_8, \varepsilon_9 > 0$ is a positive constant to be determined later. Applying Hölder's inequality to the second term on the right-hand side of (4.13), we can obtain the inequality

$$\int_\Omega v^{(m+\theta-1)/m}\,dx \le \frac{m+\theta-1}{\theta+p-1}\varepsilon_{10}\int_\Omega v^{(\theta+p-1)/m}\,dx + \frac{p-m}{\theta+p-1}\varepsilon_{10}^{(m+\theta-1)/(m-p)}|\Omega|. \tag{4.14}$$

From (4.8), (4.9), (4.13), (4.14), we get the inequality

$$\begin{aligned} \varphi'(t) &\le A_1 + A_2\varphi^{3(N-2)/(3N-8)} + (A_3 + A_4 - \theta k_1 r_5)\int_\Omega v^{(\theta+p-1)/m}\,dx \\ &\quad + (A_5 + A_6)\int_\Omega \left|\nabla v^{(\theta+m-1)/(2m)}\right|^2\,dx, \end{aligned}$$

where

$$
\begin{aligned}
A_1 &= A_7(1-\alpha_4)|\Omega| + A_7\alpha_4\frac{N\varepsilon_9(p-m)}{4(N-2)(\theta+p-1)}\varepsilon_{10}^{(m+\theta-1)/(m-p)}|\Omega|,\\
A_2 &= A_7\alpha_4\left[\frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_8^{N/(3N-8)}} + \frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_9^{N/(3N-8)}}\right],\\
A_3 &= \frac{N\theta\alpha_3k_2\varepsilon_6}{\rho_0},\\
A_4 &= A_7\alpha_4\frac{N\varepsilon_9(m+\theta-1)}{4(N-2)(\theta+p-1)}\varepsilon_{10},\\
A_5 &= A_7\alpha_4\frac{N\varepsilon_8}{4(N-2)},\\
A_6 &= \frac{2m^2\varepsilon_5\theta k_2 - 4m\theta(\theta-1)}{(\theta+m-1)^2},\\
A_7 &= \theta k_2\left[\frac{N(1-\alpha_3)}{\rho_0}\varepsilon_6^{\alpha_3/(\alpha_3-1)} + \frac{(\theta+q-1)^2d^2}{2\varepsilon_5 m^2\rho_0^2}\right].
\end{aligned}
$$

Choosing appropriate $\varepsilon_5, \varepsilon_6, \varepsilon_8, \varepsilon_9, \varepsilon_{10} > 0$, so that

$$A_3 + A_4 - \theta k_1 r_5 = 0, \quad A_5 + A_6 = 0,$$

we can have the inequality

$$\varphi'(t) \le A_1 + A_2\varphi^{3(N-2)/(3N-8)}. \tag{4.15}$$

Integrating (4.15) from 0 to t^*, we get

$$t^* \ge T_0 = \int_{\varphi(0)}^{+\infty}\frac{d\varphi}{A_1 + A_2\varphi^{3(N-2)/(3N-8)}}.$$

(2) If $2q = p+m$, making use of Hölder's and Young's inequalities, we can derive

$$
\begin{aligned}
\int_\Omega v^{(\theta+q-1)/m}\,dx &\le \left(\int_\Omega v^{(\theta+2q-m-1)/m}\,dx\right)^{1-\alpha_5}\left(\int_\Omega v^{\theta/m}\,dx\right)^{\alpha_5}\\
&\le \alpha_5\varepsilon_7\int_\Omega v^{\theta/m}\,dx + (1-\alpha_5)\varepsilon_7^{\alpha_5/(\alpha_5-1)}\int_\Omega v^{(\theta+2q-m-1)/m}\,dx,
\end{aligned} \tag{4.16}
$$

where $\alpha_5 = (q-m)/(2q-m-1) \in (0,1)$, and $\varepsilon_7 > 0$ is a positive constant to be determined later. Similarly, substituting (4.5), (4.6), (4.16) into (4.4), we get

$$
\begin{aligned}
\varphi'(t) \le{}& \frac{N\theta k_2\alpha_5\varepsilon_7}{\rho_0}\int_\Omega v^{\theta/m}\,dx - \theta k_1 r_5\int_\Omega v^{(\theta+p-1)/m}\,dx\\
&+ \theta k_2\left[\frac{N(1-\alpha_5)}{\rho_0}\varepsilon_7^{\alpha_5/(\alpha_5-1)} + \frac{(\theta+q-1)^2d^2}{2\varepsilon_5 m^2\rho_0^2}\right]\int_\Omega v^{(\theta+2q-m-1)/m}\,dx\\
&+ \frac{2m^2\varepsilon_5\theta k_2 - 4m\theta(\theta-1)}{(m+\theta-1)^2}\int_\Omega\left|\nabla v^{(m+\theta-1)/(2m)}\right|^2\,dx.
\end{aligned} \tag{4.17}
$$

By (4.9), (4.13), (4.14), (4.17), we obtain

$$\varphi'(t) \le B_1 + B_2\varphi + B_3\varphi^{3(N-2)/(3N-8)} + (B_4 - \theta k_1 r_5)\int_\Omega v^{(\theta+p-1)/m}\,dx \\ + (B_5 + B_6)\int_\Omega \left|\nabla v^{(\theta+m-1)/(2m)}\right|^2 dx,$$

where

$$\begin{aligned}
B_1 &= B_7(1-\alpha_4)|\Omega| + B_7\alpha_4 \frac{N\varepsilon_9(p-m)}{4(N-2)(\theta+p-1)}\varepsilon_{10}^{(m+\theta-1)/(m-p)}|\Omega|,\\
B_2 &= \frac{N\theta\alpha_5 k_2\varepsilon_7}{\rho_0},\\
B_3 &= B_7\alpha_4\left[\frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_8^{N/(3N-8)}} + \frac{(3N-8)C^{4(N-2)/(3N-8)}}{4(N-2)\varepsilon_9^{N/(3N-8)}}\right],\\
B_4 &= B_7\alpha_4\frac{N\varepsilon_9(m+\theta-1)}{4(N-2)(\theta+p-1)}\varepsilon_{10},\\
B_5 &= B_7\alpha_4\frac{N\varepsilon_8}{4(N-2)},\\
B_6 &= \frac{2m^2\varepsilon_5\theta k_2 - 4m\theta(\theta-1)}{(\theta+m-1)^2},\\
B_7 &= \theta k_2\left[\frac{N(1-\alpha_5)}{\rho_0}\varepsilon_7^{\alpha_5/(\alpha_5-1)} + \frac{(\theta+q-1)^2 d^2}{2\varepsilon_5 m^2\rho_0^2}\right].
\end{aligned}$$

Choosing appropriate $\varepsilon_5, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10} > 0$, so that

$$B_4 - \theta k_1 r_5 = 0, \quad B_5 + B_6 = 0,$$

we can have the inequality

$$\varphi'(t) \le B_1 + B_2\varphi + B_3\varphi^{3(N-2)/(3N-8)}. \tag{4.18}$$

Integrating (4.18) from 0 to t^*, we get

$$t^* \ge T_1 = \int_{\varphi(0)}^{+\infty} \frac{d\varphi}{B_1 + B_2\varphi + B_3\varphi^{3(N-2)/(3N-8)}}.$$

This completes the proof. □

5. Applications

In this section, we present two examples to demonstrate applications of Theorems 3.1 and 4.1.

Example 5.1. Let $u(x,t)$ be a solution of the following problem:

$$\begin{gathered}
u_t = \Delta u^3 - \frac{7}{4}e^{2-t}u^{1/6}, \quad (x,t) \in \Omega \times (0,t^*), \\
\frac{\partial u^3}{\partial \nu} = 6u^{5/2}, \quad (x,t) \in \partial\Omega \times (0,t^*), \\
u(x,0) = u_0(x) = (|x|+1)^2 > 0, \quad x \in \Omega,
\end{gathered}$$

where $\Omega = \left\{x = (x_1,x_2,x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1\right\}$. Then we have

$$k(t) = e^{2-t}, \quad f(u) = \frac{7}{4}u^{1/6}, \quad g(u) = 6u^{5/2}.$$

Now, we set $m = 3$, $p = 1/6$, $q = 5/2$, $\alpha = 0$ and $\beta = 1/4$. Then it is easy to verify that (3.1)–(3.4) hold. By (3.5), one can see that

$$\begin{aligned}
\psi(0) &= 2\int_{\partial\Omega}\int_0^{u_0} 6s^{5/2}\,dsdS - 3\int_\Omega u_0^2|\nabla u_0|^2\,dx - 2k(0)\int_\Omega\int_0^{u_0}\frac{7}{4}s^{1/6}\,dsdx \\
&= 31.4 > 0.
\end{aligned}$$

It follows from Theorem 3.1 that $u(x,t)$ must blow up in finite time t^*, and we have

$$t^* \le T = \frac{\chi(0)}{2\beta(1+\beta)\psi(0)} = 2.14,$$

where $\chi(0) = \int_\Omega u_0^2\,dx = 42$.

Example 5.2. Let $u(x,t)$ be a solution of the following problem:

$$\begin{gathered}
u_t = \Delta u^2 - 3(t+4)f(u), \quad (x,t) \in \Omega \times (0,t^*), \\
\frac{\partial u^2}{\partial \nu} = g(u), \quad (x,t) \in \partial\Omega \times (0,t^*), \\
u(x,0) = u_0(x) = |x|^4 + 9.99 \times 10^{-2} > 0, \quad x \in \Omega,
\end{gathered}$$

where $\Omega = \left\{x = (x_1,x_2,x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < (1/10)^2\right\}$.

If $2q > m+p$, $q > p$, we choosing

$$\begin{gathered}
p = \frac{5}{2}, \quad q = 3, \quad f(u) = 2u^{5/2}, \quad g(u) = 0.8u^3, \quad r_5 = 10, \\
\varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \varepsilon_{10} = 1, \quad \theta = 5, \quad \varphi(0) = \int_\Omega u_0^5\,dx = 4.18 \times 10^{-8}.
\end{gathered}$$

It can be easily seen that (4.1) hold and

$$\varepsilon_8 = 0.076, \quad \varepsilon_9 = 0.46, \quad A_1 = 0.015, \quad A_2 = 871.$$

Therefore, by the first result of Theorem 4.1, we obtain

$$t^* \ge T_0 = \int_{\varphi(0)}^{+\infty}\frac{d\varphi}{0.015 + 871\varphi^3} = 2.$$

If $2q = m + p$, we choosing

$$p = 3, \quad q = \frac{5}{2}, \quad f(u) = 2u^3, \quad g(u) = 8 \times 10^{-3/2} u^{5/2}, \quad r_5 = 1,$$
$$\varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \varepsilon_{10} = 1, \quad \theta = 3, \quad \varphi(0) = \int_\Omega u_0^5 \, dx = 4.18 \times 10^{-6}.$$

It can be easily seen that (4.1) hold and

$$\varepsilon_8 = 0.21, \quad \varepsilon_9 = 0.6, \quad B_1 = 0.01, \quad B_2 = 5.69, \quad B_3 = 34.38.$$

Therefore, by the second result of Theorem 4.1, we obtain

$$t^* \geq T_1 = \int_{\varphi(0)}^{+\infty} \frac{d\varphi}{0.01 + 5.69\varphi + 34.38\varphi^3} = 0.96.$$

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