MULTISCALE DEFORMATION ANALYSIS BY CAUCHY-NAVIER WAVELETS

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A geoscientifically relevant wavelet approach is established for the classical (inner) displacement problem corresponding to a regular surface (such as sphere, ellipsoid, and actual earth surface). Basic tools are the limit and jump relations of (linear) elastostatics. Scaling functions and wavelets are formulated within the framework of the vectorial Cauchy-Navier equation. Based on appropriate numerical integration rules, a pyramid scheme is developed providing fast wavelet transform (FWT). Finally, multiscale deformation analysis is investigated numerically for the case of a spherical boundary.

1. Introduction

First we recapitulate some results known from the theory of elasticity. We will always regard the inner space Σ_{int} of a closed surface Σ as a fixed reference configuration of a body. By a *deformation* of $\overline{\Sigma_{\text{int}}}$ we mean a one-to-one c^1-function $z : \overline{\Sigma_{\text{int}}} \to \mathbb{R}^3$ such that $\det(\nabla \otimes z) > 0$. The function $u : \overline{\Sigma_{\text{int}}} \to \mathbb{R}^3$, defined by $u(x) = z(x) - x$, $x \in \overline{\Sigma_{\text{int}}}$, is called the *displacement* of $\overline{\Sigma_{\text{int}}}$ relative to the deformation z. The tensor field $(\nabla \otimes u)(x)$ is called the *displacement gradient*. The *(infinitesimal) strain tensor* is defined by $\mathbf{e} = (1/2)((\nabla \otimes u) + (\nabla \otimes u)^T)$ as the symmetric part of the *displacement gradient*, while the antisymmetric part is used to define the *(infinitesimal) rotation tensor* $\mathbf{d}$ as $\mathbf{d} = (1/2)((\nabla \otimes u) - (\nabla \otimes u)^T)$. While $\mathbf{d}$ describes a "rigid" displacement field, $\mathbf{e}$ is responsible for the "nonrigid" displacements. According to Kirchhoff's theorem (see, e.g., [21]) if two displacement fields u and u' correspond to the same strain field, then $u = u' + w$,

where w is a rigid displacement field. One calls trace$(\mathbf{e}) = \nabla \cdot u$ the (elastic) *dilatation*. Thus the dilatations are determined by the diagonal elements of $\mathbf{e}$, the remaining elements of $\mathbf{e}$ prescribe torsions. Every displacement field can be decomposed into a pure torsion (i.e., $\nabla \cdot u = 0$) and a pure dilatation (i.e., $\nabla \wedge u = 0$).

An elastic body in a strained configuration performs by definition a tendency of recovering its original form: this tendency is materialized by a field of forces on each part of the body by the other parts. This field of internal forces, called (elastic) *stress*, is due to the interaction of the molecules of the body which have been removed from their relative position of equilibrium and to recovering this position, following the principle of action and interaction. If x is a point of a (regular) surface element in Σ_{int} with unit normal ν, then the *stress vector* $s_\nu(x) = T_\nu(u)(x)$ is the force per unit area at x exerted by the portion of Σ_{int} on the side of the surface element in Σ_{int} towards $\nu(x)$ on the portion of Σ_{int} on the other side. For time-independent behavior and in the absence of body stress fields, there exists a symmetric tensor field $\mathbf{s}$, called *stress tensor field*, such that $s_\nu = \mathbf{s}\nu$ for each vector ν and $\nabla(\mathbf{s}a) = 0$ for each fixed $a \in \mathbb{R}^3$ (for more details see, e.g., [12, 22]).

Hooke's law relates the stress to strain, that is, linear elasticity of the body implies that for each $x \in \Sigma_{\text{int}}$ there exists a linear transformation $\mathbf{C}$ from the space of all tensors into the space of all symmetric tensors such that $\mathbf{s} = \mathbf{C}\mathbf{e}$. The linear theory of elasticity is based on the *strain-displacement* relation

$$\mathbf{e} = \frac{1}{2}\left((\nabla \otimes u) + (\nabla \otimes u)^T\right), \tag{1.1}$$

the *stress-strain* relation

$$\mathbf{s} = \mathbf{C}\mathbf{e}, \tag{1.2}$$

and the *equation of equilibrium*

$$\text{div}\,\mathbf{s} + b = 0, \tag{1.3}$$

where b is the body force field in Σ_{int}. Equations (1.1), (1.2), and (1.3) imply the *displacement equation of equilibrium* in Σ_{int}

$$\text{div}\,\mathbf{C}(\nabla \otimes u) + b = 0. \tag{1.4}$$

For given $\mathbf{C}$ and b, this is a coupled linear system of partial differential equations for the fields u, $\mathbf{e}$, and $\mathbf{s}$. If the material is *isotropic*, $\mathbf{C}$ is given

by

$$\mathbf{C}\mathbf{e} = 2\mu\mathbf{e} + \lambda(\text{trace}\,\mathbf{e})\mathbf{i}, \tag{1.5}$$

where the scalars λ and μ are called the Lamé moduli. Moreover, if the material is *homogeneous*, λ and μ are constants (typical requirements imposed on λ and μ are $\mu > 0$, $3\lambda + 2\mu > 0$ (see, e.g., [16])). Therefore, in the homogeneous isotropic case, observing the identities

$$\nabla \cdot \left(\mu(\nabla \otimes u)\right) = \mu\Delta u, \qquad \nabla \cdot \left(\mu(\nabla \otimes u)^T\right) = 0,$$
$$\nabla \cdot \left(\lambda(\nabla \cdot u)\mathbf{i}\right) = \lambda\nabla(\nabla \cdot u), \tag{1.6}$$

we are led to the displacement equation of equilibrium in the form

$$\mu\Delta u + (\lambda + \mu)\nabla\nabla u + b = 0. \tag{1.7}$$

Finally, assuming that the body force field b vanishes, this equation can be reduced to the so-called *Cauchy-Navier equation* in Σ_{int}

$$\mu\Delta u + (\lambda + \mu)\nabla\nabla u = 0. \tag{1.8}$$

This equation plays the same part in the theory of elasticity as the Laplace equation in the theory of harmonic functions and it formally reduces to it for $\mu = 1$ and $\lambda = -1$. The Cauchy-Navier equation allows an equivalent formulation in Σ_{int}

$$\Delta u + \sigma\nabla\nabla \cdot u = 0, \tag{1.9}$$

where $\sigma = (1 - 2\rho)^{-1}$, $\rho = \lambda/2(\lambda + \mu)$, $\mu \neq 0$. The quantity ρ is the Poisson ratio. For simplicity, we let

$$\Diamond u = \mu\Delta u + (\lambda + \mu)\nabla\nabla u = 0 \tag{1.10}$$

in Σ_{int}. It is easy to show that the displacement field u is biharmonic and its divergence and curl are harmonic. This yields a deep relation between linear elasticity and potential theory (see, e.g., [17]).

The layout of this report on multiscale deformation analysis by Cauchy-Navier wavelets is as follows: after a brief sketch of the theory of linear elasticity given in the introduction (Section 1), we deal with some preliminary concepts of elastic potentials in Section 2. In analogy to the classical potential theoretic case we discuss (in Section 3) the limit

and jump relations within the framework of the Hilbert space of square-integrable vector fields on a regular surface Σ. The uniqueness, existence, and regularity of the solution of the displacement boundary value problem of elastostatics are discussed in Section 4. Next (in Section 5) a wavelet approach is introduced based on the layer potentials and their operator formulation in the nomenclature of the Hilbert space of square-integrable vector fields on the regular boundary Σ. We introduce the so-called (Cauchy-Navier) Σ-scaling functions and wavelets. The wavelet transform and the reconstruction formulae both in continuous and discrete formulations are explicitly written down. The geomathematically relevant (inner) three-dimensional displacement boundary value problem of elastostatics is treated within the multiscale structure of Cauchy-Navier wavelets. Finally, Section 6 is devoted to numerical applications of wavelet approximation on the sphere.

2. Preliminaries

We begin by introducing some preliminaries that will be used throughout this paper.

2.1. Notation

As usual, $\mathbb{R}^3$ denotes the three-dimensional (real) Euclidean space. We consistently write $x, y, \ldots$ for the elements of $\mathbb{R}^3$. In components, we have the representation $x = x_1 e^1 + x_2 e^2 + x_3 e^3$, where the vectors e^1, e^2, and e^3 form the canonical orthonormal basis in $\mathbb{R}^3$. The inner product, vector product, and tensor product between x and y, respectively, are defined as usual by

$$x \cdot y = x^T y = \sum_{i=1}^{3} x_i y_i,$$

$$x \wedge y = \left(x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_3 - y_3 y_1 \right)^T, \tag{2.1}$$

$$x \otimes y = xy^T = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}.$$

Furthermore, the Euclidean norm of x is denoted by $|x|$, that is, $|x| = (x \cdot x)^{1/2}$. The unit sphere in $\mathbb{R}^3$ is denoted by Ω. More explicitly,

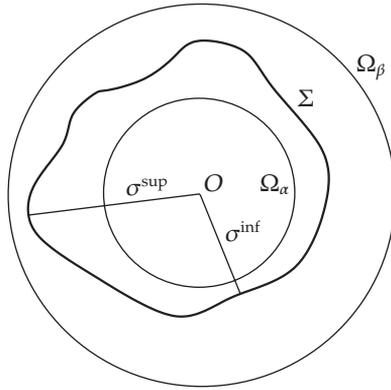$$\Omega = \left\{ \xi \in \mathbb{R}^3 \mid |\xi| = 1 \right\}. \tag{2.2}$$

FIGURE 2.1. Configuration of a regular surface.

2.2. Regular surfaces

A surface $\Sigma \subset \mathbb{R}^3$ is called *regular* if it satisfies the following properties:

(i) Σ divides $\mathbb{R}^3$ uniquely into the bounded region Σ_{int} (*inner space*) and the unbounded region Σ_{ext} (*outer space*) given by $\Sigma_{\text{ext}} = \mathbb{R}^3 \setminus \overline{\Sigma_{\text{int}}}$, $\overline{\Sigma_{\text{int}}} = \Sigma_{\text{int}} \cup \Sigma$,

(ii) Σ is a closed and compact surface free of double points,

(iii) Σ_{int} contains the origin,

(iv) Σ is locally of class $C^{(2)}$.

Given a regular surface (see Figure 2.1), then there exist positive constants α and β such that

$$\alpha < \sigma^{\text{inf}} = \inf_{x \in \Sigma} |x| \leq \sup_{x \in \Sigma} |x| = \sigma^{\text{sup}} < \beta. \tag{2.3}$$

The sets Ω_α and Ω_β denote the spheres of radii α and β, respectively. As usual, $\Omega_\beta^{\text{int}}$ and $\Omega_\beta^{\text{ext}}$ (resp., $\Omega_\alpha^{\text{int}}$ and $\Omega_\alpha^{\text{ext}}$) denote the inner and outer spaces of Ω_β (resp., Ω_α).

A vector field f possessing k continuous derivatives is said to be of class $c^{(k)}$, $0 \leq k \leq \infty$. The space $c^{(0)}(\Sigma)$ $(= c(\Sigma))$ is the class of continuous vector fields f defined on Σ. The space $c(\Sigma)$ is a complete normed space endowed with the norm $\|f\|_{c(\Sigma)} = \sup_{x \in \Sigma} |f(x)|$. In $c(\Sigma)$ we have the inner product $(\cdot, \cdot)_{\ell^2(\Sigma)}$ corresponding to the norm

$$\|f\|_{\ell^2(\Sigma)} = \left(\int_\Sigma |f(x)|^2 d\omega(x) \right)^{1/2}, \tag{2.4}$$

where $d\omega$ represents the surface element on Σ. Furthermore, for each $f \in c(\Sigma)$, we have the norm estimate

$$\|f\|_{\ell^2(\Sigma)} \leq \|\Sigma\| \|f\|_{c(\Sigma)}, \quad \|\Sigma\| = \left(\int_\Sigma d\omega \right)^{1/2}. \tag{2.5}$$

By $\ell^2(\Sigma)$ we denote the space of (Lebesgue) square-integrable vector fields on Σ. It is a Hilbert space with respect to the inner product $(\cdot, \cdot)_{\ell^2(\Sigma)}$ and a Banach space with respect to $\| \cdot \|_{\ell^2(\Sigma)}$. The space $\ell^2(\Sigma)$ is the completion of $c(\Sigma)$ with respect to the norm $\| \cdot \|_{\ell^2(\Sigma)}$, that is,

$$\ell^2(\Sigma) = \overline{c(\Sigma)}^{\| \cdot \|_{\ell^2(\Sigma)}}. \tag{2.6}$$

By $\mathrm{pot}(\Sigma_{\mathrm{int}})$ we denote the space of potentials $u \in c^{(2)}(\Sigma)$ satisfying the Cauchy-Navier equation $\Diamond u = \mu \Delta u + (\lambda + \mu) \nabla \nabla u = 0$ in Σ_{int} (with λ and μ being fixed). With $\mathrm{pot}(\overline{\Sigma_{\mathrm{int}}})$ we denote the space of all vector fields $u : \overline{\Sigma_{\mathrm{int}}} \to \mathbb{R}^3$ satisfying the properties

(1) $u \in c^{(2)}(\Sigma_{\mathrm{int}}) \cap c(\overline{\Sigma_{\mathrm{int}}})$,
(2) $u|_{\Sigma_{\mathrm{int}}} \in \mathrm{pot}(\Sigma_{\mathrm{int}})$.

3. Potential operators

Elastostatics may be formulated by a vector potential theory which closely parallels classical scalar potential theory. As a matter of fact, the displacement vector corresponds to the scalar harmonic function, whereas the traction vector corresponds to the normal derivative. Well-known integral formulae parallel the Gauss flux theorem, Betti's and Somigliana's formulae parallel Green's formulae. Moreover, vector potentials may be constructed in close orientation to the scalar single- and double-layer potentials. The resulting boundary integral equations show analogous properties to the scalar boundary integral equations. As a consequence, the fundamental existence-uniqueness theorems of classical elastostatics can be formulated in analogy to the corresponding theorems of harmonic function theory. For more details, the reader is referred to [16], which gives the theoretical treatment of the vector theory. Further theoretical aspects can be found in many books; for example, [12, 15, 20].

At each point x of a regular surface Σ we can construct a normal $\nu(x)$ pointing into the outer space Σ_{ext}. The set

$$\Sigma(\tau) = \left\{ x_\tau \in \mathbb{R}^3 \mid x_\tau = x + \tau \nu(x), \ x \in \Sigma \right\} \tag{3.1}$$

generates a *parallel surface* which is exterior for $\tau > 0$ and interior for $\tau < 0$. It is known that, if $|\tau|$ is sufficiently small, the parallel surface is regular and the normal to one parallel surface is normal to the other.

The matrix $\Gamma(x)$, $x \in \mathbb{R}^3$ with $|x| \neq 0$, given by

$$\Gamma(x) = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \left(\left(\epsilon^i \cdot \epsilon^k + \frac{(\lambda + \mu)}{\lambda + 3\mu} \frac{(x \cdot \epsilon^i)(x \cdot \epsilon^k)}{|x|^2} \right) \frac{1}{|x|} \right)_{i,k=1,2,3} \tag{3.2}$$

is constituted by the so-called fundamental solutions $\Gamma_k(x) = \Gamma(x)\epsilon^k$, $k = 1,2,3$, associated to the operator $\Diamond$ (cf. [17]).

The operator

$$N = \frac{1}{\lambda + 3\mu} \left(2\mu(\lambda + 2\mu)\frac{\partial}{\partial \nu} + (\lambda + \mu)(\lambda + 2\mu)\nu \operatorname{div} + \mu(\lambda + \mu)\nu \times \operatorname{curl} \right) \tag{3.3}$$

is called the *(pseudo-)stress operator*. Furthermore, $N_x\Gamma_k(x)$, $x \in \mathbb{R}^3$ with $|x| \neq 0$, is given by

$$N_x\Gamma_k(x) = \left(\frac{\partial}{\partial \nu(x)} \frac{1}{|x|} \right) \Lambda_k(x), \tag{3.4}$$

where

$$\Lambda_k(x) = \frac{2\mu}{\lambda + 3\mu} \left(\epsilon^k + \frac{3(\lambda + \mu)}{2\mu} \frac{(\epsilon^k \cdot x)x}{|x|^2} \right), \quad k = 1,2,3. \tag{3.5}$$

We let

$$\Lambda(x) = \left(\Lambda_i(x) \cdot \epsilon^k \right)_{i,k}, \quad i,k = 1,2,3. \tag{3.6}$$

Limit and jump relations by layer potentials such as single-layer potential, double-layer potential, and N-derivative of the single-layer potential (which parallels the normal derivative in classical potential theory) are known from the literature. For example, [17] gave their formulation in pointwise sense for a regular surface. The papers [3, 9] introduced an adequate operator notation to establish the validity of the limit and jump relations in uniform sense (i.e., in the topology of $\|\cdot\|_{C(\Sigma)}$).

In what follows, we adopt this operator notation to give a generalization of the limit and jump relations in the topology of $\|\cdot\|_{\ell^2(\Sigma)}$. The generalization will be given exclusively by functional analytic means (based on the results in uniform metric). Later on, the $\ell^2(\Sigma)$-formulations of the limit and jump relations are the essential tools for introducing vector

wavelets (associated to the Cauchy-Navier operator) on regular surfaces within $\ell^2(\Sigma)$-topology.

Assuming $|\tau|$ to be sufficiently small, we define the so-called potential operators $P(\tau)$, $P_N(\tau)$, and $N_P(\tau)$, respectively, by the following integrals:

$$P(\tau)g(x) = \int_\Sigma \Gamma(x_\tau - y)g(y)d\omega(y), \tag{3.7}$$

$P(\tau)$ is the operator of the *single-layer potential* on Σ for values on $\Sigma(\tau)$,

$$P_N(\tau)g(x) = \int_\Sigma \left(\frac{\partial}{\partial v(y)} \frac{1}{|x_\tau - y|} \right) \Lambda(x_\tau - y)g(y)d\omega(y), \tag{3.8}$$

$P_N(\tau)$ is the operator of the *double-layer potential* on Σ for values on $\Sigma(\tau)$,

$$\begin{aligned}
N_P(\tau)g(x) &= \int_\Sigma \left(\frac{\partial}{\partial v(x)} \frac{1}{|x_\tau - y|} \right) \Lambda(x_\tau - y)g(y)d\omega(y) \\
&= N_x \int_\Sigma \Gamma(x_\tau - y)g(y)d\omega(y),
\end{aligned} \tag{3.9}$$

$N_P(\tau)$ is the *N-derivative of the single-layer potential* on Σ for values on $\Sigma(\tau)$.

The operators $P(\tau)$, $P_N(\tau)$, and $N_P(\tau)$ form mappings from $\ell^2(\Sigma)$ into $c(\Sigma)$ provided that $|\tau|$ is sufficiently small. Furthermore, the integrals formally defined by

$$P(0)g(x) = \int_\Sigma \Gamma(x - y)g(y)d\omega(y),$$

$$P_N(0)g(x) = \int_\Sigma \left(\frac{\partial}{\partial v(y)} \frac{1}{|x - y|} \right) \Lambda(x - y)g(y)d\omega(y), \tag{3.10}$$

$$N_P(0)g(x) = \int_\Sigma \left(\frac{\partial}{\partial v(x)} \frac{1}{|x - y|} \right) \Lambda(x - y)g(y)d\omega(y)$$

exist and define linear bounded operators $P(0)$, $P_N(0)$, and $N_P(0)$ mapping $\ell^2(\Sigma)$ into $c(\Sigma)$.

As usual, the dual operator $P^*(\tau)$ of the operator $P(\tau)$ is defined by

$$\int_\Sigma g(x) \cdot P(\tau)f(x)d\omega(x) = \int_\Sigma P^*(\tau)g(x) \cdot f(x)d\omega(x) \tag{3.11}$$

for $f, g \in \ell^2(\Sigma)$. In connection with the property $\Gamma(x) = \Gamma(-x)$, $x \in \Sigma$, this shows that

$$P^*(\tau)g(x) = \int_\Sigma \Gamma(x - y_\tau)g(y)d\omega(y). \tag{3.12}$$

In the same way we obtain

$$\begin{aligned} P_N^*(\tau)g(x) &= \int_\Sigma \left(\frac{\partial}{\partial \nu(x)} \frac{1}{|x - y_\tau|} \right) \Lambda(x - y_\tau)g(y)d\omega(y), \\ N_P^*(\tau)g(x) &= \int_\Sigma \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x - y_\tau|} \right) \Lambda(x - y_\tau)g(y)d\omega(y). \end{aligned} \tag{3.13}$$

As mentioned before, the potential operators in elastostatics (see, e.g., [17]) behave near the boundary much like the ordinary harmonic potential operators. In particular, *limit formulae* and *jump relations* can be formulated in close orientation to the potential theoretic case. To be more explicit, let I be the identity operator in $\ell^2(\Sigma)$. For all $\tau > 0$ sufficiently small, the operators $L_i^\pm(\tau)$, $i = 1,2,3$, and $J_i(\tau)$, $i = 1,2,3,4,5$, are defined by

$$L_1^\pm(\tau) = P(\pm\tau) - P(0), \tag{3.14}$$

$$L_2^\pm(\tau) = P_N(\pm\tau) - P_N(0) \mp 2\pi I, \tag{3.15}$$

$$L_3^\pm(\tau) = N_P(\pm\tau) - N_P(0) \pm 2\pi I, \tag{3.16}$$

$$J_1(\tau) = P(\tau) - P(-\tau), \tag{3.17}$$

$$J_2(\tau) = P_N(\tau) - P_N(-\tau) - 4\pi I, \tag{3.18}$$

$$J_3(\tau) = N_P(\tau) - N_P(-\tau) + 4\pi I, \tag{3.19}$$

$$J_4(\tau) = P_N(\tau) + P_N(-\tau) - 2P_N(0), \tag{3.20}$$

$$J_5(\tau) = N_P(\tau) + N_P(-\tau) - 2N_P(0), \tag{3.21}$$

respectively. Then, for all $g \in c(\Sigma)$,

$$\begin{aligned} \lim_{\substack{\tau \to 0 \\ \tau > 0}} \left\| L_i^\pm(\tau)g \right\|_{c(\Sigma)} &= 0, \quad i = 1,2,3, \\ \lim_{\substack{\tau \to 0 \\ \tau > 0}} \left\| J_i(\tau)g \right\|_{c(\Sigma)} &= 0, \quad i = 1,2,3,4,5. \end{aligned} \tag{3.22}$$

In addition, the adjoint operators with respect to the inner product $(\cdot,\cdot)_{\ell^2(\Sigma)}$ are bounded linear operators with respect to the norm $\|\cdot\|_{c(\Sigma)}$ (see, e.g., [3, 17]).

THEOREM 3.1. *For all $g \in \ell^2(\Sigma)$,*

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left\| L_i^\pm(\tau) g \right\|_{\ell^2(\Sigma)} = 0, \quad i = 1,2,3,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left\| J_i(\tau) g \right\|_{\ell^2(\Sigma)} = 0, \quad i = 1,2,3,4,5. \tag{3.23}$$

The proof of Theorem 3.1 can be found in the appendix.

4. Uniqueness, existence, and regularity

In the notations given above, the homogeneous isotropic elastic displacement boundary value problem can be formulated as follows. Given $f \in c(\Sigma)$, find a vector field $u \in \mathrm{pot}(\overline{\Sigma_{\mathrm{int}}})$ satisfying the boundary condition $u|_\Sigma = f$. As it is well-known, the boundary value problem has a unique solution (see, e.g., [16]). In order to prove the existence, we use the double-layer potential

$$u(x) = P_N(0)g(x)$$
$$= \int_\Sigma \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) \Lambda(x-y)g(y)d\omega(y), \quad g \in c(\Sigma). \tag{4.1}$$

Observing the discontinuity of the double-layer potential, we obtain from (3.19) that

$$f(x) = -2\pi g(x) + \int_\Sigma \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) \Lambda(x-y)g(y)d\omega(y) \tag{4.2}$$

for all $x \in \Sigma$. The resulting integral equation $-f = (2\pi I - P_N(0))g$, $g \in c(\Sigma)$, fulfills all standard Fredholm theorems.

The homogeneous integral equation $(2\pi I - P_N(0))g = 0$ has no solution different from $g = 0$. Thus, the solution of the boundary value problem exists and is representable by a double-layer potential as indicated in (4.1). For details, the reader is referred to [17]. The operator $T = 2\pi I - P_N(0)$ and its adjoint operator T^* (with respect to the scalar product $(\cdot, \cdot)_{\ell^2(\Sigma)}$) form mappings from $c(\Sigma)$ into $c(\Sigma)$ which are linear and bounded with respect to the norm $\| \cdot \|_{c(\Sigma)}$. The operators T and T^* in $c(\Sigma)$ are injective and, by the Fredholm alternative, bijective in the Banach space $c(\Sigma)$. Consequently, by the open mapping theorem (see [23]), the operators T^{-1}, T^{*-1} are linear and bounded with respect to $\| \cdot \|_{c(\Sigma)}$. Moreover, $(T^*)^{-1} = (T^{-1})^*$. But this implies that both T^{-1} and $(T^*)^{-1}$ are

bounded with respect to the norm $\|\cdot\|_{\ell^2(\Sigma)}$ in $c(\Sigma)$. As we have shown, for a given $f \in c(\Sigma)$, there exists a vector field $g \in c(\Sigma)$ determined by (4.2) such that u is representable in the form (4.1). Suppose that $\mathcal{K}$ is a subset of Σ_{int} with $\text{dist}(\mathcal{K}, \Sigma) > 0$. Then Cauchy-Schwarz inequality applied to (4.1) gives, for each $x \in \mathcal{K}$,

$$
\begin{aligned}
|u(x)| \leq & \left(\int_{\Sigma} \sum_{k=1}^{3} \left| \left(\frac{\partial}{\partial v(y)} \frac{1}{|x-y|} \right) \Lambda_k(x-y) \right|^2 d\omega(y) \right)^{1/2} \\
& \times \left(\int_{\Sigma} |g(y)|^2 d\omega(y) \right)^{1/2}.
\end{aligned}
\tag{4.3}
$$

But this means that

$$
\sup_{x \in \mathcal{K}} |u(x)| \leq E \|g\|_{\ell^2(\Sigma)},
\tag{4.4}
$$

where

$$
E = \sup_{x \in \mathcal{K}} \left(\int_{\Sigma} \sum_{k=1}^{3} \left| \left(\frac{\partial}{\partial v(y)} \frac{1}{|x-y|} \right) \Lambda_k(x-y) \right|^2 d\omega(y) \right)^{1/2}.
\tag{4.5}
$$

In connection with (4.2) this implies the existence of a positive constant B (depending on Σ and $\mathcal{K}$) such that

$$
\sup_{x \in \mathcal{K}} |u(x)| \leq E \|T^{-1} f\|_{\ell^2(\Sigma)} \leq B \|f\|_{\ell^2(\Sigma)}.
\tag{4.6}
$$

Summarizing our results, we obtain the following regularity condition.

THEOREM 4.1. *Let u be a vector field of class* $\text{pot}(\overline{\Sigma_{\text{int}}})$ *and $\mathcal{K}$ a subset of Σ_{int} with* $\text{dist}(\mathcal{K}, \Sigma) > 0$. *Then*

$$
\sup_{x \in \mathcal{K}} |u(x)| \leq B \left(\int_{\Sigma} |u(x)|^2 d\omega(x) \right)^{1/2}.
\tag{4.7}
$$

In other words, $l^2(\Sigma)$-approximation implies approximation in Σ_{int} in locally compact topology.

5. Cauchy-Navier wavelets

In Sections 3 and 4, we used the operator nomenclature of layer potentials to extend the limit and jump relations to the $\|\cdot\|_{\ell^2(\Sigma)}$-topology and

to discuss the well-posedness of the inner displacement problem. In this section, we go back to the integral representations of the layer potentials. Completely written out in the $l^2(\Sigma)$-framework, Theorem 3.1 then reads as follows.

THEOREM 5.1. *For $f \in \ell^2(\Sigma)$,*

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(1)}(x,y) f(y) d\omega(y) \right.\right.$$
$$\left.\left. - \int_\Sigma \Gamma(x-y) f(y) d\omega(y) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(2)}(x,y) f(y) d\omega(y) - f(x) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(3)}(x,y) f(y) d\omega(y) - f(x) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(4)}(x,y) f(y) d\omega(y) - 0 \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(5)}(x,y) f(y) d\omega(y) - f(x) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(6)}(x,y) f(y) d\omega(y) - f(x) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(7)}(x,y) f(y) d\omega(y) \right.\right.$$
$$\left.\left. - \int_\Sigma \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) \Lambda(x-y) f(y) d\omega(y) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \left(\int_\Sigma \left| \int_\Sigma \Phi_\tau^{(8)}(x,y) f(y) d\omega(y) \right.\right.$$
$$\left.\left. - \int_\Sigma \left(\frac{\partial}{\partial \nu(x)} \frac{1}{|x-y|} \right) \Lambda(x-y) f(y) d\omega(y) \right|^2 d\omega(x) \right)^{1/2} = 0,$$

(5.1)

where

$$\Phi_\tau^{(1)}(x,y) = \Gamma(x_\tau - y),$$

$$\Phi_\tau^{(2)}(x,y) = \frac{1}{2\pi}\left(\left(\frac{\partial}{\partial v(y)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)\right.$$

$$\left. - \left(\frac{\partial}{\partial v(y)}\frac{1}{|x - y|}\right)\Lambda(x - y)\right),$$

$$\Phi_\tau^{(3)}(x,y) = -\frac{1}{2\pi}\left(\left(\frac{\partial}{\partial v(x)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)\right.$$

$$\left. - \left(\frac{\partial}{\partial v(x)}\frac{1}{|x - y|}\right)\Lambda(x - y)\right),$$

$$\Phi_\tau^{(4)}(x,y) = \Gamma(x_\tau - y) - \Gamma(x_{-\tau} - y),$$

$$\Phi_\tau^{(5)}(x,y) = \frac{1}{4\pi}\left(\left(\frac{\partial}{\partial v(y)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)\right.$$

$$\left. - \left(\frac{\partial}{\partial v(y)}\frac{1}{|x_{-\tau} - y|}\right)\Lambda(x_{-\tau} - y)\right), \qquad (5.2)$$

$$\Phi_\tau^{(6)}(x,y) = -\frac{1}{4\pi}\left(\left(\frac{\partial}{\partial v(x)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)\right.$$

$$\left. - \left(\frac{\partial}{\partial v(x)}\frac{1}{|x_{-\tau} - y|}\right)\Lambda(x_{-\tau} - y)\right),$$

$$\Phi_\tau^{(7)}(x,y) = \frac{1}{2}\left(\left(\frac{\partial}{\partial v(y)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)\right.$$

$$\left. + \left(\frac{\partial}{\partial v(y)}\frac{1}{|x_{-\tau} - y|}\right)\Lambda(x_{-\tau} - y)\right),$$

$$\Phi_\tau^{(8)}(x,y) = \frac{1}{2}\left(\left(\frac{\partial}{\partial v(x)}\frac{1}{|x_\tau - y|}\right)\Lambda(x_\tau - y)\right.$$

$$\left. + \left(\frac{\partial}{\partial v(x)}\frac{1}{|x_{-\tau} - y|}\right)\Lambda(x_{-\tau} - y)\right),$$

$\tau > 0$, $(x,y) \in \Sigma \times \Sigma$.

Briefly formulated, we obtain the following corollary.

COROLLARY 5.2. *For $f \in \ell^2(\Sigma)$,*

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \int_\Sigma \boldsymbol{\Phi}_\tau^{(i)}(x,y)f(y)d\omega(y)$$

$$= \begin{cases} \displaystyle\int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \displaystyle\int_\Sigma \left(\frac{\partial}{\partial v(y)} \frac{1}{|x-y|} \right) \Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \displaystyle\int_\Sigma \left(\frac{\partial}{\partial v(x)} \frac{1}{|x-y|} \right) \Lambda(x-y)f(y)d\omega(y), & i = 8, \end{cases} \tag{5.3}$$

holds in the sense of the $\| \cdot \|_{\ell^2(\Sigma)}$-norm.

Figure 5.1 gives graphical illustrations of matrix norm, diagonal, and nondiagonal components of $\boldsymbol{\Phi}_\tau^{(5)}(\cdot,\cdot)$.

5.1. Scaling and wavelet functions

For $\tau > 0$ and $i = 1,\ldots,8$, the family $\{\boldsymbol{\Phi}_\tau^{(i)}\}_{\tau>0}$ of kernels $\boldsymbol{\Phi}_\tau^{(i)} : \Sigma \times \Sigma \to \mathbb{R}^{3\times3}$ is called a (*Cauchy-Navier*) Σ-*scaling function of type i*. Moreover, $\boldsymbol{\Phi}_1^{(i)} : \Sigma \times \Sigma \to \mathbb{R}^{3\times3}$ (i.e., with $\tau = 1$) is called the mother kernel of the (Cauchy-Navier) Σ-scaling function of type i. Correspondingly, for $\tau > 0$ and $i = 1,\ldots,8$, the family $\{\boldsymbol{\Psi}_\tau^{(i)}\}_{\tau>0}$ of kernels $\boldsymbol{\Psi}_\tau^{(i)} : \Sigma \times \Sigma \to \mathbb{R}^{3\times3}$ given by

$$\boldsymbol{\Psi}_\tau^{(i)}(x,y) = -(\alpha(\tau))^{-1} \frac{d}{d\tau} \boldsymbol{\Phi}_\tau^{(i)}(x,y), \quad x,y \in \Sigma, \tag{5.4}$$

is called a (*Cauchy-Navier*) Σ-*wavelet function of type i*. Moreover, $\boldsymbol{\Psi}_1^{(i)} : \Sigma \times \Sigma \to \mathbb{R}^{3\times3}$ defines the so-called mother kernel of the (Cauchy-Navier) Σ-wavelet of type i. It should be noted that (5.4) is called the (scale continuous) Σ-scaling equation. The factor $\alpha(\tau)^{-1}$ can be chosen in an appropriate way. For simplicity, throughout the remainder of this paper, we will use $\alpha(\tau) = \tau^{-1}$.

Definition 5.3. Let $\{\boldsymbol{\Phi}_\tau^{(i)}\}_{\tau>0}$ be a Σ-scaling function of type i. Then the associated Σ-wavelet transform of type i $(\mathrm{WT})^{(i)} : \ell^2(\Sigma) \to \ell^2((0,\infty),\Sigma)$ of a function $f \in \ell^2(\Sigma)$ is defined by
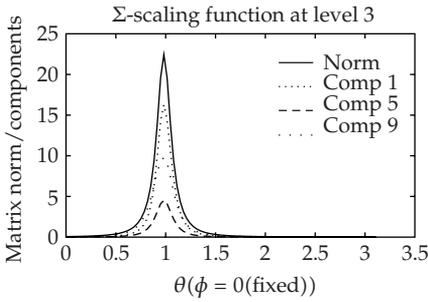
$$(\mathrm{WT})^{(i)}(f)(\tau,x) = \int_\Sigma \boldsymbol{\Psi}_\tau^{(i)}(x,y)f(y)d\omega(y). \tag{5.5}$$
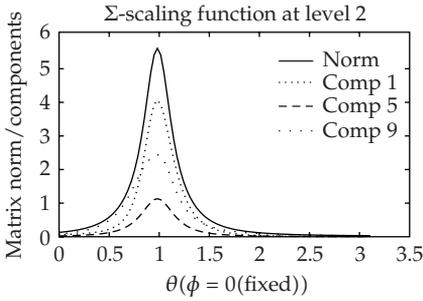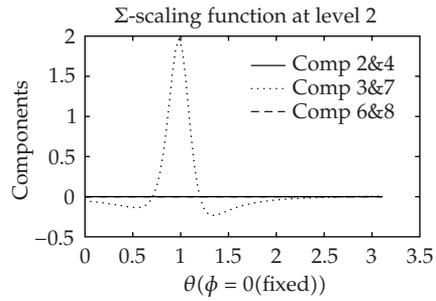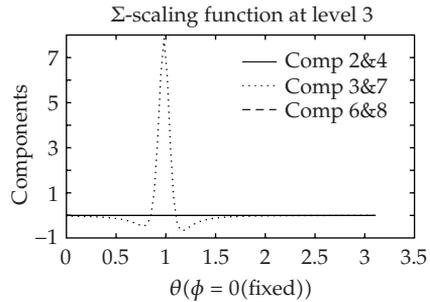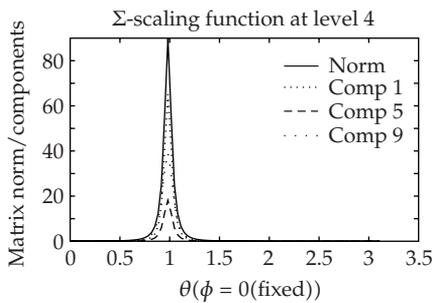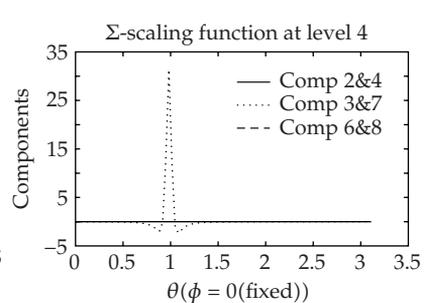
FIGURE 5.1. Matrix norm (Frobenius norm), diagonal, and non-diagonal components of the Σ-scaling function $\mathbf{\Phi}_{\tau}^{(5)}(\cdot,\cdot)$ for $\tau = 2^{-j}$, $j = 2, 3, 4$.

5.2. Scale continuous reconstruction formula

It is not difficult to show that the Σ-wavelet functions $\Psi_\tau^{(i)}$, $i = 1,\ldots,8$, behave (componentwise) like $O(\tau^{-1})$, hence the convergence of the integrals occurring in the next theorem is guaranteed.

THEOREM 5.4. *Let* $\{\Phi_\tau^{(i)}\}_{\tau>0}$ *be a Σ-scaling function of type i. Suppose that f is of class $\ell^2(\Sigma)$. Then the reconstruction formula*

$$\int_0^\infty (\text{WT})^{(i)}(f)(\tau,x)\frac{d\tau}{\tau}$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(y)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(x)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 8, \end{cases} \tag{5.6}$$

holds in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$.

Proof. Let $R > 0$ be arbitrary. Taking from (5.4) the identity

$$\Phi_R^{(i)}(x,y) = \int_R^\infty \Psi_\tau^{(i)}(x,y)\frac{d\tau}{\tau}, \quad x,y \in \Sigma, \tag{5.7}$$

we obtain

$$\int_R^\infty (\text{WT})^{(i)}(f)(\tau,x)\frac{d\tau}{\tau} = \int_R^\infty \int_\Sigma \Psi_\tau^{(i)}(x,y)f(y)d\omega(y)\frac{d\tau}{\tau}$$

$$= \int_\Sigma \left(\int_R^\infty \Psi_\tau^{(i)}(x,y)\frac{d\tau}{\tau}\right)f(y)d\omega(y) \tag{5.8}$$

$$= \int_\Sigma \Phi_R^{(i)}(x,y)f(y)d\omega(y).$$

Letting R tend to 0, we get the desired result from Theorem 5.1. $\square$

Next, we are interested in formulating the wavelet transform and the reconstruction formula by using the so-called "shift" and "dilation" operators. We define the x-shift and τ-dilation of a mother kernel, respectively, by

$$T_x : \Phi_1^{(i)} \longmapsto T_x\Phi_1^{(i)} = \Phi_{1,x}^{(i)}(\cdot) = \Phi_1^{(i)}(x,\cdot), \quad x \in \Sigma,$$

$$D_\tau : \Phi_1^{(i)} \longmapsto D_\tau\Phi_1^{(i)} = \Phi_\tau^{(i)}, \quad \tau > 0.$$

$$(5.9)$$

Consequently, we obtain by composition

$$T_xD_\tau\Phi_1^{(i)} = T_x\Phi_\tau^{(i)} = \Phi_{\tau,x}^{(i)}(\cdot) = \Phi_\tau^{(i)}(x,\cdot), \quad i = 1,\dots,8. \tag{5.10}$$

Analogously,

$$T_xD_\tau\Psi_1^{(i)} = T_x\Psi_\tau^{(i)} = \Psi_{\tau,x}^{(i)}(\cdot) = \Psi_\tau^{(i)}(x,\cdot), \quad i = 1,\dots,8. \tag{5.11}$$

We can thus state the following theorem.

THEOREM 5.5. *For $x \in \Sigma$ and $f \in \ell^2(\Sigma)$,*

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \int_\Sigma T_xD_\tau\Phi_1^{(i)}(y)f(y)d\omega(y)$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(y)} \dfrac{1}{|x-y|} \right)\Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(x)} \dfrac{1}{|x-y|} \right)\Lambda(x-y)f(y)d\omega(y), & i = 8, \end{cases}$$

$$\int_0^\infty \int_\Sigma T_xD_\tau\Psi_1^{(i)}(y)f(y)d\omega(y)\dfrac{d\tau}{\tau}$$

$$(5.12)$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(y)} \dfrac{1}{|x-y|} \right)\Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(x)} \dfrac{1}{|x-y|} \right)\Lambda(x-y)f(y)d\omega(y), & i = 8, \end{cases}$$

hold in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$.

In other words, the wavelets are used as mathematical means for breaking up a complicated structure of an $\ell^2(\Sigma)$-vector field ($i = 2,3,5,6$) or a vector potential ($i = 1,7,8$) into simple pieces at different scales and positions.

5.3. Scale discrete reconstruction formula

Until now emphasis has been put on the whole scale interval $(0,\infty)$. Next, discretization of the wavelet transform will be discussed in the form of "wavelet packets."

Let $(\tau_j)_{j\in\mathbb{Z}}$ denote a (monotonically decreasing) sequence of numbers satisfying the properties

$$\lim_{\tau\to\infty} \tau_j = 0, \qquad \lim_{\tau\to-\infty} \tau_j = \infty, \tag{5.13}$$

(e.g., $\tau_j = 2^{-j}$, $j \in \mathbb{Z}$). Given a Σ-scaling function $\{\Phi_\tau^{(i)}\}_{\tau>0}$ of type i, we define the (scale) discretized Σ-scaling function of type i by $\{\Phi_{\tau_j}^{(i)}\}_{j\in\mathbb{Z}}$. Then we are obviously led by Theorem 5.5 to the following result.

THEOREM 5.6. *For $f \in \ell^2(\Sigma)$, the limit*

$$\lim_{j\to\infty} \int_\Sigma T_x D_{\tau_j} \Phi_1^{(i)}(y)f(y)d\omega(y)$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \int_\Sigma \left(\dfrac{\partial}{\partial\nu(y)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \int_\Sigma \left(\dfrac{\partial}{\partial\nu(x)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 8 \end{cases} \tag{5.14}$$

holds in the $\|\cdot\|_{\ell^2(\Sigma)}$-sense.

Definition 5.7. Let $\{\Phi_{\tau_j}^{(i)}\}_{j\in\mathbb{Z}}$ be a discretized Σ-scaling function of type i. Then the (scale) discretized Σ-wavelet (packet) function of type i is defined by

$$\Psi_{\tau_j}^{(i)}(x,y) = \int_{\tau_{j+1}}^{\tau_j} \Psi_\tau^{(i)}(x,y)\frac{d\tau}{\tau}, \quad j \in \mathbb{Z},\ x,y \in \Sigma,\ i = 1,\dots,8. \tag{5.15}$$

With the definition of $\boldsymbol{\Psi}_{\tau_j}^{(i)}$ as given by (5.15) we immediately obtain that

$$\boldsymbol{\Psi}_{\tau_j}^{(i)}(x,y) = -\int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \boldsymbol{\Phi}_{\tau}^{(i)}(x,y) \frac{d\tau}{\tau} = \boldsymbol{\Phi}_{\tau_{j+1}}^{(i)} - \boldsymbol{\Phi}_{\tau_j}^{(i)}, \quad x,y \in \Sigma. \qquad (5.16)$$

Equation (5.16) is called the (scale) discretized Σ-scaling equation of type i. It should be remarked that, with a suitably chosen τ_j, formula (5.16) can easily be used to formulate the Σ-wavelet function in a discrete form once the Σ-scaling function has been given. To be more specific, assume that f is a vector field of class $\ell^2(\Sigma)$ and consider the discretized Σ-scaling equation of type i. Then, for $J \in \mathbb{Z}$, $N \in \mathbb{N}$, and $f \in \ell^2(\Sigma)$, we have

$$\int_{\Sigma} \boldsymbol{\Phi}_{\tau_{J+1}}^{(i)}(x,y) f(y) d\omega(y)$$
$$= \int_{\Sigma} \boldsymbol{\Phi}_{\tau_J}^{(i)}(x,y) f(y) d\omega(y) \qquad (5.17)$$
$$+ \int_{\Sigma} \boldsymbol{\Psi}_{\tau_J}^{(i)}(x,y) f(y) d\omega(y), \quad x \in \Sigma.$$

Therefore, the Jth partial wavelet reconstruction of $f \in \ell^2(\Sigma)$ may be understood as the difference between two "smoothings" at consecutive scales, $J+1$ and J. Observing the property of a telescoping sum, we see that

$$\int_{\Sigma} \boldsymbol{\Phi}_{\tau_{J+N}}^{(i)}(x,y) f(y) d\omega(y)$$
$$= \int_{\Sigma} \boldsymbol{\Phi}_{\tau_J}^{(i)}(x,y) f(y) d\omega(y) \qquad (5.18)$$
$$+ \sum_{j=J}^{J+N-1} \int_{\Sigma} \boldsymbol{\Psi}_{\tau_j}^{(i)}(x,y) f(y) d\omega(y), \quad x \in \Sigma.$$

By taking into account the property (5.18) together with Theorem 5.5, we find the following theorem.

THEOREM 5.8. *Let* $\{\boldsymbol{\Phi}_{\tau_j}^{(i)}\}_{j \in \mathbb{Z}}$ *be a (scale) discretized Σ-scaling function of type i. Then the multiscale representation of a function $f \in \ell^2(\Sigma)$*

$$\int_\Sigma \boldsymbol{\Phi}_{\tau_J}^{(i)}(x,y)f(y)d\omega(y) + \sum_{j=J}^\infty \int_\Sigma \boldsymbol{\Psi}_{\tau_j}^{(i)}(x,y)f(y)d\omega(y)$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(y)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(x)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 8, \end{cases} \quad (5.19)$$

holds for all $J \in \mathbb{Z}$ in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$.

Using the *so-called (scale) discretized Σ-wavelet transform of type i* canonically given by

$$(\mathrm{WT})^i(f)(\tau_j;x) = \int_\Sigma \boldsymbol{\Psi}_{\tau_j;x}^{(i)}(y)f(y)d\omega(y), \quad x \in \Sigma, \quad (5.20)$$

we are able to derive the following corollary.

COROLLARY 5.9. *Let* $\{\boldsymbol{\Phi}_{\tau_j}^{(i)}\}_{j \in \mathbb{Z}}$ *be a (scale) discretized Σ-scaling function of type i. Then, for all $f \in \ell^2(\Sigma)$,*

$$\sum_{j=-\infty}^\infty (\mathrm{WT})^i(f)(\tau_j;x)$$

$$= \begin{cases} \int_\Sigma \Gamma(x-y)f(y)d\omega(y), & i = 1, \\ 0, & i = 4, \\ f(x), & i = 2,3,5,6, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(y)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 7, \\ \int_\Sigma \left(\dfrac{\partial}{\partial v(x)}\dfrac{1}{|x-y|}\right)\Lambda(x-y)f(y)d\omega(y), & i = 8, \end{cases} \quad (5.21)$$

holds in the $\|\cdot\|_{\ell^2(\Sigma)}$-sense.

5.4. Scale and detail spaces

As in the spherical theory of wavelets (see [6, 7, 10] for more details), the operators $P_{\tau_j}^{(i)}$ and $R_{\tau_j}^{(i)}$ defined by

$$P_{\tau_j}^{(i)} = \int_\Sigma \Phi_{\tau_j}^{(i)}(\cdot, y) f(y) d\omega(y), \quad f \in \ell^2(\Sigma), \tag{5.22}$$

$$R_{\tau_j}^{(i)} = \int_\Sigma \Psi_{\tau_j}^{(i)}(\cdot, y) f(y) d\omega(y), \quad f \in \ell^2(\Sigma), \tag{5.23}$$

may be understood as a *lowpass filter* and a *bandpass filter*, respectively. The scale spaces $V_{\tau_j}^{(i)}$ and details spaces $W_{\tau_j}^{(i)}$ of type *i* are defined by

$$V_{\tau_j}^{(i)} = \left\{ P_{\tau_j}^{(i)}(f) \mid f \in \ell^2(\Sigma) \right\}, \tag{5.24}$$

$$W_{\tau_j}^{(i)} = \left\{ R_{\tau_j}^{(i)}(f) \mid f \in \ell^2(\Sigma) \right\}, \tag{5.25}$$

respectively. It is clear that

$$P_{\tau_{J+1}}^{(i)}(f) = P_{\tau_J}^{(i)}(f) + R_{\tau_J}^{(i)}, \quad J \in \mathbb{Z}. \tag{5.26}$$

Consequently,

$$V_{\tau_{J+1}}^{(i)} = V_{\tau_J}^{(i)} + W_{\tau_J}^{(i)}. \tag{5.27}$$

Furthermore,

$$V_{\tau_{J+1}}^{(i)} = V_{\tau_{J_0}}^{(i)} + \sum_{j=0}^{J} W_{\tau_j}^{(i)}. \tag{5.28}$$

It should be remarked that the sum (5.27), in general, is neither direct nor orthogonal. Furthermore,

$$\overline{\bigcup_{j=-\infty}^{\infty} V_{\tau_j}^{(i)}}^{\|\cdot\|_{\ell^2(\Sigma)}} = \ell^2(\Sigma), \quad i = 2,3,5,6,$$

$$\overline{\bigcup_{j=-\infty}^{\infty} V_{\tau_j}^{(1)}}^{\|\cdot\|_{\ell^2(\Sigma)}} = P(0)\left(\ell^2(\Sigma)\right),$$

$$\overline{\bigcup_{j=-\infty}^{\infty} V_{\tau_j}^{(7)}}^{\|\cdot\|_{\ell^2(\Sigma)}} = P_N(0)\left(\ell^2(\Sigma)\right),$$

$$\overline{\bigcup_{j=-\infty}^{\infty} V_{\tau_j}^{(8)}}^{\|\cdot\|_{\ell^2(\Sigma)}} = N_P(0)\left(\ell^2(\Sigma)\right).$$

(5.29)

The following scheme briefly summarizes the essential steps of our wavelet approach (for $i = 2,3,5,6$):

$$P_{\tau_0}^{(i)} f + R_{\tau_0}^{(i)} f + \cdots + R_{\tau_J}^{(i)} f + \cdots = f,$$

$$V_{\tau_0}^{(i)} + W_{\tau_0}^{(i)} + \cdots + W_{\tau_J}^{(i)} + \cdots = \ell^2(\Sigma),$$

(5.30)

where $P_{\tau_0}^{(i)} f \in V_{\tau_0}^{(i)}, R_{\tau_0}^{(i)} f \in W_{\tau_0}^{(i)}, \ldots, R_{\tau_J}^{(i)} f \in W_{\tau_J}^{(i)}, \ldots, f \in \ell^2(\Sigma)$.

5.5. A tree algorithm

In what follows, we present a particular scheme which utilizes the computational process of the reconstruction and decomposition of the wavelet approximation. This is known as tree algorithm that provides a recursive process to compute the integrals $P_{\tau_j}^{(i)}(f)$ and $R_{\tau_j}^{(i)}(f)$ on different levels, starting from an initial approximation of a given $f \in \ell^2(\Sigma)$ without falling back upon the original vector field f in each step.

Let $\{(y_k^{N_j}, w_k^{N_j})\}$ constitute an appropriate integration rule on Σ with given nodes $y_k^{N_j} \in \Sigma$ and weights $w_k^{N_j} \in \mathbb{R}$. Assume that for sufficiently large $J \in \mathbb{N}$ there exist coefficient vectors $a_k^{N_J} \in \mathbb{R}^3$, $k = 1, \ldots, N_J$, such that

$$P_{\tau_J}^{(i)}(f)(x) = \sum_{k=1}^{N_J} \Phi_{\tau_J}^{(i)}\left(x, y_k^{N_J}\right) a_k^{N_J}, \quad i = 1, \ldots, 8, \ x \in \Sigma. \tag{5.31}$$

Now we want to introduce an algorithm to obtain the coefficients $a^{N_j} = (a_1^{N_j}, \ldots, a_{N_j}^{N_j}) \in \mathbb{R}^3 \times \mathbb{R}^{N_j}$, $j = J_0, \ldots, J$, such that

(a) the vector a^{N_j} is obtainable from $a^{N_{j+1}}$, $j = J_0, \ldots, J-1$,

(b) the expressions $P_{\tau_j}^{(i)}(f)(x)$ and $R_{\tau_{j-1}}^{(i)}(f)(x)$ can be written as

$$P_{\tau_j}^{(i)}(f)(x) = \sum_{k=1}^{N_j} \Phi_{\tau_j}^{(i)}\left(x, y_k^{N_j}\right) a_k^{N_j}, \quad j = J_0, \ldots, J,$$

$$R_{\tau_{j-1}}^{(i)}(f)(x) = \sum_{k=1}^{N_{j-1}} \Psi_{\tau_{j-1}}^{(i)}\left(x, y_k^{N_j}\right) a_k^{N_{j-1}}, \quad j = J_0 + 1, \ldots, J. \tag{5.32}$$

For this scheme, we use appropriately chosen approximate integration rules such that $P_{\tau_j}^{(i)}(f)$ and $R_{\tau_j}^{(i)}(f)$ can be represented by

$$P_{\tau_j}^{(i)}(f)(x) \approx \sum_{k=1}^{N_j} w_k^{N_j} \Phi_{\tau_j}^{(i)}\left(x, y_k^{N_j}\right) f\left(y_k^{N_j}\right),$$

$$R_{\tau_{j-1}}^{(i)}(f)(x) \approx \sum_{k=1}^{N_{j-1}} w_k^{N_{j-1}} \Psi_{\tau_{j-1}}^{(i)}\left(x, y_k^{N_j}\right) f\left(y_k^{N_{j-1}}\right), \tag{5.33}$$

where $\{(y_k^{N_j}, w_k^{N_j}) \in \Sigma \times \mathbb{R}\}$ are the prescribed integration points and nodes and "$\approx$" means that the remainder is negligibly small.

The tree algorithm can be divided into two parts, namely the initial step and the pyramid step. For the *initial step* we assume as indicated by (5.31) that $J \in \mathbb{N}$ is sufficiently large. Thus we are able to start with

$$a_k^{N_J} \simeq w_k^{N_J} f\left(y_k^{N_J}\right), \quad k = 1, \ldots, N_J. \tag{5.34}$$

The aim of the pyramid step is to construct a^{N_j} from $a^{N_{j+1}}$ by recursion. At this point, it is essential to assume that there exist (tensor) kernel functions $\Xi_j^{(i)} : \Sigma \times \Sigma \to \mathbb{R}^{3 \times 3}$ such that

$$\Phi_{\tau_j}^{(i)}(x, y) \approx \int_\Sigma \Phi_{\tau_j}^{(i)}(x, z) \Xi_j^{(i)}(z, y) d\omega(z),$$

$$\Xi_j^{(i)}(x, y) \approx \int_\Sigma \Xi_j^{(i)}(x, z) \Xi_{j+1}^{(i)}(z, y) d\omega(z), \tag{5.35}$$

for $j = J_0, \ldots, J$. A reasonable choice for $\Xi_j^{(i)}$ is

$$\Xi_j^{(i)} = \Phi_{\tau_{J+L}}^{(i)}, \quad j = J_0, \ldots, J; \, i \in \{2, 3, 5, 6\}, \tag{5.36}$$

with $L \in \mathbb{N}$ suitably large. By the aforementioned approximate integration rules we obtain

$$
\int_\Sigma \Phi_{\tau_j}^{(i)}(x,y) f(y) d\omega(y) \approx \int_\Sigma \left[\int_\Sigma \Phi_{\tau_j}^{(i)}(x,z) \Xi_j^{(i)}(z,y) d\omega(z) \right] f(y) d\omega(y)
$$

$$
= \int_\Sigma \Phi_{\tau_j}^{(i)}(x,z) \left[\int_\Sigma \Xi_j^{(i)}(z,y) f(y) d\omega(y) \right] d\omega(z)
$$

$$
\approx \sum_{k=1}^{N_j} \Phi_{\tau_j}^{(i)}\left(x, y_k^{N_j} \right) a_k^{N_j},
$$

$$(5.37)$$

where

$$
a_k^{N_j} = w_k^{N_j} \int_\Sigma \Xi_j^{(i)}\left(y_k^{N_j}, y \right) f(y) d\omega(y), \quad j = J_0, \dots, J-1. \tag{5.38}
$$

Hence, in connection with (5.37), we find

$$
a_k^{N_j} = w_k^{N_j} \int_\Sigma \Xi_j^{(i)}\left(y_k^{N_j}, y \right) f(y) d\omega(y)
$$

$$
\approx w_k^{N_j} \int_\Sigma \left[\int_\Sigma \Xi_j^{(i)}\left(y_k^{N_j}, x \right) \Xi_{j+1}^{(i)}(x,y) d\omega(y) \right] f(y) d\omega(y)
$$

$$
\approx w_k^{N_j} \int_\Sigma \sum_{l=1}^{N_{j+1}} \Xi_j^{(i)}\left(y_k^{N_j}, y_l^{N_{j+1}} \right) \Xi_{j+1}^{(i)}\left(y_l^{N_{j+1}}, y \right) f(y) d\omega(y)
$$

$$(5.39)$$

$$
= w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j^{(i)}\left(y_k^{N_j}, y_l^{N_{j+1}} \right) a_l^{N_{j+1}}
$$

for $j = J-1, \dots, J_0$ and $k = 1, \dots, N_j$.

In conclusion, once the coefficients a^{N_j} are calculated, the coefficients $a^{N_{j-1}}$ are obtained by (5.39). Starting from an initial value a^{N_J}, all the coefficient vectors can be calculated recursively. Note that the coefficients a^{N_J} in the initial step do not depend on the choice of $\Xi_J^{(i)} = \Phi_{\tau_{J+L}}^{(i)}$. Furthermore, the functions $\Xi_j^{(i)}$, $j = J_0, \dots, J$, can be chosen independently of the Σ-scaling function used in the integrals $P_{\tau_j}^{(i)}(f)$ and $R_{\tau_j}^{(i)}(f)$.

Finally, with the proposed tree algorithm prescribed by (5.34) and (5.39), the decomposition and reconstruction process of the wavelet approximation can be illustrated briefly as follows:

$$f \longrightarrow a^{N_J} \longrightarrow a^{N_{J-1}} \longrightarrow \cdots \longrightarrow a^{N_{J_0+1}} \longrightarrow a^{N_{J_0}}$$

$$R^{(i)}_{\tau_J}(f) \quad R^{(i)}_{\tau_{J-1}}(f) \qquad R^{(i)}_{\tau_{J_0+1}}(f) \quad R^{(i)}_{\tau_{J_0}}(f) \qquad P^{(i)}_{\tau_{J_0}}(f)$$

$$\tag{5.40}$$

(*decomposition scheme*),

$$a^{N_{J_0}} \qquad a^{N_{J_0+1}} \qquad a^{N_{J-1}}$$

$$R^{(i)}_{\tau_{J_0}}(f) \qquad R^{(i)}_{\tau_{J_0+1}}(f) \qquad R^{(i)}_{\tau_{J-1}}(f)$$

$$P^{(i)}_{\tau_{J_0}}(f) \longrightarrow + \longrightarrow P^{(i)}_{\tau_{J_0+1}}(f) \longrightarrow + \cdots + \longrightarrow P^{(i)}_{\tau_{J-1}}(f) \longrightarrow + \longrightarrow P^{(i)}_{\tau_J}(f)$$

$$\tag{5.41}$$

(*reconstruction scheme*).

5.6. Multiscale solution of the (inner) displacement boundary value problem

Next, we discuss the solution of the (inner) displacement boundary value problem by means of the wavelet approximation techniques (derived in preceding sections). The existence, uniqueness, and regularity of the solutions of such problems are known from Section 4.

For given $f \in c(\Sigma)$, the solution $u \in \text{pot}(\overline{\Sigma}_{\text{int}})$ with $u|_\Sigma = f$ of the (inner) displacement problem can be expressed uniquely by a double-layer potential

$$u(x) = \int_\Sigma \left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) \Lambda(x-y) g(y) d\omega(y), \quad g \in c(\Sigma). \tag{5.42}$$

The corresponding integral equation reads as follows:

$$2\pi I - P_N(0)g = -f, \quad g \in c(\Sigma). \tag{5.43}$$

More explicitly,

$$2\pi g(x) - \int_\Sigma \left(\frac{\partial}{\partial v(y)} \frac{1}{|x-y|} \right) \Lambda(x-y) g(y) d\omega(y)$$
$$= -f(x), \quad g \in c(\Sigma), \ x \in \Sigma. \tag{5.44}$$

To approximate u given by the double-layer potential in (5.42), we use the concept stated in Theorem 5.1. In accordance with this approach, we are able to rewrite (5.44) approximately in the form

$$2\pi g(x) - \int_\Sigma \Phi_{\tau_L}^{(7)}(x,y) g(y) d\omega(y) = -f(x), \quad x \in \Sigma, \tag{5.45}$$

provided that $L \in \mathbb{N}$ is sufficiently large. Once the boundary integral equation (5.45) is solved, the density function g is inserted into (5.42) and, thereby the (approximate) solution u is obtained in Σ_{int}. Like in many cases of boundary integral equations, there is, in general, no straightforward way of constructing the unknown function g. It is, therefore, necessary to apply a suitable approximation method. In this respect, it is canonical for us to go back to Theorem 5.1 that enables us to formulate

$$g(y) = \lim_{j \to \infty} \int_\Sigma \Phi_{\tau_j}^{(5)}(y,z) g(z) d\omega(z). \tag{5.46}$$

Using an appropriate numerical integration technique, an approximation of g of level J, denoted by g_J, can be expressed by

$$g_J(y) = \sum_{l=1}^{N_J} w_l^{N_J} \Phi_{\tau_J}^{(5)}\left(y, y_l^{N_J}\right) g\left(y_l^{N_J}\right) = \sum_{l=1}^{N_J} \Phi_{\tau_J}^{(5)}\left(y, y_l^{N_J}\right) a_l^{N_J}, \tag{5.47}$$

where $w_l^{N_J}$, $l = 1, \ldots, N_J$, are the integration weights corresponding to the nodal points $y_l^{N_J} \in \Sigma$, $l = 1, \ldots, N_J$, and $a_l^{N_J} \in \mathbb{R}^3$, $l = 1, \ldots, N_J$.

The unknowns $a_l^{N_J} \in \mathbb{R}^3$, $l = 1, \ldots, N_J$, are deducible from (5.45) by solving a system of linear equations obtained by a suitable approximation method such as collocation, Galerkin procedure, least square approximation, and so forth (see, e.g., [5]). In consequence we are led to

the following system of equations for the unknowns $a_l^{N_J}$, $l = 1, \ldots, N_J$:

$$\sum_{l=1}^{N_J} \left(2\pi \Phi_{\tau_J}^{(5)} \left(y_m^{N_J}, y_l^{N_J} \right) - \sum_{k=1}^{N_J} w_k^{N_J} \Phi_{\tau_J}^{(5)} \left(y_m^{N_J}, y_k^{N_J} \right) \Phi_{\tau_L}^{(7)} \left(y_k^{N_J}, y_l^{N_J} \right) \right) a_l^{N_J}$$
$$= -f\left(y_m^{N_J} \right), \quad m = 1, \ldots, N_J. \tag{5.48}$$

However, such a consideration leads to a system of linear equations with a "full" matrix which seems to require much computational work for the definition of the matrix as well as for the solution. In this context, taking into consideration the localization behavior of the kernel functions, suitable accelerating techniques such as panel clustering, domain decomposition, and so forth can efficiently be used (see, e.g., [11, 13, 14]). However, further modifications of such techniques relevant to this particular consideration will be needed. This is a challenge for future work.

In this respect, a variant of our tree algorithm comes into play: once the starting values (see (5.34)) $a^{N_J} = (a_1^{N_J}, \ldots, a_{N_J}^{N_J})^T \in \mathbb{R}^{3 \times N_J}$ are given, the coefficients $a^{N_j} = (a_1^{N_j}, \ldots, a_{N_j}^{N_j})^T \in \mathbb{R}^{3 \times N_j}$, $j = J_0, \ldots, J-1$, are obtained by the recursion formula

$$a_k^{N_j} = w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j^{(i)} \left(y_k^{N_j}, y_l^{N_{j+1}} \right) a_l^{N_{j+1}}, \quad k = 1, \ldots, N_j. \tag{5.49}$$

The corresponding approximate integrals are obtained by

$$P_{\tau_j}^{(i)}(g)(x) \approx \sum_{k=1}^{N_j} \Phi_{\tau_j}^{(5)} \left(x, y_k^{N_j} \right) a_k^{N_j}, \quad x \in \Sigma, \; j = J_0, \ldots, J,$$
$$R_{\tau_{j-1}}^{(i)}(g)(x) \approx \sum_{k=1}^{N_j} \Psi_{\tau_{j-1}}^{(5)} \left(x, y_k^{N_j} \right) a_k^{N_j}, \quad x \in \Sigma, \; j = J_0 + 1, \ldots, J, \tag{5.50}$$

where

$$R_{\tau_{j-1}}^{(i)}(g)(x) = P_{\tau_j}^{(i)}(g)(x) - P_{\tau_{j-1}}^{(i)}(g)(x). \tag{5.51}$$

Hence, we finally arrive at the following theorem for the (inner) displacement boundary value problems of the Cauchy-Navier theory.

THEOREM 5.10. *Let Σ be a regular surface. For given $f \in c(\Sigma)$, let u be the potential of class $\mathrm{pot}(\overline{\Sigma}_{\mathrm{int}})$ with $u|_\Sigma = f$. Then the function $f_J \in c(\Sigma)$ given by*

$$f_J(x) = 2\pi \sum_{l=1}^{N_{J_0}} \Phi_{\tau_{J_0}}^{(i)}\left(x,y_l^{N_{J_0}}\right) a_l^{N_{J_0}} + \sum_{j=J_0}^{J-1} 2\pi \sum_{l=1}^{N_j} \Psi_{\tau_j}^{(i)}\left(x,y_l^{N_j}\right) a_l^{N_j}$$

$$- \sum_{l=1}^{N_{J_0}} \left(\int_\Sigma \Phi_{\tau_L}^{(7)}(x,y) \Phi_{\tau_{J_0}}^{(i)}\left(y,y_l^{N_{J_0}}\right) d\omega(y) \right) a_l^{N_{J_0}} \tag{5.52}$$

$$- \sum_{j=J_0}^{J-1} \sum_{l=1}^{N_j} \left(\int_\Sigma \Phi_{\tau_L}^{(7)}(x,y) \Psi_{\tau_j}^{(i)}\left(y,y_l^{N_j}\right) d\omega(y) \right) a_l^{N_j},$$

$x \in \Sigma$, represents a J-scale approximation of $f \in c(\Sigma)$ in the sense of $\|\cdot\|_{\ell^2(\Sigma)}$, where $i = 2,3,5,6$ and $L \in \mathbb{N}$ is sufficiently large. Furthermore, by virtue of *Theorem 4.1*,

$$u_J = \sum_{l=1}^{N_{J_0}} \int_\Sigma \Phi_{\tau_L}^{(7)}(\cdot,y) \Phi_{\tau_{J_0}}^{(i)}\left(y,y_l^{N_{J_0}}\right) d\omega(y) a_l^{N_{J_0}}$$

$$+ \sum_{j=J_0}^{J-1} \sum_{l=1}^{N_J} \int_\Sigma \Phi_{\tau_L}^{(7)}(\cdot,y) \Psi_{\tau_J}^{(i)}\left(y,y_l^{N_J}\right) d\omega(y) a_l^{N_j} \tag{5.53}$$

represents a J-scale approximation of u in the sense of $\|\cdot\|_{c(\mathcal{K})}$ for every $\mathcal{K} \subset \Sigma_{\text{int}}$ with $\text{dist}(\overline{\mathcal{K}},\Sigma) > 0$.

In other words, quadratic wavelet approximation on the boundary Σ implies locally uniform approximation on Σ_{int}.

6. Numerical examples

Next, we present some sample examples for the geoscientifically important case of a sphere (i.e., Σ is chosen to be the unit sphere Ω).

6.1. Spherical approximation of vector fields by layer potentials

For this purpose, we first consider the vector field $f : \Omega \to \mathbb{R}^3$ given by

$$f(x) = \begin{cases} 0e^3, & -1 \leq \xi \cdot e^3 \leq h, \\ \dfrac{3}{2}\left(\dfrac{h - \xi \cdot e^3}{h-1}\right)^2 e^3, & h \leq \xi \cdot e^3 \leq \dfrac{2+h}{3}, \\ \left(1 - 3\left(\dfrac{\xi \cdot e^3 - 1}{h-1}\right)^2\right)e^3, & \dfrac{2+h}{3} \leq \xi \cdot e^3 \leq 1, \end{cases} \tag{6.1}$$

for $x \in \Omega$, $\xi = x/|x|$, and $h = 1/2$.

The third component of the boundary function f is illustrated in Figure 6.1.
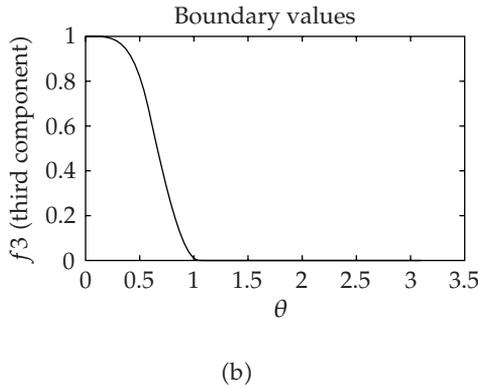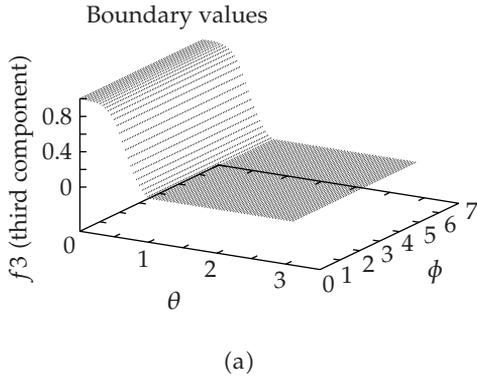
Figure 6.1. Functional values of f (third component): (a) on a longitude-latitude grid of points on Ω and (b) one-dimensional sectional illustration.

We are particularly interested in approximating "the third component" of the vector function f by our wavelet approach based on layer potentials (as prescribed above). Figure 6.2 shows the sectional illustration of approximations of the boundary function corresponding to the Ω-scaling function $\Phi_{\tau_j}^{(5)}$ for different levels, that is, $\tau_j = 2^{-j}$, $j = 1, 2, 3, 4$. Note that, in each evaluation step, a sufficiently large number of equiangular longitude-latitude grid points on the unit sphere have been used in order to avoid oscillations in the approximation process.

In accordance with Theorem 4.1, it is clear that our multiscale procedure also provides a good approximation of $u \in \text{pot}(\overline{\Omega_{\text{int}}})$ with $u|_\Omega = f$ inside Ω.
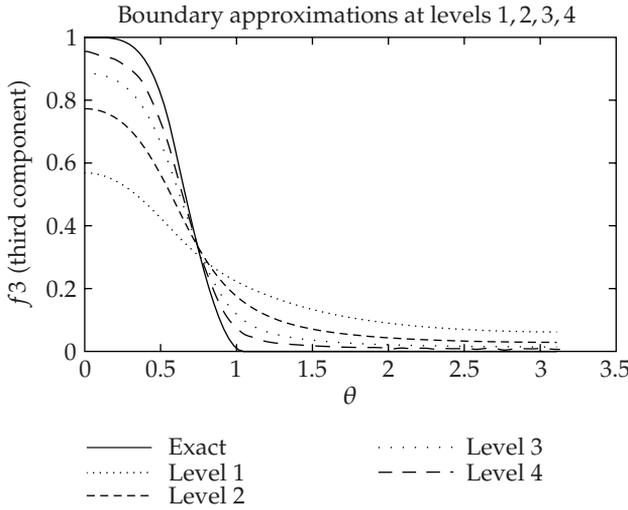
FIGURE 6.2. Sectional illustration of the scale approximations of f (third component) associated to the Ω-scaling function $\Phi_{\tau_j}^{(5)}$ for $j = 1,2,3,4$.

6.2. Solution of the displacement problem corresponding to a sphere

Finally our multiscale method will be considered for the boundary approximation for a simple displacement boundary value problem of which the boundary is assumed to be the unit sphere Ω and an analytical solution is explicitly known in $\overline{\Omega}_{\text{int}}$. In contrast to the example discussed in Section 6.1 this one enables us to check the accuracy of the scale approximation in Ω_{int}. However, we restrict ourselves to boundary approximation. The detail reconstructions are taken into account. To be more explicit, we consider the solution $u \in \text{pot}(\overline{\Omega}_{\text{int}})$ corresponding to the boundary field $u|_{\Omega} = f$ given by

$$f(x) = \frac{x_1 x_2}{2}\epsilon^1 - \frac{\sigma}{2(\sigma+3)}\left(x_1^2 + x_2^2 + x_3^2\right)\epsilon^2 + \frac{x_2 x_3}{2}\epsilon^3, \quad \sigma \neq 0. \qquad (6.2)$$

We choose, in particular, $\sigma = 2.5$ (i.e., $\lambda = 2$, $\mu = 3$). Figure 6.3 shows the exact boundary displacements under the displacement function f.

We compute the multiscale approximations of the boundary displacements on the sphere Ω for the first five levels. For the quantitative analysis of the accuracy, the radial and tangential displacements are taken into account. Figure 6.4 shows the exact radial displacement quantities by gray colormaps and tangential displacement vectors by arrows, respectively, on $\theta - \varphi$ polar coordinate plane.
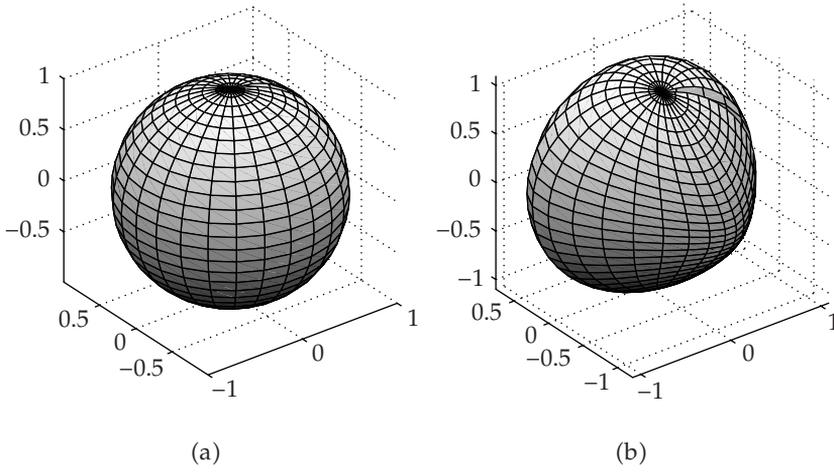
FIGURE 6.3. The reference (a) and the deformed (b) configurations of the unit sphere Ω associated to the displacement function f.
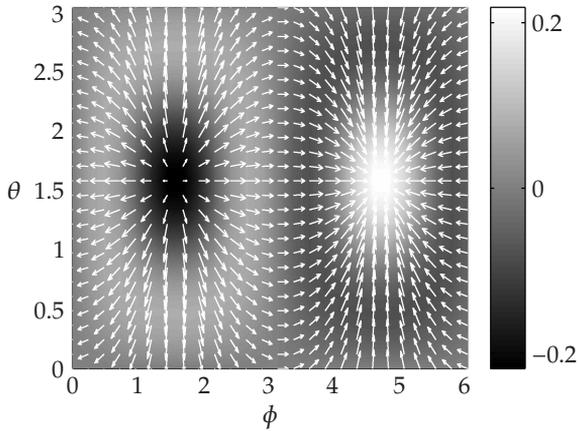
FIGURE 6.4. The (exact) radial and tangential boundary displacements on the unit sphere Ω.

Since the evaluation of wavelet coefficients naturally consists of integrations over the unit sphere Ω, we have to use an appropriate integration rule. In this respect, we use an equiangular longitude-latitude grid system where the integration weights are explicitly known (see, e.g., [4]). The following series of figures (Figures 6.5 and 6.6) show the corresponding multiscale approximations up to level 5, in which we scale the size of arrows at each level taking the exact tangential displacement
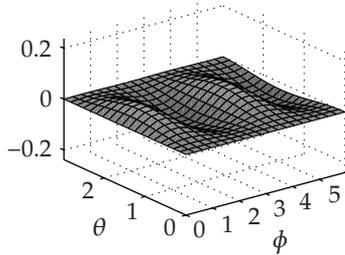
(a) Scale at $j = 1$.
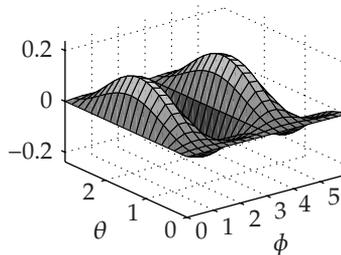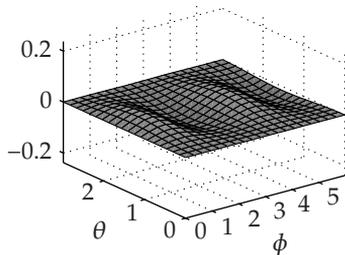
(b) Detail at $j = 1$.

(c) Scale at $j = 2$.

(d) Detail at $j = 2$.

(e) Scale at $j = 3$.

(f) Detail at $j = 3$.
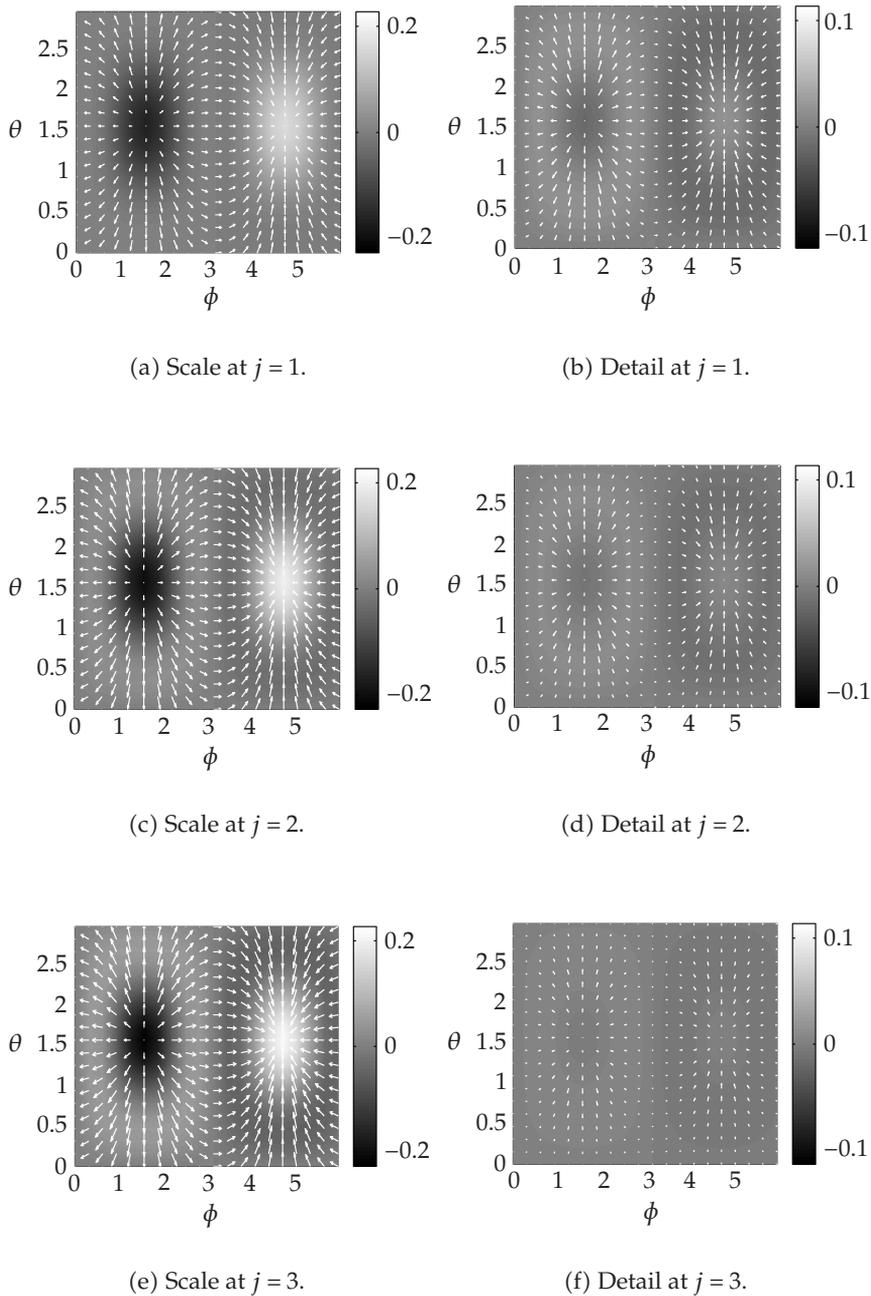
FIGURE 6.5. Scale (left) and detail (right) reconstructions at levels 1, 2, and 3.

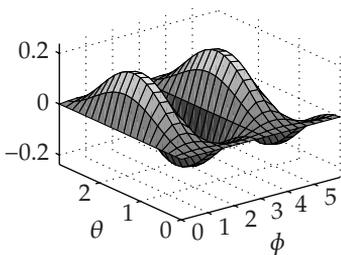(a) Scale at $j = 4$.

(b) Detail at $j = 4$.

(c) Scale at $j = 5$.

(d) Detail at $j = 5$.
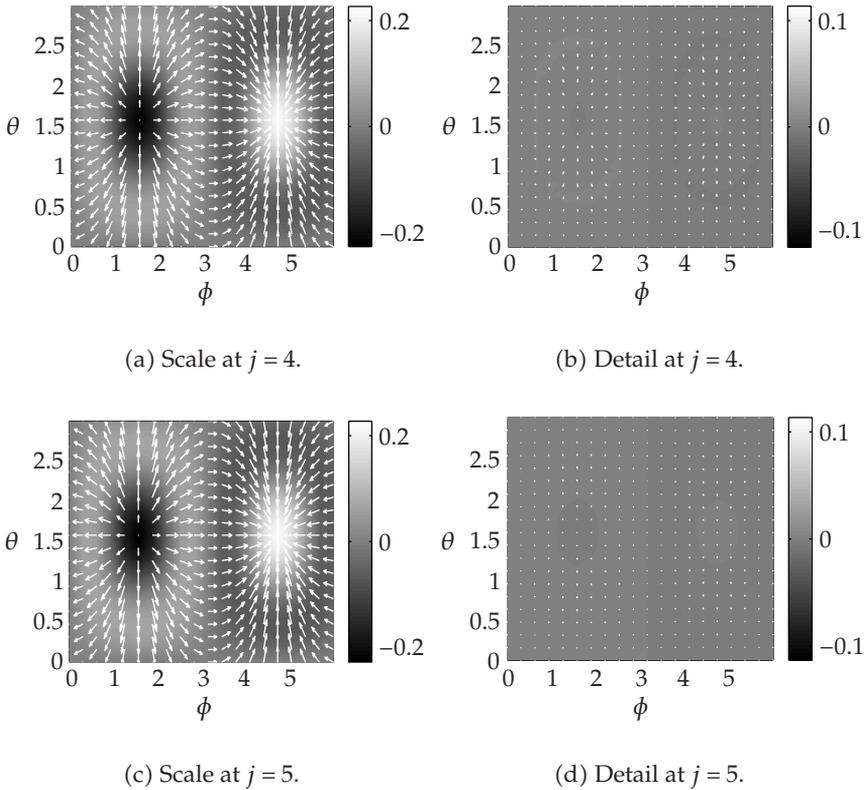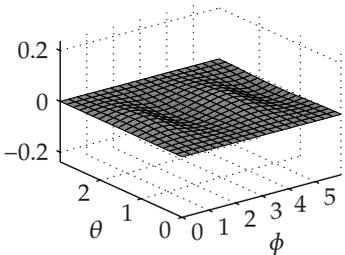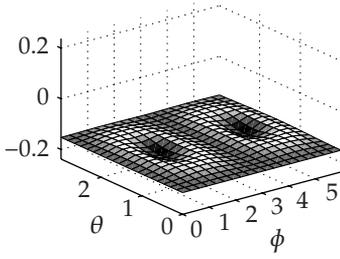
FIGURE 6.6. Scale (left) and detail (right) reconstructions at levels 4 and 5.

vectors as the reference size. Moreover, to show the results in a more detailed form we rescale all detail information quantities multiplying by 2. Furthermore, the same results are illustrated by Cartesian components in Figures 6.7, 6.8, 6.9, and 6.10.

More examples can be found in the Ph.D. thesis [1] of the first author.

7. Concluding remarks

In this paper we have introduced a multiscale method for solving the inner displacement problem using Cauchy-Navier wavelets. The method is particularly suitable for the deformation analysis corresponding to geoscientifically relevant boundaries (such as sphere, ellipsoid, actual earth surface, etc.) that involve efficient rules of numerical integration.

(a) Component $f \cdot \epsilon^1$.

(b) Component $f \cdot \epsilon^2$.
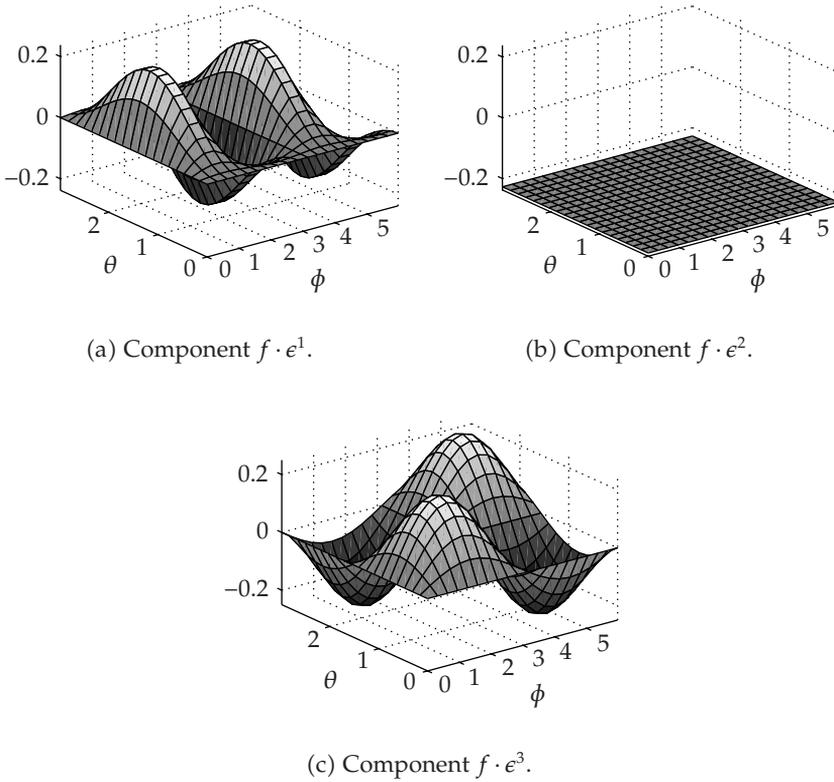
(c) Component $f \cdot \epsilon^3$.

FIGURE 6.7. Exact displacements in Cartesian components on Ω.

The principal idea of the Cauchy-Navier wavelets is based on the classical limit and jump relations of elastostatics. In conclusion, this paper can be viewed as a first attempt to "short-wavelength modelling," that is, high resolution of the fine structure of displacement fields. The method is restricted to the homogeneous and isotropic case of linear elasticity, hence it needs to be formulated under more complex (geo-)physical assumptions.

Summarizing our results, we are led to the following conclusions: Cauchy-Navier wavelets are "vectorial building blocks" that enable fast modelling of displacement data. Hence, three features are incorporated in our way of thinking about Cauchy-Navier wavelets, namely, the basis property, decorrelation, and fast computation. By virtue of the basis property, each member of the Hilbert space $\ell^2(\Sigma)$ can be expressed as a linear combination involving products of dilated and shifted copies of
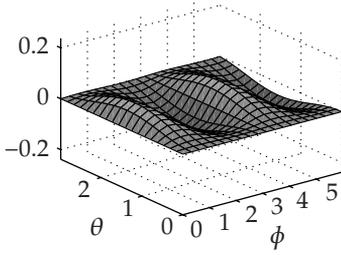
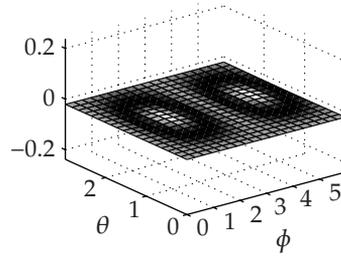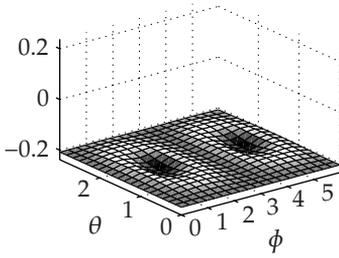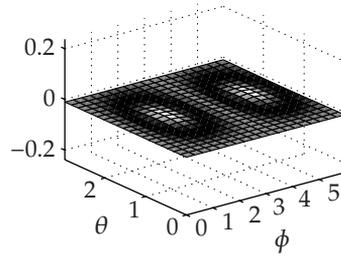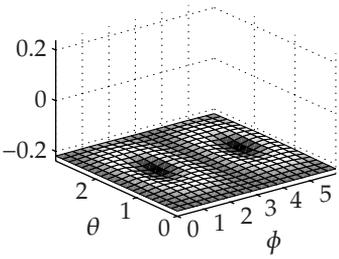(a) Scale at $j = 1$.
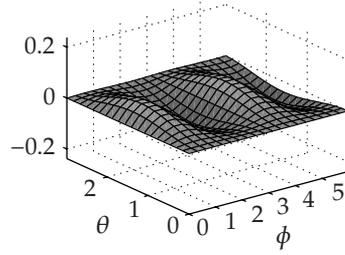
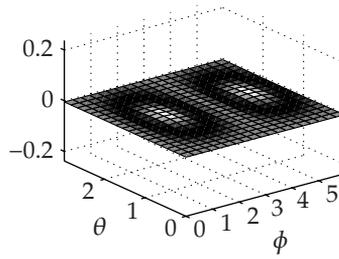(b) Detail at $j = 1$.
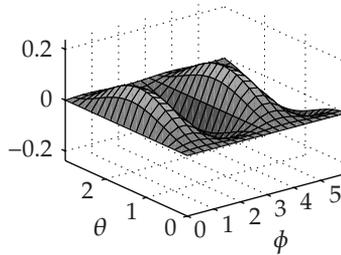
(c) Scale at $j = 2$.

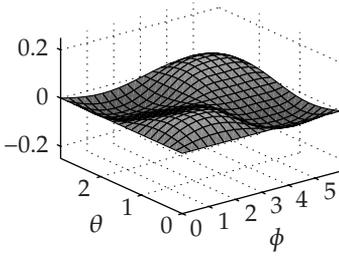(d) Detail at $j = 2$.

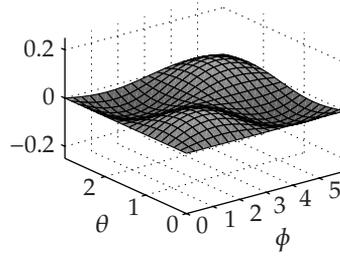(e) Scale at $j = 3$.

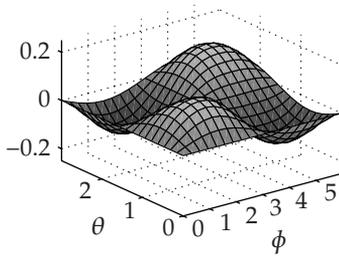(f) Detail at $j = 3$.

(g) Scale at $j = 4$.

(h) Detail at $j = 4$.

FIGURE 6.8. Scale (left) and detail (right) reconstructions (x_1-component).

(a) Scale at $j = 1$.
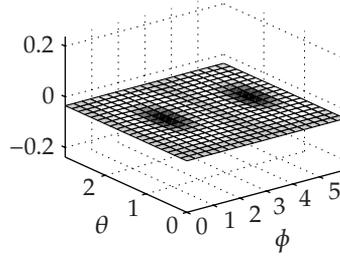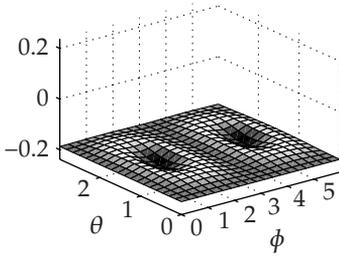
(b) Detail at $j = 1$.

(c) Scale at $j = 2$.

(d) Detail at $j = 2$.

(e) Scale at $j = 3$.

(f) Detail at $j = 3$.

(g) Scale at $j = 4$.

(h) Detail at $j = 4$.

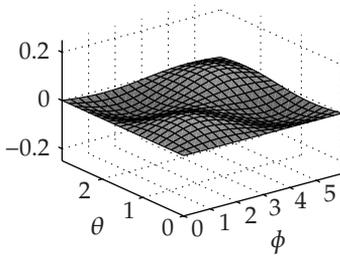FIGURE 6.9. Scale (left) and detail (right) reconstructions (x_2-component).

(a) Scale at $j = 1$.
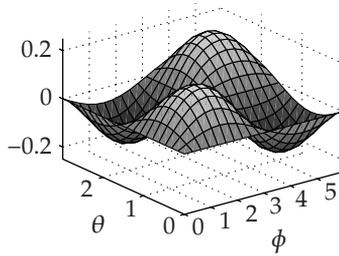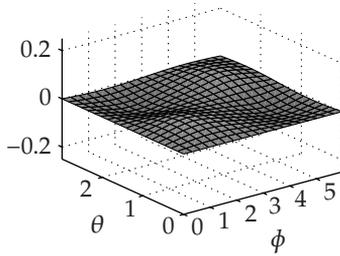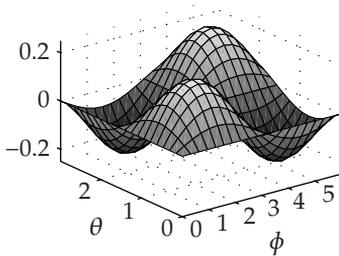
(b) Detail at $j = 1$.
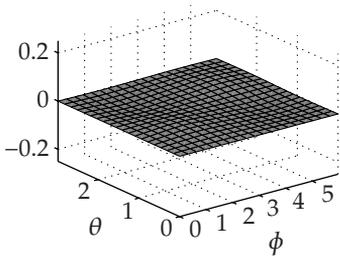
(c) Scale at $j = 2$.

(d) Detail at $j = 2$.

(e) Scale at $j = 3$.

(f) Detail at $j = 3$.

(g) Scale at $j = 4$.

(h) Detail at $j = 4$.

FIGURE 6.10. Scale (left) and detail (right) reconstructions (x_3-components).

a tensorial "mother kernel" with displacement data. The wavelet transform maps members of the Hilbert space $\ell^2(\Sigma)$ into an associated two-parameter class of scale- and space-dependent elements. Wavelets as constructed here show the power of decorrelation. As a consequence, the representation of displacement data is somehow "more compact" than the original representation, that is to say, wavelet modelling provides an accurate approximation by only using a small fraction of the original information. Typically, in the jargon of information theory, the convolutions $P_j f$ of tensorial scaling functions against the displacement field provide lowpass filtering at scale j, while the decorrelation is achieved by convolutions $R_j f$, that is, bandpass filtering. The essential question—also seen in comparison to the other solution techniques for the displacement boundary value problem—is how to decompose a displacement field into wavelet coefficients and how to reconstruct efficiently the field from the wavelet coefficients. As a matter of fact, the investigations of our paper have led to a tree algorithm, that makes all steps simple and fast. Moreover, it is worth mentioning that the fast decorrelation power of Cauchy-Navier wavelets is the key to (future) applications such as data compression, fast data transmission, noise cancellation, and so forth.

Altogether, the multiscale procedure as presented here is an essential step for flexible and economical microstructural modelling of (isotropic homogeneous) elastic fields on (geoscientifically relevant) regular boundaries such as sphere, ellipsoid, or (actual) Earth surface.

Appendix

For the proof of Theorem 3.1 we use a modification of a technique due to [18]. Denote by $T(\tau)$ one of the operators $L_i^\pm(\tau)$, $i = 1,2,3$, and $J_i(\tau)$, $i = 1,2,3,4,5$. Let $T^*(\tau)$ be the adjoint operator with respect to the inner product $(\cdot,\cdot)\ell^2(\Sigma)$. According to the Cauchy-Schwarz inequality we find

$$\left\|T(\tau)g\right\|_{\ell^2(\Sigma)}^2 \leq \|g\|_{\ell^2(\Sigma)}\left\|T^*(\tau)T(\tau)\right\|_{\ell^2(\Sigma)}. \tag{A.1}$$

Therefore it follows that

$$\begin{aligned}
\left\|T(\tau)g\right\|_{\ell^2(\Sigma)}^{2^2} &\leq \|g\|_{\ell^2(\Sigma)}^2\left\|T^*(\tau)T(\tau)\right\|_{\ell^2(\Sigma)}^2 \\
&\leq \|g\|_{\ell^2(\Sigma)}^2\|g\|_{\ell^2(\Sigma)}\left\|\left(T^*(\tau)T(\tau)\right)^2 g\right\|_{\ell^2(\Sigma)}^2.
\end{aligned} \tag{A.2}$$

Induction states that for all $n \geq 2$,

$$\left\|T(\tau)g\right\|_{\ell^2(\Sigma)}^{2^n} \leq \|g\|_{\ell^2(\Sigma)}^{2^n-1}\left\|\left(T^*(\tau)T(\tau)\right)^{2^{n-1}} g\right\|_{\ell^2(\Sigma)}. \tag{A.3}$$

Because of the boundedness of the operators $T^*(\tau)$ and $T(\tau)$ with respect to $\|\cdot\|_{c(\Sigma)}$ there exists a positive constant D such that

$$\|T(\tau)g\|_{\ell^2(\Sigma)}^{2^n} \leq \|\Sigma\| D^{2^n} \|g\|_{\ell^2(\Sigma)}^{2^n-1} \|g\|_{c(\Sigma)}, \quad \|\Sigma\| = \left(\int_\Sigma d\omega \right)^{1/2}. \quad \text{(A.4)}$$

Thus, for all $n \geq 2$ and $g \in c(\Sigma)$ with $g \neq 0$, we obtain

$$\frac{\|T(\tau)g\|_{\ell^2(\Sigma)}}{\|g\|_{\ell^2(\Sigma)}} \leq D \left(\frac{\|\Sigma\| \|g\|_{c(\Sigma)}}{\|g\|_{\ell^2(\Sigma)}} \right)^{2^{-n}}. \quad \text{(A.5)}$$

Letting n tend to infinity, we obtain for all $g \neq 0$

$$\lim_{n\to\infty} \left(\frac{\|g\|_{c(\Sigma)}}{\|g\|_{\ell^2(\Sigma)}} \right)^{2^{-n}} = 1. \quad \text{(A.6)}$$

This shows that $\|T(\tau)\|_{\ell^2(\Sigma)} \leq D$ for all $g \in c(\Sigma)$. Since $c(\Sigma)$ is a dense linear subspace of $\ell^2(\Sigma)$, we are able to extend the operator $T(\tau)$ from $c(\Sigma)$ to $\ell^2(\Sigma)$ without enlarging its norm (cf. [2, 8, 19]). Therefore, $T(\tau)$ is bounded with respect to $\|\cdot\|_{\ell^2(\Sigma)}$ and we have

$$\|T(\tau)\|_{\ell^2(\Sigma)} \leq \sqrt{\|T(\tau)\|_{c(\Sigma)} \|T^*(\tau)\|_{c(\Sigma)}}. \quad \text{(A.7)}$$

Hence it follows that $\|T(\tau)\|_{\ell^2(\Sigma)} \to 0$ as $\tau \to 0$, $\tau > 0$.

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