PERTURBED SPECTRA OF DEFECTIVE MATRICES

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Received 22 May 2002 and in revised form 7 October 2002

This paper is devoted to the perturbation theory for defective matrices. We consider the asymptotic expansions of the perturbed spectrum when a matrix A is changed to $A + tE$, where $E \neq 0$ and $t > 0$ is a small parameter. In particular, we analyse the rational exponents that may occur when the matrix E varies over the sphere $\|E\| = \rho > 0$. We partially characterize the leading exponents noting that the description of the set of all leading exponents remains an open problem.

1. Introduction

We consider the perturbation theory for eigenvalues of real or complex matrices. Obtaining perturbation bounds (including asymptotic estimates) for the eigenvalues of A under perturbations is one of the most studied problems in matrix perturbation theory [9]. In particular, the eigenvalue problem for defective matrices is of great interest in view of some theoretical and significant numerical difficulties accompanying its solution, see, for example, [1, 4]. We recall that an $n \times n$ matrix A is defective if it has less than n linearly independent eigenvectors, or equivalently, if it has at least one Jordan block of second or higher order in its complex Jordan canonical form.

There are numerical problems in implementing computer codes for solving the eigenvalue problem for defective matrices in finite arithmetic. Reliable codes for this purpose have been proposed in [2, 3] and further developed in [8]. Also, symbolic computations may be used in some cases to solve the eigenvalue problem. However, the computer computation of the eigenstructure of defective matrices is far from its

Journal of Applied Mathematics 2003:3 (2003) 115–140
2000 Mathematics Subject Classification: 15A18, 65F15
URL: http://dx.doi.org/10.1155/S1110757X0320512X

satisfactory solution. Thus, obtaining asymptotic perturbation bounds for perturbed spectra of defective matrices is of great theoretical as well as practical importance.

In this paper, we analyse the following eigenvalue perturbation problem. Let A be a real or complex matrix and let it be perturbed to $A+tE$, where $\|E\| = \rho > 0$ and $t > 0$. Supposing that we know the partial algebraic multiplicites of the eigenvalues of A, then characterize, and possibly compute all (for varying E) leading fractional exponents p/q in the asymptotically small eigenvalue perturbations of order $O(t^{p/q})$ for $t \to 0$.

We propose a simple procedure to determine some of the leading fractional exponents based on a correspondence between the lattice of integer partitions and certain sets of fractional exponents called *fractional intervals*.

Note that we vary E so as to successively determine the possible fractional exponents in the asymptotic expansion of the eigenvalues of $A+tE$. Thus, our problem is different from the classical problem, where the matrix E is fixed.

The main construction is based on specific perturbations which "fill in" the zeros in the superdiagonal of the Jordan form of A to create larger Jordan blocks.

We characterize subsets of the set of leading exponents and give the complete set for special cases. The general problem remains a challenging open problem.

2. Notation and preliminaries

We denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Q}_1 = \{p/q : p,q \in \mathbb{N},\ p < q\}$ the set of positive rational numbers less than 1, and by $\mathcal{R}_n = \{p/q : p,q \in \mathbb{N},\ p < q \le n\} \subset \mathbb{Q}_1$ the set of proper rational fractions with denominator not exceeding $n \in \mathbb{N}$.

The entire part of a real number $x > 0$ is denoted by $\mathrm{Ent}(x) \in \mathbb{N} \cup \{0\}$ and $[n]$ is the least common multiplier of the integers $1,\dots,n \in \mathbb{N}$. Thus, $[n]$ is the least positive integer which is divisible by all numbers from the set $\{1,\dots,n\}$. For example, $[3] = 6$, $[4] = 12$, and $[5] = [6] = 60$.

We denote the set of unordered integer partitions $\nu = (\nu_1,\dots,\nu_n)$ of $n \in \mathbb{N}$ by Π_n, that is, $\nu_1 \ge \cdots \ge \nu_n \ge 0$ and $n = \nu_1 + \cdots + \nu_n$, and the partial sums by $\sigma_i(\nu) = \nu_1 + \cdots + \nu_i$. On Π_n, we consider the partial oder $\preceq$ such that $\mu \preceq \nu$ if $\sigma_i(\mu) \le \sigma_i(\nu)$ for $i = 1,\dots,n$.

In $\mathbf{M}_n(\mathbb{F})$, the space of $n \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we denote by $\|\cdot\|$ the spectral norm, by $\mathbf{GL}_n(\mathbb{F}) \subset \mathbf{M}_n(\mathbb{F})$ the group of invertible matrices, and by $\mathbf{S}(\mathbb{F}) = \{X \in \mathbf{M}_n(\mathbb{F}) : \|X\| = \rho\}$—the sphere of radius $\rho > 0$ in $\mathbf{M}_n(\mathbb{F})$.

Single nilpotent Jordan blocks are denoted by $N_n = \left[\begin{smallmatrix} 0 & I_{n-1} \\ 0 & 0 \end{smallmatrix}\right] \in \mathbf{M}_n(\mathbb{R})$, where for $n = 1$, we set $N_1 = 0$. The unit basis matrices from $\mathbf{M}_n(\mathbb{R})$, with a single nonzero element 1 in position (i,j), are denoted by $E_n(i,j)$, and $J(\lambda,n) = \lambda I_n + N_n$ is an $n \times n$ Jordan block with eigenvalue λ. The m pairwise distinct eigenvalues of a matrix A, with algebraic multiplicities $k_1,\dots,k_m$, are denoted by $\lambda_1(A),\dots,\lambda_m(A)$ where $k_1 + \cdots + k_m = n$.

From now on, we introduce fractional intervals. For $p,q,s \in \mathbb{N}$, define the set $\mathcal{B}_p(q,s)$ as follows.

(1) If $p < s \le q$, then

$$\mathcal{B}_p(q,s) := \left\{ \frac{p}{q}, \frac{p}{q-1}, \dots, \frac{p}{s} \right\}. \tag{2.1}$$

The set $\mathcal{B}_p(q,s)$ is referred to as a *fractional interval of first kind* with endpoints p/q and p/s. It has $q - s + 1$ elements.

(2) If $s \le p < q$, then

$$\mathcal{B}_p(q,s) := \left\{ \frac{p}{q}, \frac{p}{q-1}, \dots, \frac{p}{p+1} \right\}. \tag{2.2}$$

The set $\mathcal{B}_p(q,s)$ is referred to as a *fractional interval of second kind* with endpoints p/q and $p/(p+1)$. It has $q - p$ elements.

(3) If $p \ge q$ and/or $s > q$, then

$$\mathcal{B}_p(q,s) := \emptyset. \tag{2.3}$$

Hence, the fractional interval $\mathcal{B}_p(q,s)$ either has elements less than 1 (if it is of first or second kind) or is empty. Also, if the interval $\mathcal{B}_p(q,s)$ is of second kind, as a set, it is equal to the interval $\mathcal{B}_p(q,p+1)$ of first kind.

Note that if the fractional intervals $\mathcal{B}_p(q_1,s_1)$ and $\mathcal{B}_p(q_2,s_2)$ are of first kind, then

$$\mathcal{B}_p(q_1,s_1) \cup \mathcal{B}_p(q_2,s_2) = \mathcal{B}_p(q_1,s_2) \tag{2.4}$$

if and only if $s_2 \le s_1 \le q_2 + 1$ and $q_2 \le q_1$. In particular,

$$\mathcal{B}_p(q,s_1) \cup \mathcal{B}_p(s_1 - 1,s) = \mathcal{B}_p(q,s) \tag{2.5}$$

provided the intervals in the left-hand side are of first kind. Similar results are valid for intervals of second kind as well.

The following lemma will be used in the construction of specific perturbations.

Lemma 2.1. *Let the n numbers $t_1,\dots,t_n$ be given. Then*

$$\det\left(\lambda I_n - \sum_{i=1}^{n-1} t_i E_n(i,i+1) - t_n E_n(n,j)\right) = \lambda^{j-1}\left(\lambda^{n+1-j} - \prod_{i=j}^{n} t_i\right) \tag{2.6}$$

for all $j=1,\dots,n$. In particular, taking $t_1=\cdots=t_{n-1}=1$ and $t_n=t$, we have

$$\det\left(\lambda I_n - N_n - tE_n(n,j)\right) = \lambda^{j-1}\left(\lambda^{n+1-j} - t\right). \tag{2.7}$$

Proof. Relations (2.6) and (2.7) follow immediately from the fact that the left-hand side of (2.6) is the determinant of the matrix

$$\begin{bmatrix} \lambda & -t_1 & 0 & \dots & 0 \\ 0 & \lambda & -t_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -t_{n-1} \\ -t_n & 0 & 0 & \dots & \lambda \end{bmatrix} \tag{2.8}$$

for $j=1$, or of the block upper-triangular matrix

$$\begin{bmatrix} \lambda & -t_1 & 0 & \dots & 0 \\ 0 & \lambda & -t_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -t_{n-1} \\ -t_n & 0 & 0 & \dots & \lambda \end{bmatrix}$$

for $j=1$, or of the block upper-triangular matrix

$$\left[\begin{array}{ccccc|ccccc} \lambda & -t_1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & -t_2 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -t_{j-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda & -t_{j-1} & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & \lambda & -t_j & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda & -t_{j+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -t_{n-1} \\ 0 & 0 & 0 & \dots & 0 & -t_n & 0 & 0 & \dots & \lambda \end{array}\right] \tag{2.9}$$

if $j \geq 2$. □

3. Fractional exponents in asymptotic spectral perturbations

In this section, we consider the problem of determining the possible and the leading fractional exponents in asymptotically small spectral perturbations.

3.1. Problem statement

Let a given matrix $A \in \mathbf{M}_n(\mathbb{F})$ of unit norm be perturbed to $A + tE$ where $t > 0$ and $E \in \mathbf{S}(\mathbb{F})$. The aim of *asymptotic spectral perturbation analysis* is to find asymptotic bounds of the quantities $|\lambda_s(A + tE) - \lambda_s(A)|$ as functions of the real parameter t for $t \to 0$, where $\lambda_i(B)$ are the eigenvalues of B, and E varies over the sphere $\mathbf{S}(\mathbb{F})$.

Under the (small) perturbation $A \to A + tE$, each eigenvalue $\lambda_i(A)$ of algebraic multiplicity k_i perturbs into k_i (not necessarily different) eigenvalues of $A + tE$, say

$$\lambda_{k_1+\cdots+k_{i-1}+1}(A + tE), \ldots, \lambda_{k_1+\cdots+k_i}(A + tE) \tag{3.1}$$

(for $i = 1$, the sum from k_1 to k_0 is considered void). The corresponding functions

$$t \longmapsto l_s(t) := \lambda_{k_1+\cdots+k_{i-1}+s}(A + tE) \in \mathbb{C} \tag{3.2}$$

are *algebraic*. Indeed, their values are the roots of the characteristic polynomial

$$\det\left(\lambda I_n - A - tE\right) = \lambda^n - c_1(t)\lambda^{n-1} + \cdots + (-1)^n c_n(t) \tag{3.3}$$

of $A + tE$ with coefficients $c_j(t)$ which, being sums of the principal jth-order minors of $A + tE$, are polynomials of degree up to j in t. In particular, the functions l_s are continuous on $\mathbb{R}$. In fact, l_s are the branches of algebraic functions which are piecewise differentiable and hence piecewise analytic, see [4]. At the isolated set $\Theta := \{t_1, \ldots, t_\sigma\}$ of *exceptional points* in an open interval $T \subset \mathbb{R}$, where the functions l_s are not differentiable, there exists $\varepsilon = \varepsilon(\tau, \Theta) > 0$, such that differentiability holds on the *pierced neighbourhood*

$$N(\tau, \varepsilon) := (\tau - \varepsilon, \tau) \cup (\tau, \tau + \varepsilon) = (\tau - \varepsilon, \tau + \varepsilon) \backslash \{\tau\} \subset T. \tag{3.4}$$

Indeed, given $\tau \in T$, there are two alternatives. If $\tau \notin \Theta$, then l_s is differentiable on each open subinterval $T_0 \ni \tau$ of T, which does not contain exceptional points and, in particular, on each $N(\tau, \varepsilon) \subset T_0$. On the other hand, if $\tau \in \Theta$, then l_s is not differentiable at τ but on $T_0 \backslash \{\tau\}$, where $T_0 \subset T$ is an open interval such that $T_0 \cap \Theta = \{\tau\}$.

We are interested in the local behaviour of l_s in a small neighbourhood of $t = 0$. This behaviour is completely determined by the matrices A and E if E is fixed. In particular, not only the fractional powers in the asymptotics of the eigenvalues of $A + tE$ but also the corresponding coefficients are uniquely determined. But, often the matrix E is not known, for example, the perturbation tE may be due to rounding errors during the computation of the eigenvalues of A in finite arithmetic. When a numerically stable algorithm is implemented to compute the eigenvalues, we may assume that t is of order eps$\|A\|/\rho$, where eps is the rounding unit.

Our goal is to determine the fractional exponents in the asymptotic eigenvalue expansions. When E varies over $\mathbf{S}(\mathbb{F})$, then the fractional exponents describe the behaviour of l_s modulo $\mathbf{S}(\mathbb{F})$ and this is determined by A only. In this case, the whole information is coded in the Jordan form of A and, in particular, in the arithmetic invariant of A under the similarity action $(X, A) \mapsto X^{-1}AX$, $X \in \mathbf{GL}_n(\mathbb{F})$ of the general linear group $\mathbf{GL}_n(\mathbb{F})$. The arithmetic invariant consists of the partial algebraic multiplicities of the eigenvalues of A which are exactly the orders of the diagonal blocks of the complex Jordan form $J_A = R^{-1}AR$ of A relative to $\mathbf{GL}_n(\mathbb{C})$. In order to treat, in an uniform way, the cases of real ($\mathbb{F} = \mathbb{R}$) and complex ($\mathbb{F} = \mathbb{C}$) matrices, we will always use the complex Jordan form. Of course, using the real Jordan form for real matrices would give analogous results.

If A is nondefective (J_A is diagonal), then all functions l_s are analytic in a sufficiently small neighbourhood of $t = 0$. This case is not interesting from the point of view of asymptotic analysis, since then $|l_s(t) - l_s(0)| = O(t)$, $t \to 0$, see [9].

If A is defective, then the functions l_s are analytic in a small pierced neighbourhood $N(0, \varepsilon)$ of $t = 0$, see [4]. This means that for some $\varepsilon > 0$ and some $t_0 \in N_\varepsilon$, all derivatives $l_s^{(j)}(t_0)$ of l_s at $t = t_0$ exist, and

$$l_s(t) = \sum_{j=0}^{\infty} \frac{l_s^{(j)}(t_0)}{j!}(t - t_0)^j \tag{3.5}$$

for all $t \in N_\varepsilon$. For $t = 0$, however, some of the functions l_s may not be differentiable since the limit $\lim_{t\to 0} |l_s^{(j)}(t)| = \infty$ is possible. In this case, l_s has the asymptotic expansion

$$l_s(t) = l_s(0) + O\left(t^{p_s/q_s}\right) + O\left(t^{p_s/q_s + \varepsilon_s}\right), \quad t \longrightarrow 0, \tag{3.6}$$

where $p_s/q_s \in \mathbb{Q}_1$ and $\varepsilon_s > 0$.

Consider, for example, the case $i = 1$ and $s = 1$, and set

$$\delta_E(t) := \left| l_1(t) - l_1(0) \right| = \left| \lambda_1(A + tE) - \lambda_1(A) \right|. \tag{3.7}$$

Note that, here, l_1 may be *any* of the continuous functions l_s satisfying $l_s(0) = \lambda_1(A)$ and that l_1 is continuous on $\mathbb{R}$ and analytic on some $N(0,\varepsilon)$.

If the elementary divisors of $\lambda I_n - A$ corresponding to $\lambda_1 = \lambda_1(A)$ are linear, we have

$$\delta_E(t) \le at + O(t^2), \quad t \longrightarrow 0. \tag{3.8}$$

Here, $a > 0$ may be taken as

$$a = \min\left\{ \|X\| \left\|X^{-1}\right\| : X \in \mathbf{GL}_n(\mathbb{C}),\ X^{-1}AX = \operatorname{diag}\left(\lambda_1 I_{k_1}, *\right) \right\}, \tag{3.9}$$

where $*$ denotes an unspecified matrix block.

If the matrix A is defective and there are nonlinear elementary divisors corresponding to λ_1, then the asymptotics of $\delta_E(t)$ is more involved and may include fractional exponents, namely, [4, 5]

$$\delta_E(t) = \sum_i a_i t^{p_i/q_i} + O(t), \quad t \longrightarrow 0. \tag{3.10}$$

Here, a_i are constants and $p_i/q_i \in \mathbb{Q}_1$ are fractional exponents, where p_i and q_i are coprime integers with $p_i < q_i \le n$. Note that if we perturb A to $A + t^{\alpha}E$ with $\alpha = [n]$, then the fractional powers of t^{α} will become integer powers of t.

The fractions p_i/q_i are necessarily elements of $\mathcal{R}_n$, for example,

$$\mathcal{R}_2 = \left\{ \frac{1}{2} \right\}, \qquad \mathcal{R}_3 = \left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \right\}, \qquad \mathcal{R}_4 = \left\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \right\}, \ldots. \tag{3.11}$$

This follows from (3.3) and a simple application of the technique of Newton diagrams.

Let $\mathcal{P} \subset \mathcal{R}_n$ be the set of *possible fractional exponents* p_i/q_i in (3.10) when E varies over $\mathbf{S}(\mathbb{F})$. A fraction $p/q \in \mathcal{R}_n$ is an element of $\mathcal{P}$ if and only if there are $E \in \mathbf{S}(\mathbb{F})$ and $j \in \mathbb{N}$ such that $p = p_j$, $q = q_j$, and $a_j > 0$ in the expression (3.10).

We will also introduce the subset $\mathcal{L} \subset \mathcal{P}$ of *leading fractional exponents* as follows. A fraction $p/q \in \mathcal{R}_n$ is an element of $\mathcal{L}$ if and only if there are $E \in \mathbf{S}_\rho(\mathbb{F})$, $a > 0$, and $\varepsilon > 0$, such that the expression (3.10) may be written as

$$\delta_E(t) = at^{p/q} + O\left(t^{p/q+\varepsilon}\right), \quad t \longrightarrow 0. \tag{3.12}$$

We stress that the sets $\mathcal{P}$ and $\mathcal{L}$ correspond to a given eigenvalue of A, in this case, the eigenvalue $\lambda_1(A)$, and we could denote them by $\mathcal{P}_{\lambda_1}$ and $\mathcal{L}_{\lambda_1}$, respectively. For the other eigenvalues of A, there are other (possibly) different sets $\mathcal{P}_{\lambda_i}$ and $\mathcal{L}_{\lambda_i}$, $i = 2,\dots,m$, of possible and leading exponents.

It is obvious that we have

$$\mathcal{L} \subset \mathcal{P} \subset \mathcal{R}_n. \tag{3.13}$$

Whether some of these inclusions is proper depends on the Jordan structure of A. In general, to determine the sets $\mathcal{L}$ and $\mathcal{P}$ on the basis of the arithmetic invariant of A is an open problem. We will present partial results for this problem below.

Example 3.1. For the matrix $A = N_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, we have $\mathcal{L} = \mathcal{P} = \mathcal{R}_2$. Indeed, for $E = E_2(2,1)$, the characteristic polynomial of $A + tE$ is $\lambda^2 - t$ with roots $\pm t^{1/2}$. Thus, the only element $1/2$ of $\mathcal{R}_2$ is an element of $\mathcal{L}$ as well.

Example 3.2. Consider the matrix $A = N_3$. For $E = E_3(3,1)$, the characteristic polynomial of $A + tE$ is $\lambda^3 - t$, and the eigenvalues are the cube roots of t, for example,

$$\lambda_1(A + tE) = t^{1/3}. \tag{3.14}$$

For $E = E_3(3,2)$, the characteristic polynomial of $A + tE$ is $\lambda(\lambda^2 - t)$ and has roots $\pm t^{1/2}$ and 0. Hence,

$$\lambda_1(A + tE) = t^{1/2}. \tag{3.15}$$

For $E = E_3(3,1) + E_3(3,2)$, the characteristic polynomial is $\lambda^3 - t\lambda - t$ and has a root with Pisseaux series [4]

$$\lambda_1(A + tE) = t^{1/3} + \frac{t^{2/3}}{3} + 0 \cdot t^{3/3} - \frac{t^{4/3}}{81} + O\left(t^{5/3}\right), \quad t \longrightarrow 0, \tag{3.16}$$

in $t^{1/3}$. Hence, $1/3, 1/2, 2/3 \in \mathcal{P}$ and, since these are all elements of $\mathcal{R}_3$, we see that $\mathcal{P} = \mathcal{R}_3$.

Relations (3.14) and (3.15) show that $1/3$ and $1/2$ are also elements of $\mathcal{L}$. To show that $2/3$ is also an element of $\mathcal{L}$ is more complicated. For $E = E_3(2,1) - E_3(2,2) - E_3(3,2) + E_3(3,3)$, the characteristic polynomial of the matrix

$$A + tE = \begin{bmatrix} 0 & 1 & 0 \\ t & -t & 1 \\ 0 & -t & t \end{bmatrix} \tag{3.17}$$

is $\lambda^3 - t^2\lambda + t^2$ and has a root

$$\lambda_1(A+tE) = -t^{2/3} - \frac{t^{4/3}}{3} + O(t^2), \quad t \longrightarrow 0. \tag{3.18}$$

Hence, 2/3 is also an element of $\mathcal{L}$ which gives $\mathcal{L} = \mathcal{R}_3$. Hence, in this particular example, we also have $\mathcal{L} = \mathcal{D} = \mathcal{R}_3$.

Example 3.3. For the matrix $A = \text{diag}(N_2, N_1)$, taking $E = E_3(2,1)$, we obtain the characteristic polynomial $\lambda(\lambda^2 - t)$, that is, 1/2 is an element of $\mathcal{L}$. Taking $E = E_3(2,3) + E_3(3,1)$, we get the characteristic polynomial $\lambda^3 - t^2$. Hence, 2/3 is also an element of $\mathcal{L}$. But 1/3 is not an element of $\mathcal{L}$. Indeed, in this case, the characteristic polynomial of $A + tE$ must be

$$\lambda^3 - \cdots - (c_{30}t^3 + c_{31}t^2 + c_{32}t) \tag{3.19}$$

with $c_{32} \neq 0$. But $\det(A+tE)$ is a polynomial in t of degree not less than 2, so $c_{32} = 0$ and the exponents 1/3 cannot be leading. Thus, $\mathcal{L} = \mathcal{D} = \{1/2, 2/3\}$ are proper subsets of $\mathcal{R}_3 = \{1/3, 1/2, 2/3\}$.

Note that, in general, if $p/q \in \mathcal{L}$, then we can find E so that

$$\delta_E(t) = \sum_{i=1}^{\infty} b_i t^{ip/q}, \quad t \longrightarrow 0. \tag{3.20}$$

So the numbers ip/q are elements of $\mathcal{D}$ for $i = 1, \ldots, \text{Ent}(q/p)$.

Example 3.4. Let $A \in \mathbf{M}_n(\mathbb{R})$ be zero except for the first superdiagonal, where $a_{1,2} = 1$ and $a_{i,i+1}$ is equal to 1 or 0 in such a way that there are $j-1$ zero elements $(1 \leq j \leq n-1)$. Take a matrix $E = F + G$ so that F has all its elements equal to zero with the exception of $j-1$ elements on its superdiagonal equal to 1 in the positions of the zero elements $a_{i,i+1}$, and $G = E_n(n,1) + E_n(n,2)$. Then the matrix $\lambda I_n - (A+tE)$ has the form

$$\begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & -t_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -t_{n-1} \\ -t & -t & 0 & \ldots & \lambda \end{bmatrix}, \tag{3.21}$$

where $j-1$, among the numbers $t_2, \ldots, t_{n-1}$, are equal to t and the other $n-j-1$ are equal to 1. Then (2.6) yields $\det(\lambda I_n - (A+tE)) = \lambda^n - t^j\lambda - t^j$.

The roots of this polynomial have Pisseaux series [4] in $t^{j/n}$, for example,

$$\lambda_1(A+tE)=t^{j/n}+\frac{t^{2j/n}}{n}-\frac{n-3}{2n^2}t^{3j/n}+O(t^{4j/n}),\quad t\longrightarrow 0. \tag{3.22}$$

This example shows that the set $\mathcal{L}$ of leading fractional exponents is of major interest.

The computation (and even the estimation) of a_i, p_i, and q_i in (3.10) is usually a very difficult task. However, if the orders of the blocks in the complex Jordan form J_A of A are known, then some direct calculations, together with an implementation of the technique of Newton diagrams, allow to determine a subset of the set of leading fractional exponents $\mathcal{L}$ for a general defective matrix. In [6, 7], the problem is solved for some of the leading exponents of a defective matrix with only one eigenvalue. Here we extend these results, but still the complete analysis is an open problem.

Of course, we have the fractional exponents

$$\frac{1}{k_{1,1}},\frac{1}{k_{1,1}-1},\dots,\frac{1}{2}\in\mathcal{L} \tag{3.23}$$

corresponding to the largest block $J(\lambda_1,k_{1,1})$ of J of order $k_{1,1}$ with eigenvalue λ_1. Similarly, we have the exponents

$$\frac{1}{k_{1,j}},\frac{1}{k_{1,j}-1},\dots,\frac{1}{2}\in\mathcal{L} \tag{3.24}$$

corresponding to the smaller blocks $J(\lambda_1,k_{1,j})$ of J_A, $k_{1,j}<k_{1,1}$, but these exponents are already in the list (3.23). We also have that for each $1/\alpha$ from the list (3.23), the integer multiples i/α with $i<\alpha$ belong to $\mathcal{P}$. At the same time, there may be other fractional exponents with denominators up to n and numerators up to $n-1$ as in the following example of Wilkinson [10].

Example 3.5. Consider the nilpotent matrix $A=\operatorname{diag}(J(0,3),J(0,2))\in\mathbf{M}_5(\mathbb{R})$ with Jordan blocks of order $k_{1,1}=3$, $k_{1,2}=2$, which is in Jordan form. Choosing the matrix E as $E_5(3,1)$, we get the fractional exponent $1/3$, while choosing E as $E_5(3,2)$ or $E_5(5,4)$, we get the exponent $1/2$, both being in the list (3.23) for $k_{1,1}=3$. However, we have also the leading exponents $2/5$, corresponding to $E=E_5(3,4)+E_5(5,1)$, and $2/3$, corresponding to $E=E_5(3,4)+E_5(5,3)$, which are not in the list (3.23). Choosing $E=E_5(2,4)+E_5(3,5)-E_5(4,1)-E_5(4,4)+E_5(5,3)+E_5(5,5)$, we also find the leading exponent $3/4$ since the characteristic polynomial of $A+tE$ in this case is $\lambda(\lambda^4-t^3\lambda-t^3)$.

Furthermore, for $E = E_5(1,4) - E_5(2,1) - E_5(2,4) + E_5(3,2) + E_5(3,5) + E_5(5,1)$, we have $\det(\lambda I_5 - A - tE) = \lambda^5 + t^3$ which gives $3/5 \in \mathcal{L}$. Similarly, taking $E = E_5(1,1) + E_5(1,4) + E_5(2,1) - E_5(2,4) - E_5(2,5) - E_5(3,2) + E_5(3,5) + E_5(4,1) - E_5(5,2)$, we see that $\det(\lambda I_5 - A - tE) = \lambda^5 - t\lambda^4 - 2t^2\lambda^3 + t^3\lambda^2 + t^4\lambda - t^4$ and, hence, $4/5 \in \mathcal{L}$.

On the other hand, $1/5$ is not an element of $\mathcal{L}$ since the lowest power of t in $\det(A + tE)$ is 2. Whether the other candidate $1/4$ is a member of $\mathcal{L}$ is not known. Thus, the list of leading exponents for this case satisfies

$$\left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{5}, \frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \frac{4}{5}\right\} \subset \mathcal{L}. \tag{3.25}$$

A key factor in obtaining the fractions $2/5$ and $2/3$ from $\mathcal{L}$ is that for $t > 0$, the matrix $J_A + tE_5(3,4)$ has nonzero superdiagonal and its Jordan form consists of a single Jordan block $J(0,5)$. Thus an additional perturbation $tE_5(5,j)$ gives the fractions $2/(6-j)$ with $j = 1,2,3$ of which $2/5$ and $2/3$ are "new", that is, not in the list (3.23). This idea is systematically developed later on.

In turn, the exponents $3/4, 3/5, 4/5 \in \mathcal{L}$ were found after some trials, having in mind the desired forms of the characteristic polynomial of $A + tE$ capable of producing these particular exponents as leading.

3.2. Calculation of fractional exponents

In this subsection, we present a simple algorithm to compute a subset $\mathcal{F}$ of $\mathcal{L}$. This method is based on the idea of perturbing J_A to $J_A + t(F + G)$ so that the Jordan form of the matrix $J_A + tF$ consists of larger blocks than those of J_A. This is done using perturbations tF which contain matrices of the form $tE_n(i, i+1)$ whenever the ith row of J_A is zero. After that, the perturbation tG leads to a matrix $J_A + t(F + G)$ with a simple characteristic polynomial with roots which are the desired leading fractional powers of t.

Suppose that A has m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ with algebraic multiplicities $k_1 \geq \cdots \geq k_m$, $k_1 + \cdots + k_m = n$, which means that the characteristic polynomial of A is $\prod_{i=1}^{m}(\lambda - \lambda_i)^{k_i}$. Suppose next that each λ_i takes part in n_i Jordan blocks of orders $k_{i,1} \geq \cdots \geq k_{i,n_i}$, $k_{i,1} + \cdots + k_{i,n_i} = k_i$. Thus the minimal polynomial of A is $\prod_{i=1}^{m}(\lambda - \lambda_i)^{k_{i,1}}$.

The numbers $k_{i,j}$ are the *partial algebraic multiplicities* of λ_i. The number n_i is the *geometric multiplicity* of λ_i, that is, the number of linearly independent eigenvectors of A, corresponding to λ_i. If at least one geometrical multiplicity is larger than 1, then the matrix A is *derogatory*. We recall that $k_i - n_i$ is the *defect* of the eigenvalue λ_i, while $n - n_1 - \cdots - n_m$ is the *defect* of the matrix A and is denoted by $\operatorname{def}(A)$. With this notation, the Jordan form of A may be written as

$$J_A = \operatorname{diag}\left(J(\lambda_1, k_{1,1}), \ldots, J(\lambda_1, k_{1,n_1}), \ldots, J(\lambda_m, k_{m,1}), \ldots, J(\lambda_m, k_{m,n_m})\right). \tag{3.26}$$

The matrices $A + tE$ and $J_A + tR^{-1}ER$ have the same fractional powers of t in their asymptotic eigenvalue expansions. Since we are interested in the fractional exponents p_i/q_i and not in the coefficients a_i in (3.10), we assume, in the following, that the matrix A is already transformed to Jordan form, that is, $A = J_A$. We also assume that $k_{1,1} > 1$, which is necessary and sufficient for the existence of fractional exponents in the asymptotic expansion of $\delta_E(t)$, $t \to 0$ for some E.

To simplify the notation, we set

$$k := k_1; \qquad r := n_1; \qquad \kappa_i := k_{1,i}, \quad i = 1, \ldots, r. \tag{3.27}$$

Then, we have the (unordered) partition $\kappa := (\kappa_1, \ldots, \kappa_r)$, $\kappa_1 \geq \cdots \geq \kappa_r$, $k = \kappa_1 + \cdots + \kappa_r$ of k with $\kappa_1 > 1$ and $r < k$. If necessary, we set $\kappa_i = 0$ for $i > r$ and write the partition as $\kappa = (\kappa_1, \ldots, \kappa_k) = (\kappa_1, \ldots, \kappa_r, 0, \ldots, 0)$.

Denote $\sigma_i(\kappa) := \kappa_1 + \cdots + \kappa_i$. The set Π_k of unordered partitions of k is a lattice with a partial order relation $\preceq$ such that $\alpha \preceq \beta$ if and only if $\sigma_i(\alpha) \leq \sigma_i(\beta)$, $i = 1, \ldots, k$. This order relation is not linear since for $k \geq 6$, there are incomparable partitions such as $(4,1,1)$ and $(3,3)$.

The set $\mathcal{F} \subset \mathcal{L}$, that we will determine later on via (3.30) and (3.31), depends not only on the partition κ of k but also on $\operatorname{def}(A)$. As a set $\mathcal{F}$ will consist of pairwise disjoint fractions p/q in which p and q are coprime. However, in the description of $\mathcal{F}$ given below, in order to simplify the notation, we allow p_i and q_i to have common divisors, for example, sometimes we write $2/6$ instead of $1/3$, and so forth. For instance, we use sets of rational fractions as

$$\left\{\frac{2}{q}, \frac{2}{q-1}, \ldots, \frac{2}{3}\right\} \tag{3.28}$$

with $q \geq 3$ in which some of the fractions may not be prime.

Finally, introduce the number

$$\pi_i := \min\left\{\sigma_i(\kappa) - i - 1, n - r - \operatorname{def}(A)\right\} \tag{3.29}$$

(we recall that $n - r = n_2 + \cdots + n_m$) which depends not only on κ but also on the defect of A.

Now we can state our main result.

THEOREM 3.6. *Consider the fractional intervals*

$$\mathcal{F}_{ij} \coloneqq \mathcal{B}_{i+j}\big(\sigma_i(\kappa), 1+k-\sigma_{r+1-i}(\kappa)\big) = \mathcal{B}_{i+j}\big(\sigma_i(\kappa), 1+\kappa_{r+2-i}+\cdots+\kappa_r\big) \quad (3.30)$$

for $i = 1,\ldots,r$ and $j = 0,\ldots,\pi_i$ (here the sum from $r+1$ to r for $i = 1$ above is considered void).

Then, for the set of leading fractional exponents $\mathcal{L}$ corresponding to the eigenvalue λ_1 of A, we have

$$\mathcal{F} := \bigcup_{i=1,\,j=0}^{r,\,\pi_i} \mathcal{F}_{ij} \subset \mathcal{L}. \quad (3.31)$$

Furthermore, for $n = 2$, we have $\mathcal{F} = \mathcal{L}$, while for $n > 2$, the inclusion in (3.31) may be proper.

Proof. When studying the absolute perturbations in λ_1, we may always assume, without loss of generality, that $\lambda_1 = 0$ (otherwise, we may consider the matrix $A - \lambda_1 I_n$ with the same Jordan structure and a zero eigenvalue of multiplicity k). For the proof, we consider two cases. First, we consider the case when A has a single zero eigenvalue and second, the case when A has two or more pairwise distinct eigenvalues.
Case 1. If A has a single zero eigenvalue, that is, $m = 1$, then $k = k_1 = n$ and $\pi_1 = \cdots = \pi_r = 0$. In this case, we have to show that $\mathcal{L}$ contains the union of the sets

$$\begin{aligned}
\mathcal{F}_{10} &= \mathcal{B}_1\big(\sigma_1(\kappa), 2\big) = \left\{\frac{1}{\kappa_1}, \frac{1}{\kappa_1 - 1}, \ldots, \frac{1}{2}\right\},\\
\mathcal{F}_{20} &= \mathcal{B}_2\big(\sigma_2(\kappa), 1+\kappa_r\big) = \left\{\frac{2}{\kappa_1+\kappa_2}, \frac{2}{\kappa_1+\kappa_2-1}, \ldots, \frac{2}{1+\kappa_r}\right\}\setminus\{1\},\\
&\vdots\\
\mathcal{F}_{i0} &= \mathcal{B}_i\big(\sigma_i(\kappa), 1+k-\sigma_{r+1-i}(\kappa)\big)\\
&= \left\{\frac{i}{\sigma_i(\kappa)}, \frac{i}{\sigma_i(\kappa)-1}, \ldots, \frac{i}{1+\kappa_{r+2-i}+\cdots+\kappa_r}\right\}\setminus\{1\},\\
&\vdots\\
\mathcal{F}_{r0} &= \mathcal{B}_r\big(k, 1+k-\sigma_1(\kappa)\big) = \left\{\frac{r}{k}, \frac{r}{k-1}, \ldots, \frac{r}{1+\kappa_2+\cdots+\kappa_r}\right\}\setminus\{1\}.
\end{aligned} \quad (3.32)$$

Here we have used the fact that $\kappa_{r+2-i}+\cdots+\kappa_r = k - \sigma_{r+1-i}(\kappa)$, $i \geq 2$.

Recall that for $m = 1$, the Jordan form of A is

$$J_A = \operatorname{diag}\big(J(0,\kappa_1),\ldots,J(0,\kappa_r)\big). \tag{3.33}$$

We show successively how each member p/q of $\mathcal{F}_{10},\ldots,\mathcal{F}_{r0}$ appears as an actual leading exponent in a term $O(t^{p/q})$ in the asymptotic expansion of $\delta_E(t)$, $t \to 0$, for an appropriate choice of E. Note that to have an exponent p/q with p and q coprime, we must have at least p nonzero elements $e_{i_1,j_1},\ldots,e_{i_p,j_p}$ of E in different rows and columns, that is, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_p$. Only in this case, a term t^p may appear in the characteristic polynomial of $J_A + tE$.

The set $\mathcal{F}_{10}$

Consider perturbations which affect only one block $J(0,\kappa_j)$ of J_A from (3.33) with $\kappa_j \geq 2$. Since $\kappa_1 \geq 2$, then at least one such block exists. Taking the matrix $E = E_n(\sigma_j(\kappa),\sigma_{j-1}(\kappa)+s)$ for $s = 1,\ldots,\kappa_j$ and using relation (2.7), we see that $\det(\lambda I_n - J_A - tE) = \lambda^{n-s}(\lambda^s - t)$. Hence, the fractions

$$\frac{1}{\kappa_j}, \frac{1}{\kappa_j - 1}, \ldots, \frac{1}{2} \tag{3.34}$$

are among the leading exponents. All these exponents are in the fractional interval $\mathcal{F}_{10} = \mathcal{B}_1(\kappa_1, 2)$, while for $j = 1$, they are exactly the elements of $\mathcal{F}_{10}$.

The set $\mathcal{F}_{i0}$, $1 < i < r$ (this case exists for $r \geq 3$)

Take first a perturbation $E = E_1$ which affects the superdiagonal over the first i blocks $J(0,\kappa_1),\ldots,J(0,\kappa_i)$ of the matrix J. Let $E_1 = F_1 + G_1$ where

$$F_1 := \sum_{j=1}^{i-1} E_n\big(\sigma_j(\kappa),\sigma_j(\kappa)+1\big), \quad G_1 := E_n\big(\sigma_i(\kappa), s\big), \quad 1 \leq s \leq \kappa_1. \tag{3.35}$$

According to relation (2.6), we have

$$\det\big(\lambda I_n - J_A - tE_1\big) = \lambda^{n-1+s-\sigma_i(\kappa)}\big(\lambda^{1-s+\sigma_i(\kappa)} - t^i\big). \tag{3.36}$$

This gives us exponents from the set

$$\begin{aligned} &\left\{\frac{i}{\sigma_i(\kappa)}, \frac{i}{\sigma_i(\kappa)-1}, \ldots, \frac{i}{1+\kappa_2+\cdots+\kappa_i}\right\} \backslash \{1\} \\ &\quad = \mathcal{B}_i\big(\sigma_i(\kappa), 1+\kappa_2+\cdots+\kappa_i\big) = \mathcal{B}_i\big(\sigma_i(\kappa), 1+\sigma_i(\kappa)-\sigma_1(\kappa)\big), \end{aligned} \tag{3.37}$$

some of which are in the fractional interval $\mathcal{F}_{i0}$. The set $\mathcal{F}_{i0}$ may contain larger elements than the elements of the set (3.37). Now we show that these larger elements are also leading exponents by the construction of suitable perturbations E_2, E_3, and so forth.

Determine the perturbation $E = E_2$ which affects the superdiagonal over the next i blocks $J(0,\kappa_2),\dots,J(0,\kappa_{i+1})$ of J_A as $E_2 = F_2 + G_2$, where

$$F_2 := \sum_{j=2}^{i} E_n\big(\sigma_j(\kappa),\sigma_j(\kappa)+1\big), \quad G_2 := E_n\big(\sigma_{i+1}(\kappa),s\big), \quad \kappa_1+1 \le s \le \sigma_2(\kappa). \tag{3.38}$$

We have

$$\det\big(\lambda I_n - J_A - tE_2\big) = \lambda^{n-1+s-\sigma_{i+1}(\kappa)+\kappa_1}\big(\lambda^{1-s+\sigma_{i+1}(\kappa)-\kappa_1} - t^i\big) \tag{3.39}$$

which gives exponents from the set

$$\begin{aligned}\left\{\frac{i}{\kappa_2+\cdots+\kappa_{i+1}},\frac{i}{\kappa_2+\cdots+\kappa_{i+1}-1},\dots,\frac{i}{1+\kappa_3+\cdots+\kappa_{i+1}}\right\}\setminus\{1\}&\\ = \mathcal{B}_i\big(\sigma_{i+1}(\kappa)-\sigma_1(\kappa),1+\sigma_{i+1}(\kappa)-\sigma_2(\kappa)\big).&\end{aligned} \tag{3.40}$$

Since $\kappa_{i+1} \ge 1$, we have

$$\frac{i}{\kappa_2+\cdots+\kappa_{i+1}} \le \frac{i}{\kappa_2+\cdots+\kappa_i}. \tag{3.41}$$

Hence the first (smallest) element of (3.40) is not strictly larger than the last (largest) element of (3.37). Therefore, the union of the fractional intervals (3.37) and (3.40) is

$$\begin{aligned}\left\{\frac{i}{\sigma_i(\kappa)},\frac{i}{\sigma_i(\kappa)-1},\dots,\frac{i}{1+\kappa_3+\cdots+\kappa_{i+1}}\right\}\setminus\{1\}&\\ = \mathcal{B}_i\big(\sigma_i(\kappa),1+\sigma_{i+1}(\kappa)-\sigma_2(\kappa)\big).&\end{aligned} \tag{3.42}$$

Continuing this process with a matrix E_3, affecting the superdiagonal over the blocks $J(0,\kappa_3),\dots,J(0,\kappa_{i+2})$ of J_A, and so forth, we see that all exponents from the set

$$\begin{aligned}\left\{\frac{i}{\sigma_i(\kappa)},\frac{i}{\sigma_i(\kappa)-1},\dots,\frac{i}{1+\kappa_{r+1-i}+\cdots+\kappa_r}\right\}\setminus\{1\}&\\ = \mathcal{B}_i\big(\sigma_i(\kappa),1+\sigma_r(\kappa)-\sigma_{r+1-i}(\kappa)\big)&\end{aligned} \tag{3.43}$$

are leading. But the set (3.43) is exactly the fractional interval $\mathcal{F}_{i0}$.

The set $\mathcal{F}_{r0}$

Here we may take $E = F + G$ with

$$F := \sum_{j=1}^{r-1} E_n\big(\sigma_j(\kappa), \sigma_j(\kappa)+1\big), \quad G := E_n(n,s), \quad 1 \le s \le \kappa_1. \tag{3.44}$$

The characteristic polynomial of $J_A + tE$ is $\lambda^{n-k-1+s}(\lambda^{k+1-s} - t^r)$ and therefore, all exponents from $\mathcal{F}_{r0}$ are leading.

Thus, we have proved the inclusions $\mathcal{F}_{10}, \dots, \mathcal{F}_{r0} \subset \mathcal{L}$.

Case 2. If A has more than one eigenvalue, or $m \ge 2$, then we take a perturbation tF that puts a quantity t in the places of the zero elements of the superdiagonal of J_A over i blocks $J(0,\kappa_{\alpha_1}), \dots, J(0,\kappa_{\alpha_i})$ of J_A with zero eigenvalue, and over j blocks $J(\lambda_{\beta_1},\kappa_{\beta_1}), \dots, J(\lambda_{\beta_j},\kappa_{\beta_j})$ with eigenvalues $\lambda_{\beta_1}, \dots, \lambda_{\beta_j} \ne 0$ (we recall that in J_A, the first blocks have eigenvalue zero).

Add a perturbation tG with

$$G := E_n\big(k_{\alpha_1} + \cdots + k_{\alpha_i} + k_{\beta_1} + \cdots + k_{\beta_j}, \gamma\big), \quad 1 \le \gamma \le \alpha_1. \tag{3.45}$$

Then, the characteristic polynomial of $J_A + tE$ is

$$\lambda^{n-1+\gamma-k_{\alpha_1}-\cdots-k_{\alpha_i}-k_{\beta_1}-\cdots-k_{\beta_j}}\left(\lambda^{k_{\alpha_1}+\cdots+k_{\alpha_i}+1-\gamma}\prod_{\rho=1}^{j}(\lambda-\lambda_{\beta_\rho})^{\kappa_{\beta_\rho}} - t^{i+j}\right). \tag{3.46}$$

This polynomial has $q := k_{\alpha_1} + \cdots + k_{\alpha_i} + 1 - \gamma$ zeros of the form

$$e_\omega b t^{(i+j)/q}\Big(1 + O\big(t^{1/q}\big)\Big), \quad t \longrightarrow 0,\ \omega = 0, \dots, q-1, \tag{3.47}$$

where $e_\omega := \exp(2\pi\omega\imath/q)$ are the q roots of 1, and

$$b := \left(\prod_{\rho=1}^{j}(-\lambda_{\beta_\rho})^{\kappa_{\beta_\rho}}\right)^{-1/q}. \tag{3.48}$$

Thus, we have the leading exponents $(i+j)/q$ for $j < q - i$. Also, j cannot exceed $n - r - \operatorname{def}(A)$, which is the number of eigenvalues of A, which are different from zero. Hence, we have proved that the fractional intervals

$$\mathcal{B}_{i+j}\big(k_{\alpha_1} + \cdots + k_{\alpha_i} + 1 - \gamma, 1 + k_{\alpha_2} + \cdots + k_{\alpha_i}\big), \quad 1 \le \gamma \le \alpha_1, \tag{3.49}$$

are subsets of $\mathcal{L}$. Taking all combinations of $\alpha_1, \dots, \alpha_i$ from the partition κ and repeating the argument from Case 1, we see that the union of the intervals (3.49) is $\mathcal{F}_{ij}$, that is, $\mathcal{F}_{ij} \subset \mathcal{L}$.

The case $n = 2$ is trivial, see Example 3.1, which gives $\mathcal{F} = \mathcal{L} = \mathcal{R}_2 = \{1/2\}$. That the inclusion $\mathcal{F} \subset \mathcal{L}$ may be proper for $n > 2$ is demonstrated below by examples. □

Remark 3.7. In view of (2.1), (2.2), (2.3), (2.6), (2.7), (3.3), (3.10), (3.12), (3.13), (3.14), (3.15), (3.23), and (3.30), the sets $\mathcal{F}_{ij}$ and their union $\mathcal{F}$ are in fact easily computable despite of their not very pleasant appearance. Thus, Theorem 3.6 gives a simple algorithm to compute some of the leading fractional exponents in the eigenvalues of the matrix $A + tE$, $t \to 0$, given the partial algebraic multiplicities $\kappa_{i,j}$ of the eigenvalues $\lambda_i(A)$ of A.

Remark 3.8. The key step in the constructive proof of Theorem 3.6 is the splitting of the matrix E as $E = F + G$. The perturbation tF puts a quantity t in positions $(i, i+1)$ whenever the ith row of J_A is zero, that is, it fills in the superdiagonal of J_A. Thus, the partition of n containing the partial multiplicities of the eigenvalues of $J_A + tF$ is larger (relative to the partial order $\preceq$) than the partition containing the partial multiplicities of the eigenvalues of J_A. Finally, a perturbation tG gives the necessary fractional exponents from the set $\mathcal{F}$.

To see that $\mathcal{F}$ may be a proper subset of $\mathcal{L}$, consider the following examples.

Example 3.9. For $n = 3$, the first possible case (producing fractional exponents) is $A = \operatorname{diag}(N_2, N_1)$. Here we have $\mathcal{F} = \mathcal{L} = \{1/2, 2/3\}$. Indeed, if $\mathcal{F} \neq \mathcal{L}$, then $\mathcal{L}$ must contain the fraction $1/3$. The only possibility for $1/3$ to appear as a leading exponent is when the λ-free term $-\det(A + tE)$ in the characteristic polynomial of $A + tE$ has t in first power (see also the Appendix). But this is impossible since the lowest possible degree of t in the polynomial $\det(A + tE)$ is 2. Hence $1/3$ is not an element of $\mathcal{L}$ and hence $\mathcal{F} = \mathcal{L}$. The second possible case is $A = N_3$. But Example 3.2 shows that here we have $\mathcal{F} = \{1/3, 1/2\}$ and $\mathcal{L} = \mathcal{R}_3 = \{1/3, 1/2, 2/3\}$, see also [6]. Thus $\mathcal{F}$ is a proper subset of $\mathcal{L}$.

Example 3.10. For 4×4 matrices, we have 4 subcases.

(1) For $A = \operatorname{diag}(N_2, N_1, N_1)$, we have $\mathcal{F} = \mathcal{L} = \{1/2, 2/3, 3/4\}$. Indeed, the leading fractions $1/2$, $2/3$, and $3/4$ from $\mathcal{F}$ are achieved for E equal to $E_4(2,1)$, $E_4(2,3) + E_4(3,1)$, and $E_4(2,3) + E_4(3,4) + E_4(4,1)$, respectively. We will show that $\mathcal{L}$ has no other elements than these of $\mathcal{F}$. The other elements from $\mathcal{R}_4$ are $1/4$ and $1/3$. The exponent $1/4$ cannot be leading since it requires the polynomial $\det(A + tE)$ to contain a multiple of t. But the lowest possible degree of t in this polynomial is 3. The fraction $1/3$

also cannot be a leading exponent because there are no principal minors of $A+tE$ of order 3 containing t in first power (see also the next subcase).

(2) For $A = \mathrm{diag}(N_2, N_2)$, we have $\mathcal{F} = \{1/2, 2/3\}$ but $\mathcal{L} = \{1/2, 2/3, 3/4\}$ and hence $\mathcal{F}$ is a proper subset of $\mathcal{L}$. Indeed, $\mathcal{F}$ does not contain the elements $1/4$, $1/3$, and $3/4$ from $\mathcal{R}_4$ which are candidates for members of $\mathcal{L}$. We prove first that $1/4$ and $1/3$ can not be elements of $\mathcal{L}$ and then we show that $3/4 \in \mathcal{L}$ by a special choice of E.

Note that for each E, the lowest power of t in the polynomial $c_4(t) := \det(A+tE)$ is 2, that is,

$$c_4(t) = \gamma_{40}t^4 + \gamma_{41}t^3 + \gamma_{42}t^2. \tag{3.50}$$

If $1/4 \in \mathcal{L}$, then the characteristic polynomial of $A+tE$ must have the form $\lambda^4 + \cdots + c_4(t)$ with $c_4(t) = \gamma_{40}t^4 + \gamma_{41}t^3 + \gamma_{42}t^2 + \gamma_{43}t$ and $\gamma_{43} \neq 0$. The last inequality is impossible in view of (3.50).

Suppose now that $1/3 \in \mathcal{L}$. Then the characteristic polynomial $A+tE$ must have the form $(\lambda - \tau t)(\lambda^3 + b_1(t)\lambda^2 + b_2(t)\lambda + b_3(t))$, where $b_3(t) = \beta_{30}t^3 + \beta_{31}t^2 + \beta_{32}t$ and $\beta_{32} \neq 0$. Hence $b_3(t) - \tau t b_2(t) = -c_3(t)$ and at least one principal minor of order 3 of the matrix $A+tE$ must contain t in first degree. But all such minors contain t in second or higher degree.

To show that $3/4$ is an element of $\mathcal{L}$, take E as $E = E_4(1,3) + E_4(2,3) - E_4(3,1) - E_4(3,3) + E_4(4,2) + E_4(4,4)$. Then the characteristic polynomial of the matrix

$$A + tE = \begin{bmatrix} 0 & 1 & t & 0 \\ 0 & 0 & t & 0 \\ -t & 0 & -t & 1 \\ 0 & t & 0 & t \end{bmatrix} \tag{3.51}$$

is $\lambda^4 - t^3\lambda - t^3$ and has roots of the form $\lambda = t^{3/4} + O(t^{3/2})$, $t \to 0$.

(3) For $A = \mathrm{diag}(N_3, N_1)$, we have $\mathcal{F} = \{1/3, 1/2, 2/3\}$. That $1/4$ is not element of $\mathcal{L}$ is clear from the fact that the lowest power of t in the polynomial $\det(A+tE)$ is 2 (see subcase 2). Whether $3/4$ is a member of $\mathcal{L}$, and hence $\mathcal{F}$ is a proper subset of $\mathcal{L}$, is an open question.

(4) For $A = N_4$, we have $\mathcal{F} = \{1/4, 1/3, 1/2\}$. The set $\mathcal{L}$ contains the fraction $2/3$. Indeed, according to Example 3.2, we can take $E = E_4(2,1) - E_4(2,2) - E_4(3,2) + E_4(3,3)$, which gives the characteristic polynomial $\lambda(\lambda^3 - t^2\lambda + t^2)$ of $A+tE$ with one zero root and three roots of order $t^{2/3}$, $t \to 0$. Thus $\mathcal{F}$ is a proper subset of $\mathcal{L}$. Whether $\mathcal{L}$ contains $3/4$ is not known.

3.3. Use of Newton diagram

The set $\mathcal{L}$ can be studied via the characteristic polynomial of $A + tE$ and the technique of Newton diagrams [7]. The coefficients $c_j(t)$ of this polynomial are polynomials in t of degrees not exceeding j, $c_j(t) = c_{j0}t^j + c_{j1}t^{j-1} + \cdots + c_{j,j-1}t$ and $c_{\alpha\beta} \in \mathbb{F}$.

Define the numbers d_j, $j = 1,\ldots,n$, as follows. If $c_j(t)$ is the zero polynomial set $d_j = 0$ and if $c_j(t) \neq 0$, set $d_j = c_{j,\alpha_j}$, where $\alpha_j := \max\{s : c_{js} \neq 0\}$. Thus d_j is the coefficient corresponding to the lowest degree α_j of t in $c_j(t)$. Keeping only the lowest powers of t in each $c_j(t)$, we get the polynomial $\lambda^n - d_1 t^{\alpha_1}\lambda^{n-1} + \cdots + (-1)^n d_n t^{\alpha_n}$ which is, in general, different from the characteristic polynomial of $A + tE$ but has roots of the same low asymptotic order for $t \to 0$. We call two such polynomials *asymptotically equivalent*. Such asymptotically equivalent polynomials are used in the Appendix. For low-order matrices, they allow to calculate directly some of the leading exponents. This was demonstrated in the examples presented above.

4. Fractional exponents for low-order matrices

In this section, we present the fractional exponents from the set $\mathcal{F}$ according to Theorem 3.6 for matrices of order up to 7.

4.1. Matrices with a single eigenvalue

In this case, $m=1$, $k = k_1 = n$, and κ is a partition of n; namely, $n=\kappa_1+\cdots+\kappa_r$.

Table 4.1 gives the fractional exponents from $\mathcal{F}$ for different partitions of n. Note that the Wilkinson's example, see [7, 10] and Example 3.5, corresponds to $5 = 3 + 2$.

The smallest partition $n = 1 + \cdots + 1$ does not produce fractional exponents. The largest partition $n = n$ gives exponents $1/p$, $p = 2,\ldots,n$, which are exactly the elements of $\mathcal{B}_1(n,2)$. The partition $n = 2 + 1 + \cdots + 1$ gives the exponents $p/(p+1)$ which are exactly the elements of the union of one-element sets $\mathcal{B}_p(p+1,p+1)$ for $p = 1,\ldots,n-1$.

4.2. Matrices with two or more eigenvalues

Table 4.1 describes the union of the sets $\mathcal{F}_{i0}$. If the matrix A has more than one eigenvalue, then the sets $\mathcal{F}_{ij}$ with $j \geq 1$ also "contribute" to $\mathcal{F}$.

If the number m, of different eigenvalues of A, is close to n, the set $\mathcal{F}$ is small as shown below.

(i) For $m = n$, $n \geq 2$, we have $k = 1$, there are no fractional exponents and $\mathcal{F} = \emptyset$.

(ii) For $m = n - 1$ and $n \geq 3$, we have $k = 2$ and hence $\mathcal{F} = \{1/2\}$.

Table 4.1

Partition	Fractional exponents from $\mathcal{F}$
2 = 2	1/2
3 = 3	1/3, 1/2
3 = 2 + 1	1/2; 2/3
4 = 4	1/4, 1/3, 1/2
4 = 3 + 1	1/3, 1/2; 2/3
4 = 2 + 2	1/2; 2/3
4 = 2 + 1 + 1	1/2; 2/3; 3/4
5 = 5	1/5, 1/4, 1/3, 1/2
5 = 4 + 1	1/4, 1/3, 1/2; 2/5, 2/3
5 = 3 + 2	1/3, 1/2; 2/5, 2/3
5 = 3 + 1 + 1	1/3, 1/2; 2/3; 3/5, 3/4
5 = 2 + 2 + 1	1/2; 2/3; 3/5, 3/4
5 = 2 + 1 + 1 + 1	1/2; 2/3; 3/4; 4/5
6 = 6	1/6, 1/5, 1/4, 1/3, 1/2
6 = 5 + 1	1/5, 1/4, 1/3, 1/2; 2/5, 2/3
6 = 4 + 2	1/4, 1/3, 1/2; 2/5, 2/3;
6 = 4 + 1 + 1	1/4, 1/3, 1/2; 2/5, 2/3; 3/5, 3/4
6 = 3 + 3	1/3, 1/2; 2/5
6 = 3 + 2 + 1	1/3, 1/2; 2/5, 2/3; 3/5, 3/4
6 = 3 + 1 + 1 + 1	1/3, 1/2; 2/3; 3/5, 3/4; 4/5
6 = 2 + 1 + 1 + 1 + 1	1/2; 2/3; 3/4; 4/5; 5/6
7 = 7	1/7, 1/6, 1/5, 1/4, 1/3, 1/2
7 = 6 + 1	1/6, 1/5, 1/4, 1/3, 1/2; 2/7, 2/5, 2/3
7 = 5 + 2	1/5, 1/4, 1/3, 1/2; 2/7, 2/5, 2/3
7 = 5 + 1 + 1	1/5, 1/4, 1/3, 1/2; 2/5, 2/3; 3/7, 3/5, 3/4
7 = 4 + 3	1/4, 1/3, 1/2; 2/7, 2/5
7 = 3 + 3 + 1	1/3, 1/2; 2/5, 2/3; 3/7, 3/5
7 = 4 + 2 + 1	1/4, 1/3, 1/2; 2/5, 2/3; 3/7, 3/5, 3/4
7 = 4 + 1 + 1 + 1	1/4, 1/3, 1/2; 2/5, 2/3; 3/5, 3/4; 4/7, 4/5
7 = 3 + 2 + 2	1/3, 1/2; 2/5, 2/3; 3/7, 3/5
7 = 3 + 2 + 1 + 1	1/3, 1/2; 2/5, 2/3; 3/5, 3/4; 4/7, 4/5
7 = 2 + 2 + 1 + 1 + 1	1/2; 2/3; 3/5, 3/4; 4/5; 5/7, 5/6
7 = 2 + 1 + 1 + 1 + 1 + 1	1/2; 2/3; 3/4; 4/5; 5/6; 6/7

(iii) For $m = n-2$ and $n \geq 4$, we have $k = 3$ and there are two possible subcases: $\kappa = (3)$ and $\kappa = (2,1)$. For both of them, $\mathcal{F} = \{1/3, 1/2; 2/3\}$.

(iv) For $m = n-3$, $n \geq 5$, there are 7 subcases as follows.

(1) If $k = 4$, $\kappa = (4)$, and $n = 5$, then $\mathcal{F} = \{1/4, 1/3, 1/2; 2/3\}$.

(2) If $k = 4$ and $n \geq 6$, then $\mathcal{F} = \{1/4, 1/3, 1/2; 2/3; 3/4\}$.

(3) If $k = 4$ and $\kappa = (3,1)$, then $\mathcal{F} = \{1/3, 1/2; 2/3; 3/4\}$.

(4) If $k = 4$ and $\kappa = (2,2)$ or $\kappa = (2,2,1)$, then $\mathcal{F} = \{1/2; 2/3; 3/4\}$.

(5) If $k = 3$ and $\kappa = (3)$, then $\mathcal{F} = \{1/3, 1/2; 2/3\}$.

(6) If $k = 3$ and $\kappa = (2,1)$, then $\mathcal{F} = \{1/2; 2/3\}$.

(7) If $k = 2$, then $\mathcal{F} = \{1/2\}$.

Now consider the possible fractional exponents for $n \leq 7$ and $m \leq n-4$ since the cases with $m \geq n-3$ have been already analysed. Here we have additional exponents to those given in Table 4.1. We will consider the cases $n = 6$ with $m = 2$, $n = 7$ with $m = 2$, and $n = 7$ with $m = 3$. For each of these case, there are subcases depending on the multiplicity k of λ_1 and its partitions κ. We do not consider the partition $k = 2 + 1 + \cdots + 1$ since it gives no additional exponents other than $1/2, 2/3, \ldots, (k-1)/k$.

(1) *Case* $n = 6$, $m = 2$, *and* $k = 5$. Here we have 5 subcases according to the partitions of 5 in Table 4.1 except the last one. In addition to the exponents from the second column of Table 4.1, we have new exponents as in Table 4.2.

Table 4.2

Partition	Additional fractional exponents
$5 = 5$	$2/5, 2/3$
$5 = 4 + 1$	$3/5, 3/4; 4/5$
$5 = 3 + 2$	$3/5, 3/4$
$5 = 3 + 1 + 1$	$4/5$
$5 = 2 + 2 + 1$	$4/5$

(2) *Case* $n = 6$, $m = 2$, *and* $k = 4$. There are 3 subcases according to the partitions of 4 in Table 4.1. In addition to the exponents from the second column of Table 4.1, we have new exponents as in Table 4.3.

Table 4.3

Partition	Additional fractional exponents
$4 = 4$	$2/3$
$4 = 3 + 1$	$3/4$
$4 = 2 + 2$	$3/4$

(3) *Case* $n = 6$, $m = 2$, *and* $k = 3$ *or* $k = 2$. No additional exponents are added to the exponents already mentioned in the second column of Table 4.1 for the partitions of 3 and 2.

(4) *Case* $n = 7$, $m = 2$, *and* $k = 6$. We have 7 subcases according to the partitions of 6 in Table 4.1. In addition to the exponents from the second column of Table 4.1, the new exponents are shown in Table 4.4.

TABLE 4.4

Partition	Additional fractional exponents
$6=6$	$2/5, 2/3$
$6=5+1$	$3/5, 3/4$
$6=4+2$	$3/5, 3/4$
$6=4+1+1$	$4/5$
$6=3+3$	$2/3; 3/5, 3/4$
$6=3+2+1$	$4/5$
$6=3+1+1+1$	$5/6$

(5) *Case* $n = 7$, $m = 2$, *and* $k = 5$. There are 5 subcases according to the partitions of 5 in Table 4.1. In addition to the exponents from the second column of Table 4.1, there are new exponents as in Table 4.5.

TABLE 4.5

Partition	Additional Fractional Exponents
$5=5$	$2/5, 2/3$
$5=4+1$	$3/5, 3/4$
$5=3+2$	$3/5, 3/4$
$5=3+1+1$	$4/5$
$5=2+2+1$	$4/5$

(6) *Case* $n = 7$, $m = 2$, *and* $k = 4$. There are 3 subcases according to the partitions of 4 in Table 4.1. In addition to the exponents from the second column of Table 4.1, we have new exponents as in Table 4.6.

TABLE 4.6

Partition	Additional fractional exponents
$4=4$	$2/3$
$4=3+1$	$3/4$
$4=2+2$	$3/4$

Table 4.7

Partition	Additional fractional exponents
$5=5$	$2/5, 2/3; 3/5, 3/4$
$5=4+1$	$3/5, 3/4; 4/5$
$5=3+2$	$3/5, 3/4; 4/5$
$5=3+1+1$	$4/5$
$5=2+2+1$	$4/5$

(7) *Case* $n=7$, $m=2$, *and* $k=3$ *or* $k=2$. No additional exponents are added to the exponents already listed in Table 4.1 for the partitions of 3 and 2.

(8) *Case* $n=7$, $m=3$, *and* $k=5$. Here we have 5 subcases according to the partitions of 5 in Table 4.1. In addition to the exponents from the second column of Table 4.1, the new exponents are shown in Table 4.7.

(9) *Case* $n=7$, $m=3$, *and* $k=4$. Here we have 3 subcases according to the partitions of 4 in Table 4.1. In addition to the exponents from the second column of Table 4.1, there are new exponents as in Table 4.8.

Table 4.8

Partition	Additional fractional exponents
$4=4$	$2/3; 3/4$
$4=3+1$	$3/4$
$4=2+2$	$3/4$

(10) *Case* $n=7$, $m=3$, *and* $k=3$ *or* $k=2$. No additional exponents are added.

Consider finally a particular example of higher order. Suppose that $n=19$ and the matrix A has three eigenvalues $\lambda_1,\lambda_2,\lambda_3$ ($m=3$) with partial multiplicities 7,4,2 for the first, 3,1 for the second, and 1,1 for the third. Then $k=13=7+4+2$, $r=3$, the defect of A is 12 and $n-r-\operatorname{def}(A)=4$ (the number of Jordan blocks with eigenvalues λ_2 or λ_3). The sets $\mathcal{F}_{i0}$ are

$$\begin{aligned}
\mathcal{F}_{10} &= \left\{\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2}\right\},\\
\mathcal{F}_{20} &= \left\{\frac{2}{11},\frac{2}{10}=\frac{1}{5},\frac{2}{9},\frac{2}{8}=\frac{1}{4},\frac{2}{7},\frac{2}{6}=\frac{1}{3},\frac{2}{5},\frac{2}{4}=\frac{1}{2},\frac{2}{3}\right\},\\
\mathcal{F}_{30} &= \left\{\frac{3}{11},\frac{3}{10},\frac{3}{9}=\frac{1}{3},\frac{3}{8},\frac{3}{7}\right\}
\end{aligned} \tag{4.1}$$

and their union is the 15-element set

$$\left\{\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2};\frac{2}{11},\frac{2}{9},\frac{2}{7},\frac{2}{5},\frac{2}{3},\frac{3}{11},\frac{3}{10},\frac{3}{8},\frac{3}{7}\right\}. \tag{4.2}$$

The consideration of $\mathcal{F}_{ij}$ for $i = 1,2,3$ and $j = 1,2,3,4$ gives 22 new exponents 3/5, 3/4; 4/13, 4/11, 4/9, 4/7, 4/5; 5/13, 5/12, 5/9, 5/8, 5/7, 5/6, 6/13, 6/11, 6/7; 7/13, 7/12, 7/11, 7/10, 7/9, 7/8. Thus $\mathcal{F}$ has 37 elements in this case.

5. Conclusions

In this paper, we have presented a simple description of a subset $\mathcal{F}$ of the set $\mathcal{L}$ of leading fractional exponents in the asymptotic expansion of the eigenvalue perturbations of a defective matrix. The set $\mathcal{L}$ corresponding to a given eigenvalue depends on its partial algebraic multiplicities and on the defect of the whole matrix. For each case, we have explicitly constructed the perturbation which produces the corresponding fractional exponent.

Computationally, the problem of determining the Jordan structure of a general matrix is very difficult, see [2, 3, 8]. The standard codes for eigenvalue analysis based on the QR algorithm in general do not produce reliable results in this case. Only the codes based on regularisation, techniques for root localization and clustering and, other sophisticated tools [2, 3, 8] may give satisfactory results for defective matrices. A preliminary knowledge of possible and leading fractional exponents, based on theoretical or experimental considerations (e.g., computation of the pseudospectra of the matrix), may be very helpful in this area. As a whole, the numerical solution of the eigenvalue problem for general dense (and even very low order) matrices remains a challenging problem in numerical linear algebra scientific computing.

Another problem is the determination of the whole set $\mathcal{L}$. Even the completion of the list of leading exponents for low-order matrices (of order say up to 7) is still an open problem.

Appendix

Here we present conditions for the characteristic polynomial of the matrix $A + tE$ to have a root λ_0 with an asymptotic expansion

$$\lambda_0 = at^{p/q} + O(t^{2p/q}), \quad t \longrightarrow 0, \tag{A.1}$$

where $p < q$ and p, q are coprime.

Keeping only the leading terms $d_j t^{\alpha_j}$ in the coefficients $c_j(t)$ of the characteristic polynomial of $A+tE$, we obtain asymptotically equivalent polynomial. Now λ_0 is a root of this latter polynomial satisfying (A.1) only if it has a multiplier of the form

$$\lambda^{rq} + \sum_{j=1}^{rq-1} \gamma_j t^{\alpha_j} \lambda^j + \gamma_0 t^{rp}, \quad \gamma_0 \neq 0, \tag{A.2}$$

where $r \in \mathbb{N}$ and $rq \leq n$. Substituting (A.1) in (A.2), we obtain

$$a^{rq} + \gamma_0 + \sum_{j=1}^{rq-1} \gamma_j t^{\alpha_j + jp/q - rp} + O(t^{p/q}), \quad t \longrightarrow 0. \tag{A.3}$$

Hence for each $j = 1, \ldots, rq-1$, we have either $\gamma_j = 0$ or $\gamma_j \neq 0$ and $jp \geq q(rp - \alpha_j)$.

Suppose that $\gamma_{j_1}, \ldots, \gamma_{j_s}$ are the nonzero coefficients in the sum from (A.2) for which $jp = q(rp - \alpha_j)$. Then the coefficient a in (A.1) may be determined from

$$a = \left(-\gamma_0 - \gamma_{j_1} - \cdots - \gamma_{j_s}\right)^{1/(rq)}. \tag{A.4}$$

These considerations are illustrated by the next example.

Example A.1. Let $A = N_3$ and suppose we are interested in the leading exponent $2/3$. The characteristic polynomial of $N_3 + tE$ may be written as

$$\lambda^3 - c_{10} t \lambda^2 + \left(c_{20} t^2 + c_{21} t\right) \lambda - \left(c_{30} t^3 + c_{31} t^2 + c_{32} t\right). \tag{A.5}$$

We obtain that $c_{10} = c_{21} = c_{32} = 0$, $c_{31} \neq 0$.

If we are interested only in the first two coefficients in the asymptotic expansion $\lambda = a_1 t^{2/3} + a_2 t^{4/3} + O(t^2)$, $t \to 0$, then we may omit the term containing t^3. Hence we have the asymptotically equivalent polynomial $\lambda^3 + c_{20} t^2 \lambda - c_{31} t^2$ which gives $a_1 = c_{31}^{1/3}$, $a_2 = -c_{20}/(3c_{31}^{1/3})$.

Acknowledgment

We thank the referee for his helpful comments.

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