MIXED VARIATIONAL INEQUALITIES AND ECONOMIC EQUILIBRIUM PROBLEMS

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We consider rather broad classes of general economic equilibrium problems and oligopolistic equilibrium problems which can be formulated as mixed variational inequality problems. Such problems involve a continuous mapping and a convex, but not necessarily differentiable function. We present existence and uniqueness results of solutions under weakened P-type assumptions on the cost mapping. They enable us to establish new results for the economic equilibrium problems under consideration.

1. Introduction

Variational inequalities (VIs) are known to be a very useful tool to formulate and investigate various economic equilibrium problems. In particular, they allow one to obtain existence and uniqueness results and construct iterative solution methods for finding equilibrium points; for example, see [10, 18, 19] and the references therein. The most general results were established for the case where the cost mapping of the corresponding VI is multivalued. At the same time, the single-valued formulation enables one to simplify essential statements and derivation of these results in comparison with those in the multivalued case. This is also the case for constructing iterative solution methods. However, such a formulation covers rather a narrow class of equilibrium problems in economics.

The usual VI formulation admits various modifications and extensions which also can be in principle applied to economic equilibrium problems. Consider the *mixed variational inequality problem* (MVI) which

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is to find a point $x^* \in K$ such that

$$\langle G(x^*), x - x^* \rangle + f(x) - f(x^*) \geq 0 \quad \forall x \in K, \tag{1.1}$$

where K is a nonempty convex set in the real Euclidean space $\mathbb{R}^n$, $G : V \to \mathbb{R}^n$ is a mapping, $f : V \to \mathbb{R}$ is a convex, but not necessarily differentiable function, and V is a nonempty subset of $\mathbb{R}^n$ such that $K \subseteq V$. Problem (1.1) was originally considered by Lescarret [14] and Browder [3] in connection with its numerous applications in mathematical physics and afterwards studied by many authors; for example, see [2, 6]. It clearly reduces to the usual (single-valued) VI if $f \equiv 0$ and to the usual convex nondifferentiable optimization problem if $G \equiv 0$, respectively. Thus it can be considered as an intermediate problem between single-valued and multivalued VIs. Note that most of works on MVIs are traditionally devoted to the case where G possesses certain strict (strong) monotonicity properties, which enable one to present various existence and uniqueness results for problem (1.1) and suggest various solution methods, including descent methods with respect to a so-called merit function; for example, see [22]. However, these properties seem too restrictive for economic applications, where order monotonicity type conditions are used. For this reason, we will consider problem (1.1) under other assumptions. Namely, we will suppose that the cost mapping G possesses P-type properties, f is separable, and K is defined by box-type constraints. In this paper, we first present two rather broad classes of perfectly and nonperfectly competitive economic equilibrium models which are involved in this class of MVIs. It should be noted that such MVIs have also a great number of other applications in mathematical physics, engineering, and operations research; for example, see [13, 20, 21]. It suffices to recall mesh schemes for obstacle and dam problems, Nash equilibrium problems in game theory, and equilibrium problems for network flows. Nevertheless, theory and solution methods of such MVIs are developed mainly for several particular cases of MVI (1.1), which for instance involve the case where either $f \equiv 0$ or G is an affine M-mapping and $K = \mathbb{R}^n$; for example, see [10, 13, 21]. However, this technique cannot be extended directly to the general nonlinear and nondifferentiable case. Next, in [12], several existence and uniqueness results were presented for the general MVI (1.1), but they were proved under additional conditions on G which could be too restrictive for economic equilibrium problems under consideration. In this paper, we give new existence and uniqueness results for the general MVI (1.1) under weaker assumptions on G which are suitable for its economic applications. In fact, we show that these assumptions hold in the general economic equilibrium model if the demand mapping satisfies rather natural conditions such as gross substitutability and homogeneity of degree zero. We also show that these assumptions hold in the oligopolistic equilibrium problem. We thus obtain various existence and uniqueness results for both classes of economic equilibrium problems. Moreover, these results allow us to apply the D-gap function approach, which was suggested and developed for MVIs in [11, 12], to find equilibrium points. We recall that the D-gap function approach consists in replacing the initial MVI, which contains a nondifferentiable function f and the feasible set K, with the problem of finding a stationary point of a differentiable merit function. In other words, we thus can find equilibrium points with the help of the usual differentiable optimization methods, such as the steepest descent and conjugate gradient methods. This approach to find equilibria seems more effective and suitable than the usual simplicial based one; for example, see [26, 27, 28].

In what follows, for a vector $x \in \mathbb{R}^n$, $x \geq 0$ (resp., $x > 0$) means $x_i \geq 0$ (resp., $x_i > 0$) for all $i = 1, \ldots, n$; $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}; \tag{1.2}$$

$\mathbb{R}^n_>$ denotes the interior of $\mathbb{R}^n_+$, that is,

$$\mathbb{R}^n_> = \{x \in \mathbb{R}^n \mid x > 0\}. \tag{1.3}$$

We denote by I_n the identity map in $\mathbb{R}^n$, that is, the $n \times n$ unit matrix. For a set E, $\Pi(E)$ denotes the family of all subsets of E. Also, $\partial f(x)$ denotes the subdifferential of a function f at x, that is,

$$\partial f(x) = \{g \in \mathbb{R}^n \mid f(y) - f(x) \geq \langle g, y - x\rangle \ \forall y \in \mathbb{R}^n\}. \tag{1.4}$$

We also recall definitions of convexity properties for functions and monotonicity properties for mappings.

Definition 1.1 (see [23]). Let U be a convex subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is said to be

(a) *strongly convex with constant* $\tau > 0$, if for all $u', u'' \in U$ and $\lambda \in [0,1]$, we have

$$f(\lambda u' + (1-\lambda)u'') \leq \lambda f(u') + (1-\lambda) f(u'') - 0.5\tau\lambda(1-\lambda)\|u' - u''\|^2; \tag{1.5}$$

(b) *strictly convex*, if for all $u', u'' \in U$, $u' \neq u''$ and $\lambda \in (0,1)$, we have

$$f(\lambda u' + (1-\lambda)u'') < \lambda f(u') + (1-\lambda) f(u''); \tag{1.6}$$

(c) *convex*, if for all $u', u'' \in U$ and $\lambda \in [0,1]$, we have

$$f\left(\lambda u' + (1-\lambda)u''\right) \le \lambda f(u') + (1-\lambda)f(u''). \tag{1.7}$$

Also, the function $f : U \to \mathbb{R}$ is said to be *concave* (resp., *strictly concave, strongly concave with constant* $\tau > 0$) if the function $-f$ is convex (resp., strictly convex, strongly convex with constant $\tau > 0$).

Definition 1.2 (see [2, 10, 22]). Let U be a convex subset of $\mathbb{R}^n$. A mapping $Q : U \to \Pi(\mathbb{R}^n)$ is said to be

(a) *strongly monotone with constant* $\tau > 0$, if for all $u', u'' \in U$ and $q' \in Q(u')$, $q'' \in Q(u'')$, we have

$$\langle q' - q'', u' - u'' \rangle \ge \tau \|u' - u''\|^2; \tag{1.8}$$

(b) *strictly monotone*, if for all $u', u'' \in U$, $u' \ne u''$ and $q' \in Q(u')$, $q'' \in Q(u'')$, we have

$$\langle q' - q'', u' - u'' \rangle > 0; \tag{1.9}$$

(c) *monotone*, if for all $u', u'' \in U$ and $q' \in Q(u')$, $q'' \in Q(u'')$, we have

$$\langle q' - q'', u' - u'' \rangle \ge 0. \tag{1.10}$$

It is well known that the subdifferential $\partial f(x)$ of any convex function $f : \mathbb{R}^n \to \mathbb{R}$ is nonempty at each point $x \in \mathbb{R}^n$. We now recall the known relationships between convexity properties of functions and monotonicity properties of their subdifferentials.

Lemma 1.3 (see [23]). *A function* $f : U \to \mathbb{R}$ *is*

(a) *convex if and only if* ∂f *is monotone;*
(b) *strictly convex if and only if* ∂f *is strictly monotone;*
(c) *strongly convex with constant* $\tau > 0$ *if and only if* ∂f *is strongly monotone with constant* $\tau > 0$.

2. Economic equilibrium models

In this section, we briefly outline two economic equilibrium models which can be formulated as MVI of form (1.1). Note that both models involve the possibility for producers to change the technology of production.

Model 2.1 (Walrasian equilibrium). We consider a market structure with perfect competition. The model deals in n commodities. Then, given a price vector $p \in \mathbb{R}^n_+$, we can define the value $E(p)$ of the excess demand mapping $E : \mathbb{R}^n_+ \to \Pi(\mathbb{R}^n)$, which is multivalued in general. Traditionally (see, e.g., [10, 18, 19]), a vector $p^* \in \mathbb{R}^n$ is said to be an equilibrium price vector if it solves the following complementarity problem:

$$p^* \geq 0, \quad \exists q^* \in E(p^*) : q^* \leq 0, \quad \langle p^*, q^* \rangle = 0, \tag{2.1}$$

or equivalently, the following VI: find $p^* \geq 0$ such that

$$\exists q^* \in E(p^*), \quad \langle -q^*, p - p^* \rangle \geq 0 \quad \forall p \geq 0. \tag{2.2}$$

We now specialize our model from this very general one. First, we suppose that each price of a commodity which is involved in the market structure has a lower positive bound and may have an upper bound. It follows that the feasible prices are assumed to be contained in the box-constrained set

$$K = \prod_{i=1}^{n} K_i, \quad K_i = \{t \in \mathbb{R} \mid 0 < \tau_i' \leq t \leq \tau_i'' \leq +\infty\}, \; i = 1, \dots, n. \tag{2.3}$$

Next, as usual, the excess demand mapping is represented as follows:

$$E(p) = D(p) - S(p), \tag{2.4}$$

where D and S are the demand and supply mappings, respectively. We suppose that the demand mapping is single-valued and set $G = -D$. Then, the problem of finding an *equilibrium price* can be formulated as follows: find $p^* \in K$ such that

$$\exists s^* \in S(p^*), \quad \langle G(p^*), p - p^* \rangle + \langle s^*, p - p^* \rangle \geq 0 \quad \forall p \in K. \tag{2.5}$$

In addition, we impose the condition that each producer supplies a single commodity. This condition does not seem too restrictive. Clearly, it follows that there is no loss of generality to suppose that each jth producer supplies the single jth commodity for each $j = 1, \dots, n$. Then, given a price vector $p \in \mathbb{R}^n_+$, the supply mapping is of the form $S(p) = \prod_{i=1}^n S_i(p_i)$. Next, it is rather natural to suppose that each S_i is monotone, but not necessarily single-valued, that is, $S_i : \mathbb{R}_+ \to \Pi(\mathbb{R})$ for $i = 1, \dots, n$. In fact, these assumptions are rather standard even for general supply mappings; for example, see [18, 20] and the references therein. Here they mean that the individual supply is nondecreasing with respect to the price and that there exist prices which imply more than one optimal value of production. For instance, these prices can be treated as switching points between different technologies of production. Under the above assumptions, each supply mapping is nothing but the subdifferential one, that is, $S_j = \partial f_j$, where $f_j : \mathbb{R}_+ \to \mathbb{R}$ is a general convex function for each $j = 1,\dots,n$; for example, see [25]. Thus, our VI (2.5), (2.3) can be then rewritten as follows: find $p^* \in K$ such that

$$\exists s_i^* \in S_i(p_i^*),\ i = 1,\dots,n; \quad \langle G(p^*), p - p^* \rangle + \sum_{i=1}^{n} s_i^* (p_i - p_i^*) \geq 0 \quad \forall p \in K; \tag{2.6}$$

or equivalently (see Proposition 3.1),

$$\langle G(p^*), p - p^* \rangle + \sum_{i=1}^{n} [f_i(p_i) - f_i(p_i^*)] \geq 0 \quad \forall p \in K. \tag{2.7}$$

However, this problem is nothing but MVI (1.1). Moreover, we can use the same problem (2.7) in order to model the more general case where the market structure involves additional consumers with nonincreasing single commodity demand mappings. Then S_i serves as a partial excess supply mapping for the ith commodity.

Model 2.2 (oligopolistic equilibrium). Now consider an oligopolistic market structure in which n firms supply a homogeneous product. Let $p(\sigma)$ denote the inverse demand function, that is, it is the price at which consumers will purchase a quantity σ. If each ith firm supplies q_i units of the product, then the total supply in the market is defined by

$$\sigma_q = \sum_{i=1}^{n} q_i. \tag{2.8}$$

If we denote by $f_i(q_i)$ the ith firm's total cost of supplying q_i units of the product, then the ith firm's profit is defined by

$$\varphi_i(q) = q_i p(\sigma_q) - f_i(q_i). \tag{2.9}$$

As usual, each output level is nonnegative, that is, $q_i \geq 0$ for $i = 1,\dots,n$. In addition, we suppose that it can be in principle bounded from above, that is, there exist numbers $\beta_i \in (0,+\infty]$ such that $q_i \leq \beta_i$ for $i = 1,\dots,n$. In order to define a solution in this market structure we use the Nash equilibrium concept for noncooperative games.

Definition 2.3 (see [17]). A feasible vector of output levels $q^* = (q_1^*, q_2^*,\dots,q_n^*)$ for firms $1,\dots,n$ is said to constitute a *Nash equilibrium* solution for the oligopolistic market, provided q_i^* maximizes the profit function φ_i of the ith firm given that the other firms produce quantities q_j^*, $j \neq i$, for each $j = 1,\dots,n$.

That is, for $q^* = (q_1^*, q_2^*, \dots, q_n^*)$ to be a Nash equilibrium, q_i^* must be an optimal solution to the problem

$$\max_{0 \le q_i \le \beta_i} \longrightarrow \{q_i p(q_i + \sigma_i^*) - f_i(q_i)\}, \tag{2.10}$$

where $\sigma_i^* = \sum_{j=1,\, j\neq i}^n q_j^*$ for each $i = 1,\dots,n$. This problem can be transformed into an equivalent MVI of the form (1.1) if each ith profit function φ_i in (2.9) is concave in q_i (see, e.g., [9, Chapter 5] and [17]). This assumption conforms to the usually accepted economic behaviour and implies that (2.10) is a concave maximization problem. In addition, we assume that the price function $p(\sigma)$ is continuously differentiable. At the same time, the concavity of φ_i in q_i implies usually the convexity of the cost function f_i but it need not be differentiable in general. For instance, the cost function can be piecewise-smooth, and each smooth part then corresponds to a single technological process, so that there exist quantities which can be treated as switching points between different technologies of production. Under the assumptions above, we can define the multivalued mapping $F : \mathbb{R}_+^n \to \Pi(\mathbb{R}^n)$ by

$$F(q) = \left(\partial_{q_1}[-\varphi_1(q)], \dots, \partial_{q_n}[-\varphi_n(q)]\right), \tag{2.11}$$

where

$$F_i(q) = \partial_{q_i}[-\varphi_i(q)] = G_i(q) + \partial f_i(q_i), \tag{2.12}$$

and $G_i(q) = -p(\sigma_q) - q_i p'(\sigma_q)$ for $i = 1,\dots,n$. Next, we set

$$K = \prod_{i=1}^n K_i, \quad K_i = \{t \in \mathbb{R} \mid 0 \le t \le \beta_i\},\ i = 1,\dots,n. \tag{2.13}$$

Then (see, e.g., [9, Chapter 5] and [17]), the problem of finding a Nash equilibrium in the oligopolistic market can be rewritten as the following VI: find $q^* \in K$ such that

$$\exists d_i^* \in \partial f_i(q_i^*),\ i = 1,\dots,n; \quad \langle G(q^*), q - q^*\rangle + \sum_{i=1}^n d_i^*(q_i - q_i^*) \ge 0 \quad \forall q \in K; \tag{2.14}$$

or equivalently (see Proposition 3.1),

$$\langle G(q^*), q-q^*\rangle + \sum_{i=1}^{n}\left[f_i(q_i)-f_i(q_i^*)\right] \geq 0 \quad \forall q \in K. \tag{2.15}$$

Again, this problem is nothing but MVI of the form (1.1).

We intend to obtain existence and uniqueness results of solutions of both models under certain additional assumptions which are rather natural for these models. Since the equilibrium problems in both cases are rewritten as MVI of form (1.1), we first establish new existence and uniqueness results for this general problem.

3. Technical preliminaries

In this section, we recall some definitions and give some properties which will be used in our further considerations. We consider MVI (1.1) under the following standing assumptions:

(A1) $G: V \to \mathbb{R}^n$ is a continuous mapping and V is a convex subset of $\mathbb{R}^n_+$;

(A2) f is of the form $f(x) = \sum_{i=1}^n f_i(x_i)$, where $f_i: \mathbb{R}_+ \to \mathbb{R}$ is a convex continuous function for every $i = 1,\dots,n$;

(A3) K is a box constrained set, that is,

$$K = \prod_{i=1}^{n} K_i, \tag{3.1}$$

where $K_i = \{t \in \mathbb{R} \mid \alpha_i \leq t \leq \beta_i\}$, $[\alpha_i, \beta_i] \subseteq [0, +\infty]$ for every $i = 1,\dots,n$.

These assumptions have been discussed in Section 1 and problems (2.7), (2.3) and (2.15), (2.13) clearly satisfy them. Also, note that K in (A3) is obviously convex and closed. In the case where $\alpha_i = 0$ and $\beta_i = +\infty$ for all $i = 1,\dots,n$, we obtain $K = \mathbb{R}^n_+$, hence MVI (1.1) involves complementarity problems with the multivalued cost mapping $G + \partial f$. First we give an equivalence result for MVI (1.1).

Proposition 3.1 (see [12, Proposition 1]). *The following assertions are equivalent:*

(i) x^* *is a solution to MVI (1.1);*

(ii) $x^* \in K$ *and*

$$G_i(x^*)(x_i - x_i^*) + f_i(x_i) - f_i(x_i^*) \geq 0 \quad \forall x_i \in K_i,\ i = 1,\dots,n; \tag{3.2}$$

(iii) $x^* \in K$ *and*

$$\exists g_i^* \in \partial f_i(x_i^*) : G_i(x^*)(x_i - x_i^*) + g_i^*(x_i - x_i^*) \geq 0 \quad \forall x_i \in K_i,\ i = 1,\dots,n; \tag{3.3}$$

Now we recall definitions of several properties of matrices.

Definition 3.2 (see [8, 21]). An $n \times n$ matrix A is said to be

(a) a *P-matrix* if it has positive principal minors;
(b) a *P_0-matrix* if it has nonnegative principal minors;
(c) a *Z-matrix* if it has nonpositive off-diagonal entries;
(d) an *M-matrix* if it has nonpositive off-diagonal entries and its inverse A^{-1} exists and has nonnegative entries.

It is well known that an $n \times n$ matrix A is P if and only if, for every vector $x \neq 0$, there exists an index k such that $x_k y_k > 0$ where $y = Ax$. Similarly, A is P_0 if and only if, for every vector x, there exists an index k such that $x_k y_k \geq 0$, $x_k \neq 0$ where $y = Ax$. Also, it is well known that A is M if and only if $A \in P \cap Z$; see [8, 21]. Hence, each M-matrix is P, but the reverse assertion is not true in general.

Definition 3.3 (see [8, 21]). An $n \times n$ matrix A is said to be an *M_0-matrix* if it is both P_0- and Z-matrix.

The following assertion gives a criterion for a matrix A to be an M- or M_0-matrix.

Proposition 3.4 (see [8]). *Suppose A is a Z-matrix. If there exists a vector $x > 0$ such that $Ax > 0$ (resp., $Ax \geq 0$), then A is an M-matrix (resp., M_0-matrix).*

Now we recall some extensions of these properties for mappings.

Definition 3.5. Let U be a convex subset of $\mathbb{R}^n$. A mapping $F : U \to \mathbb{R}^n$ is said to be

(a) a *P-mapping* [16], if $\max_{1 \leq i \leq n}(x_i - y_i)(F_i(x) - F_i(y)) > 0$ for all $x, y \in U$, $x \neq y$;
(b) a *strict P-mapping* [12], if there exists $\gamma > 0$ such that $F - \gamma I_n$ is a P-mapping;
(c) a *uniform P-mapping* (see, e.g., [16]), if there exists $\tau > 0$ such that

$$\max_{1 \leq i \leq n}(x_i - y_i)(F_i(x) - F_i(y)) \geq \tau \|x - y\|^2 \tag{3.4}$$

for all $x, y \in U$;

(d) a P_0*-mapping* [16], if for all $x, y \in U$, $x \neq y$, there exists an index i such that $x_i \neq y_i$ and $(x_i - y_i)(F_i(x) - F_i(y)) \geq 0$.

In fact, if F is affine, that is, $F(x) = Ax + b$, then F is a P-mapping (P_0-mapping) if and only if its Jacobian $\nabla F(x) = A$ is a P-matrix (P_0-matrix). In the general nonlinear case, if the Jacobian $\nabla F(x)$ is a P-matrix, then F is a P-mapping, but the reverse assertion is not true in general. At the same time, F is a P_0-mapping if and only if its Jacobian $\nabla F(x)$ is a P_0-matrix. Next, if F is a strict P-mapping, then its Jacobian is a P-matrix; for example, see [7, 12, 16]. Moreover, if a single-valued mapping $F : U \to \mathbb{R}^n$ is monotone (resp., strictly monotone, strongly monotone), then, by definition, it is a P_0-mapping (resp., P-mapping, uniform P-mapping), but the reverse assertions are not true in general. Thus, P-type properties are usually weaker than the corresponding monotonicity properties.

We give an additional relationship between P_0- and strict P-mappings.

Lemma 3.6. *If $F : U \to \mathbb{R}^n$ is a P_0-mapping, then, for any $\varepsilon > 0$, $F + \varepsilon I_n$ is a strict P-mapping.*

Proof. First we show that $F^{(\varepsilon)} = F + \varepsilon I_n$ is a P-mapping for each $\varepsilon > 0$. Choose $x', x'' \in U$, $x' \neq x''$, set $I = \{i \mid x'_i \neq x''_i\}$ and fix $\varepsilon > 0$. Since F is a P_0-mapping, there exists an index $k \in I$ such that

$$\left[F_k(x') - F_k(x'')\right]\left(x'_k - x''_k\right) = \max_{1 \leq i \leq n}\left[F_i(x') - F_i(x'')\right]\left(x'_i - x''_i\right). \tag{3.5}$$

Then, by definition,

$$\begin{gathered} \left[F_k(x') - F_k(x'')\right]\left(x'_k - x''_k\right) \geq 0, \quad x'_k \neq x''_k, \\ \varepsilon\left(x'_k - x''_k\right)\left(x'_k - x''_k\right) > 0. \end{gathered} \tag{3.6}$$

Adding these inequalities yields

$$\left[F_k^{(\varepsilon)}(x') - F_k^{(\varepsilon)}(x'')\right]\left(x'_k - x''_k\right) > 0. \tag{3.7}$$

Hence, $F^{(\varepsilon)}$ is a P-mapping. Since $F^{(\varepsilon'')} = F^{(\varepsilon')} - (\varepsilon' - \varepsilon'')I_n = F + \varepsilon'' I_n$ is a P-mapping, if $0 < \varepsilon'' < \varepsilon'$, we conclude that $F^{(\varepsilon)}$ is a strict P-mapping. □

Note that each uniform P-mapping is a strict P-mapping, but the reverse assertion is not true in general. Thus, although most existence and uniqueness results for VIs were established for uniform P-mappings (see, e.g., [10, 16, 21]), this concept is not convenient for various Tikhonov regularization procedures which involve mappings of the form $F + \varepsilon I_n$; for example, see [5, 7, 24]. At the same time, such mappings are strict P, if F is P_0 because of Lemma 3.6 and this fact can serve as a motivation for developing the theory of VIs (MVIs) with strict P-mappings. Also, this concept is very useful in investigation of MVIs arising from economic applications.

4. General existence and uniqueness results

In this section, we consider the general MVI (1.1) under assumptions (A1), (A2), and (A3).

PROPOSITION 4.1. (i) *If G is a P-mapping, then MVI (1.1) has at most one solution.*

(ii) *If G is a strict P-mapping, then MVI (1.1) has a unique solution.*

The proofs of these assertions follow directly from Propositions 2 and 3 in [12], respectively.

However, the assumptions on G in Proposition 4.1 seem too restrictive for economic equilibrium problems. For instance, the mapping G in (2.7) and (2.15) need not be (strict) P in general. Now we present new existence and uniqueness results under weaker assumptions on G. The basic idea consists in replacing the (strict) P property of G with (strong) strict convexity of f. For the convenience of the reader, we give their proofs in the appendix.

We begin our considerations from the simplest case where K is bounded and G only satisfies (A1).

PROPOSITION 4.2. *Suppose that K is a bounded set. Then MVI (1.1) has a solution.*

Combining this result with Proposition 4.1(i) yields the following result.

COROLLARY 4.3. *Let G be a P-mapping and let K be a bounded set. Then MVI (1.1) has a unique solution.*

The following uniqueness result illustrates also the dependence between the properties of G and f if we compare it with Proposition 4.1(i).

THEOREM 4.4. *Let G be a P_0-mapping and let f_i be strictly convex for each $i = 1, \dots, n$. Then MVI (1.1) has at most one solution.*

Again, combining Theorem 4.4 and Proposition 4.2 yields the following result immediately.

Corollary 4.5. *In addition to the assumptions of Theorem 4.4, suppose that K is a bounded set. Then MVI (1.1) has a unique solution.*

We now present an existence and uniqueness result on unbounded sets under the P_0 condition. This result can be viewed as a counterpart of that in Proposition 4.1(ii).

Theorem 4.6. *Let G be a P_0-mapping and let f_i be a strongly convex function for each $i = 1,\dots,n$. Then MVI (1.1) has a unique solution.*

Thus, it is possible to obtain existence and uniqueness results if we replace (strict) P properties of the cost mapping G with strengthened convexity properties of all the functions f_i. However, if even a part of G possesses such (strict) P properties, we can obtain similar results in the case where the functions f_i corresponding to the other part of G possess the strengthened properties.

For the index set $L = \{1,\dots,l\}$, we will write $x_L = (x_i)_{i\in L}$ and $A_l(x) = \nabla_{x_L} G_L(x)$. Hence, $A_n(x) = \nabla G(x)$. First we give an existence and uniqueness result for unbounded sets.

Theorem 4.7. *Let G be a differentiable P_0-mapping. Suppose that, for every $x \in K$, $\nabla G(x)$ is a Z-matrix, and there exists $\varepsilon > 0$ such that $A_k(x) - \varepsilon I_k$ is a P-matrix for a fixed k. Suppose also that f_i, $i = k+1,\dots,n$ are strongly convex functions. Then MVI (1.1) has a unique solution.*

We now give a specialization of the previous result in the bounded case.

Theorem 4.8. *Let G be a differentiable P_0-mapping. Suppose that, for every $x \in K$, $\nabla G(x)$ is a Z-matrix and $A_k(x)$ is a P-matrix for a fixed k. Suppose also that f_i, $i = k+1,\dots,n$, are strongly convex functions and that K is bounded. Then MVI (1.1) has a unique solution.*

It should be noted that the assertions of Theorems 4.7 and 4.8 remain true if we replace the index set $\{1,\dots,k\}$ with an arbitrary subset of $\{1,\dots,n\}$. Moreover, Theorems 4.7 and 4.8 also justify the partial regularization approach for MVI (1.1), whereas Proposition 4.1 also justifies the full Tikhonov type regularization. For instance, we first consider MVI (1.1) under assumptions (A1), (A2), and (A3) and in addition let G be a P_0-mapping. We then can replace G with the following mapping:

$$\tilde{G}^{(\varepsilon)} = G + \varepsilon I_n, \tag{4.1}$$

where $\varepsilon > 0$ is an arbitrary sufficiently small number. On account of Lemma 3.6, $\tilde{G}^{(\varepsilon)}$ is a strict P-mapping, hence, due to Proposition 4.1(ii), such a perturbed MVI with the cost mapping $\tilde{G}^{(\varepsilon)}$ will have a unique solution which is close to that of the initial problem. Now suppose that we have MVI (1.1) which satisfies (A1), (A2), and (A3), the Jacobian ∇G is an M_0-matrix and f_i, $i = k+1, \dots, n$, are strongly convex for a fixed k. Then we can replace G with $G^{(\varepsilon)}$ whose components are defined by

$$G_i^{(\varepsilon)}(x) = \begin{cases} G_i(x) + \varepsilon x_i, & \text{if } i \leq k; \\ G_i(x), & \text{if } i > k, \end{cases} \tag{4.2}$$

where $\varepsilon > 0$ is an arbitrary sufficiently small parameter. On account of Theorem 4.7, such a perturbed MVI with the cost mapping $G^{(\varepsilon)}$ will also have a unique solution which is close to that of the initial problem. This situation seems rather natural for economic applications, nevertheless, we see that now the full regularization is not necessary.

5. Application to the Walrasian equilibrium model

We now specialize the results above for the models considered in Section 2. We first consider the general Walrasian equilibrium model from Section 2 which can be reformulated as MVI (2.7), (2.3). For the sake of convenience, we rewrite it here. Namely, the problem is to find $p^* \in K$ such that

$$\langle G(p^*), p - p^* \rangle + \sum_{i=1}^{n} [f_i(p_i) - f_i(p_i^*)] \geq 0 \quad \forall p \in K, \tag{5.1}$$

where

$$K = \prod_{i=1}^{n} K_i, \quad K_i = \{t \in \mathbb{R} \mid 0 < \tau_i' \leq t \leq \tau_i'' \leq +\infty\},\ i = 1, \dots, n; \tag{5.2}$$

τ_i' and τ_i'' are the lower and upper bounds for the price of the ith commodity.

We also recall that $D = -G$ is the demand mapping, $S_i = \partial f_i$ is the supply mapping of the ith producer which is supposed to be monotone, hence f_i is then convex, but not necessarily differentiable. In addition, we set $V = \mathbb{R}^n_>$ and suppose that $G : V \to \mathbb{R}^n$ is continuous. Clearly, f_i, $i = 1, \dots, n$, are also continuous on V. Therefore, our problem then satisfies all the assumptions (A1), (A2), and (A3). For this reason, we can establish the first existence result directly from Proposition 4.2.

Proposition 5.1. *If $\tau_i'' < +\infty$ for each $i = 1, \dots, n$, then problem (5.1) has a solution.*

Of course, the assumption of this proposition implies the boundedness of K and the result follows.

In order to apply the other results from Section 4 to problem (5.1) we have to impose certain additional conditions on G and f_i which should conform to the usually accepted economic behaviour.

Definition 5.2 (see [19]). A mapping $Q : V \to \mathbb{R}^n$ is said to

(a) satisfy the *gross substitutability* property, if $\partial Q_j / \partial p_i \geq 0$, $j \neq i$;
(b) be *positive homogeneous of degree m*, if $Q(\alpha x) = \alpha^m Q(x)$ for every $\alpha \geq 0$.

The gross substitutability of demand is one of the most popular conditions on market structures; see, for example, [1, 19, 20] and the references therein. It means that all the commodities in the market are *substitutable* in the sense that if the price of the ith commodity increases, then the demand of other commodities does not decrease. Next, the positive homogeneity of degree 0 of demand is also rather a standard condition. It follows usually from insatiability of consumers; see, for example, [1, 15, 19]. For this reason, throughout this section we will suppose that *the demand mapping D is continuously differentiable, positive homogeneous of degree* 0*, and possesses the gross substitutability property*.

From the gross substitutability of D it follows that

$$\frac{\partial G_i(p)}{\partial p_j} \leq 0, \quad i \neq j. \tag{5.3}$$

Hence $\nabla G(p)$ is a Z-matrix. Next, since $G_i(p)$ is homogeneous of degree zero, it follows from the Euler theorem (see, e.g., [19, Lemma 18.4]) that

$$\sum_{j=1}^{n} \frac{\partial G_i(p)}{\partial p_j} p_j = 0 \quad \forall i = 1, \ldots, n. \tag{5.4}$$

Applying now Proposition 3.4, we conclude that $\nabla G(p)$ is an M_0-matrix, hence G is also a P_0-mapping and we thus have obtained the following assertions.

Lemma 5.3. *The following statements are true:*

(i) *G is a P_0-mapping;*
(ii) *$\nabla G(p)$ is an M_0-matrix for each $p \in V$.*

Note that (5.4) implies that G cannot be a (strict) P-mapping, hence the results of Proposition 4.1 are not applicable in this case. At the same time, we do not suppose for the supply mapping to be homogeneous, although this condition is rather usual for most known economic equilibrium models. If this is the case, then, using the standard technique of fixing the price of the nth commodity (numéraire), that is, setting $p_n^* = 1$, one can consider the reduced (normalized) mapping $\tilde{G} : \mathbb{R}_+^{n-1} \to \mathbb{R}^{n-1}$, defined by $\tilde{G}(p) = G(p_1, \dots, p_{n-1}, 1)$, whose Jacobian is an M-matrix if the nth column of $\nabla G(p)$ contains only negative entries. Thus, in this case the price of the nth commodity, which is considered as money, can be arbitrary in the initial model, that is, money is neutral in such a model. It also means that both supply and demand do not depend on the level of prices. Therefore, homogeneity of both supply and demand implies the additional P-type properties of the cost mapping. We intend to investigate our model under weaker assumptions with the help of the results of Section 4, and money need not be neutral in our model.

Proposition 5.4. (i) *Let K be a bounded set and let f_i, $i = 1, \dots, n$, be strictly convex. Then problem (5.1) has a unique solution.*

(ii) *Let f_i, $i = 1, \dots, n$, be strongly convex. Then problem (5.1) has a unique solution.*

On account of Lemma 5.3, the proofs of assertions (i) and (ii) follow now from Corollary 4.5 and Theorem 4.6, respectively.

We recall that, due to Lemma 1.3, strict (strong) convexity of f_i is equivalent to strict (strong) monotonicity of the ith supply mapping $S_i = \partial f_i$. Although G need not be a (strict) P-mapping, its part can possess such properties. In this case, we can apply Theorems 4.7 and 4.8 to our problem.

Proposition 5.5. *Suppose that there exists $\varepsilon > 0$ such that for every $p \in K$, $A_{n-1}(p) - \varepsilon I_{n-1}$ is an M-matrix and that f_n is strongly convex. Then problem (5.1) has a unique solution.*

The proof follows from Theorem 4.7. We can specialize the result above for the bounded case.

Proposition 5.6. *Suppose that K is bounded and that, for every $p \in K$, $A_{n-1}(p)$ is an M-matrix. Suppose also that f_n is strongly convex. Then problem (5.1) has a unique solution.*

The proof follows from Theorem 4.8.

We now give additional examples of sufficient conditions for (5.1) to have a unique solution.

THEOREM 5.7. *Suppose that K is bounded and that for every $p \in K$,*

$$\frac{\partial G_i(p)}{\partial p_n} < 0 \quad \forall i = 1,\dots,n-1. \tag{5.5}$$

Suppose also that f_n is strongly convex. Then problem (5.1) has a unique solution.

Proof. By (5.3), (5.4), and (5.5), we have

$$\sum_{j=1}^{n-1} \frac{\partial G_i(p)}{\partial p_j} p_j > \sum_{j=1}^{n} \frac{\partial G_i(p)}{\partial p_j} p_j = 0 \tag{5.6}$$

for each $i = 1,\dots,n-1$. Therefore, $A_{n-1}(p)$ is an M-matrix. The result follows now from Proposition 5.6. □

Consider the case where the functions f_i, $i = 1,\dots,n$, are not strongly convex but K is bounded and (5.5) holds. Then we can replace the cost mapping G in (5.1) by $G^{(\varepsilon)}$, whose components are defined by

$$G_i^{(\varepsilon)}(p) = \begin{cases} G_i(p), & \text{if } i < n; \\ G_i(p) + \varepsilon p_i, & \text{if } i = n, \end{cases} \tag{5.7}$$

where $\varepsilon > 0$ is small enough. Then, following the proof of Theorem 5.7 and using the properties of M-matrices, we see that $\nabla G^{(\varepsilon)}$ is M, hence the perturbed problem will have a unique solution due to Proposition 4.1(i), this solution being close to that of the initial problem.

It should be noted that all the considerations above, in particular, Propositions 5.5 and 5.6 and Theorem 5.7, remain valid if we replace n with an arbitrary index from $\{1,\dots,n\}$. Moreover, we can replace a single index with an arbitrary subset of $\{1,\dots,n\}$, thus extending the results above.

PROPOSITION 5.8. *Suppose that K is bounded and that there exists an index k such that for every $p \in K$,*

$$\sum_{j=k+1}^{n} \frac{\partial G_i(p)}{\partial p_j} < 0 \quad \forall i = 1,\dots,k. \tag{5.8}$$

Suppose also that f_j, $j = k+1,\dots,n$, are strongly convex. Then problem (5.1) has a unique solution.

The proof is the same as that of Theorem 5.7, using Theorem 4.8. We can state the similar result in the unbounded case.

THEOREM 5.9. *Suppose that there exist $\delta > 0$ and an index k such that for every $p \in K$,*

$$\sum_{j=k+1}^{n} \frac{\partial G_i(p)}{\partial p_j} p_j < -\delta p_i \quad \forall i = 1,\dots,k. \tag{5.9}$$

Suppose also that f_j, $j = k+1,\dots,n$, are strongly convex. Then problem (5.1) has a unique solution.

Proof. Fix $\gamma \in (0,\delta)$, then, by (5.3), (5.4), and (5.9), we have

$$\sum_{j=1}^{k} \frac{\partial G_i(p)}{\partial p_j} p_j - \gamma p_i > \sum_{j=1}^{n} \frac{\partial G_i(p)}{\partial p_j} p_j = 0 \tag{5.10}$$

for each $i = 1,\dots,k$. Therefore, $A_k(p) - \gamma I_k$ is an M-matrix. The result follows now from Theorem 4.7. □

Again, if all the functions f_i, $i = 1,\dots,n$, are not strongly convex, we can use the partial regularization of G (see (5.7)). Note that the results of Proposition 5.8 and Theorem 5.9 remain true if we replace the subset $\{1,\dots,k\}$ with an arbitrary subset of $\{1,\dots,n\}$.

6. Application to the oligopolistic equilibrium model

In this section, we consider the oligopolistic equilibrium model from Section 2 which was shown to be equivalent to problem (2.15), (2.13). For the sake of convenience, we also rewrite it here. Namely, the problem is to find $q^* \in K$ such that

$$\langle G(q^*), q - q^* \rangle + \sum_{i=1}^{n} \left[f_i(q_i) - f_i(q_i^*) \right] \geq 0 \quad \forall q \in K, \tag{6.1}$$

where

$$\begin{gathered} K = \prod_{i=1}^{n} K_i, \quad K_i = \left\{ t \in \mathbb{R} \mid 0 \leq t \leq \beta_i \right\},\ i = 1,\dots,n; \\ G_i(q) = -p(\sigma_q) - q_i p'(\sigma_q), \quad i = 1,\dots n; \\ \sigma_q = \sum_{i=1}^{n} q_i, \end{gathered} \tag{6.2}$$

where p is the price (inverse demand) function, which is supposed to be continuously differentiable, and f_i is the cost function of the ith firm, which is supposed to be convex, but it is not necessarily differentiable. If we set $V = \mathbb{R}^n_+$, then we see that our problem coincides with (1.1) and that assumptions (A1), (A2), and (A3) hold here. Therefore, we can deduce the existence result for the bounded case from Proposition 4.2.

Proposition 6.1. *If $\beta_i < +\infty$ for each $i = 1,\dots,n$, then problem (6.1) has a solution.*

In order to establish additional existence and uniqueness results for problem (6.1) we have to derive P-type properties for the cost mapping G. To this end, throughout this section we suppose that the price function $p(\sigma)$ is nonincreasing and that the industry revenue function $\mu(\sigma) = \sigma p(\sigma)$ is concave for $\sigma \geq 0$. These assumptions conform to the usual economic behaviour and provide the concavity in q_i of the each ith profit function $q_i p(\sigma) - f_i(q_i)$ (see, e.g., [17]). It was indicated in Section 2 that the oligopolistic equilibrium problem (2.10) and MVI (6.1) become equivalent under these assumptions. We now give additional properties of G which also follow from these assumptions.

Lemma 6.2. *It holds that* $\det A_k(q) = [-(k-1)p'(\sigma_q) - \mu''(\sigma_q)](-p'(\sigma_q))^{k-1}$.

The proof of this technical result will be given in the appendix.

Proposition 6.3. *The following statements are true:*

(i) *$\nabla G(q)$ is a P_0-matrix for every $q \in V$;*
(ii) *let $p'(\sigma) < 0$ and either $\mu''(\sigma) < 0$ or $p''(\sigma) \leq 0$ for all $\sigma \geq 0$. Then $\nabla G(q)$ is a P-matrix for every $q \in V$.*

Proof. Since $p'(\sigma) \leq 0$ and $\mu''(\sigma) \leq 0$, it follows from Lemma 6.2 that all the principal minors of the matrix $\nabla G(q)$ are nonnegative. Hence, assertion (i) is true. Next, by Lemma 6.2, all the principal minors of $\nabla G(q)$ will be positive under the assumptions of (ii). It follows that $\nabla G(q)$ is a P-matrix. □

Now we obtain new existence and uniqueness results for MVI (6.1) with the help of those in Section 4.

Proposition 6.4. (i) *Let $\beta_i < +\infty$ and let f_i be strictly convex for each $i = 1,\dots,n$. Then problem (6.1) has a unique solution.*

(ii) *Let f_i be strongly convex for each $i = 1,\dots,n$. Then problem (6.1) has a unique solution.*

Proof. Due to Proposition 6.3(i), G is a P_0-mapping. The assertions (i) and (ii) follow directly from Corollary 4.5 and Theorem 4.6, respectively. □

PROPOSITION 6.5. *Let $p'(\sigma) < 0$ and either $\mu''(\sigma) < 0$ or $p''(\sigma) \leq 0$ for all $\sigma \geq 0$. Then MVI (6.1) has at most one solution. If, in addition, $\beta_i < +\infty$ for all $i = 1, \ldots, n$, then problem (6.1) has a unique solution.*

Proof. Due to Proposition 6.3(ii), G is now a P-mapping. We conclude, from Proposition 4.1(i), that the first assertion is true, whereas the second assertion follows now from Proposition 6.1. □

We also present a similar result in the general unbounded case.

PROPOSITION 6.6. *Suppose that there exists $\delta > 0$ such that $-p'(\sigma) \geq \delta$ and either $-\mu''(\sigma) \geq \delta$ or $p''(\sigma) \leq 0$ for all $\sigma \geq 0$. Then MVI (6.1) has a unique solution.*

The proof of this assertion will be given in the appendix. Thus, the specialization of the general results for MVIs from Section 4 allowed us to obtain new existence and uniqueness results for oligopolistic equilibrium problems in comparison with the known ones (see [4, 17, 18] and the references therein).

7. Concluding remarks

In this paper, we have considered the class of mixed variational inequalities (MVIs) which is intermediate between classes of VIs with single-valued and multivalued cost mappings. We have established new existence and uniqueness results of solutions of MVIs under rather general assumptions and presented perfectly and nonperfectly competitive economic equilibrium models which satisfy these assumptions.

Taking this observation as a basis, we have obtained also new existence and uniqueness results for these economic equilibrium problems. We emphasize that all the results are similar to those for single-valued problems, but they have been in fact obtained for multivalued ones.

The results above also enable us to develop effective solution methods for such economic equilibrium problems. For instance, we can convert MVI into the problem of finding a stationary point of a continuously differentiable function with the help of the D-gap function approach (see [11, 12]). Hence, the usual differentiable optimization methods become applicable to economic equilibrium problems containing multivalued mappings or nonsmooth functions. In addition, if the cost mapping does not possess strengthened P-type properties, it is possible to apply the full or partial regularization approach (see (4.1), (4.2), and (5.7)) and obtain an approximate solution with any prescribed accuracy.

Appendix

In this section, we give proofs of the assertions from Sections 4 and 6.

Proof of Proposition 4.2. Consider the function

$$\varphi_\alpha(x) = \max_{y\in K}\sum_{i=1}^{n}\Phi_i^\alpha(x,y_i) = \sum_{i=1}^{n}\max_{y_i\in K_i}\Phi_i^\alpha(x,y_i), \tag{A.1}$$

where

$$\Phi_i^\alpha(x,y_i) = G_i(x)(x_i-y_i) - 0.5\alpha(x_i-y_i)^2 + f_i(x_i) - f_i(y_i) \tag{A.2}$$

for $i = 1,\dots,n$, and $\alpha > 0$. The function $\Phi_i^\alpha(x,\cdot)$ is strongly concave, hence, there exists a unique solution to each inner problem in (A.1), that is, there exist elements $y_i^\alpha(x) \in K_i$ such that

$$\max_{y_i\in K_i}\Phi_i^\alpha(x,y_i) = \Phi_i^\alpha(x,y_i^\alpha(x)) \tag{A.3}$$

for $i = 1,\dots,n$. Using the necessary and sufficient condition of optimality for each problem, we see that $y_i^\alpha(x)$ can be redefined as follows:

$$\begin{aligned}\exists g_i \in \partial f_i(y_i^\alpha(x)) : \left[G_i(x) + \alpha(y_i^\alpha(x) - x_i)\right](y_i - y_i^\alpha(x))&\\ + g_i(y_i - y_i^\alpha(x)) \geq 0 \quad \forall y_i \in K_i,&\end{aligned} \tag{A.4}$$

or equivalently (see Proposition 3.1),

$$\left[G_i(x) + \alpha(y_i^\alpha(x) - x_i)\right](y_i - y_i^\alpha(x)) + f_i(y_i) - f_i(y_i^\alpha(x)) \quad \forall y_i \in K_i \tag{A.5}$$

for all $i = 1,\dots,n$. Set $y_\alpha(x) = (y_1^\alpha(x),\dots,y_n^\alpha(x))$. It was shown in [12, Lemma 2] that the mapping $x \mapsto y_\alpha(x)$ is continuous. Applying now Brouwer's fixed point theorem, we conclude that there exists $x^* = y_\alpha(x^*)$. Setting $x = x^*$ in (A.5), we deduce that x^* is a solution to MVI (1.1). The proof is complete. □

Proof of Theorem 4.4. Suppose for contradiction that there exist x' and x'', $x' \neq x''$, which are solutions to MVI (1.1). By Proposition 3.1, we have

$$\begin{aligned}\exists g_i' \in \partial f_i(x_i') : G_i(x')(x_i'' - x_i') + g_i'(x_i'' - x_i') \geq 0,\\ \exists g_i'' \in \partial f_i(x_i'') : G_i(x'')(x_i' - x_i'') + g_i''(x_i' - x_i'') \geq 0\end{aligned} \tag{A.6}$$

for all $i = 1, \dots, n$. Adding these inequalities yields

$$[G_i(x') - G_i(x'')](x_i'' - x_i') + (g_i' - g_i'')(x_i'' - x_i') \geq 0, \tag{A.7}$$

for each $i = 1, \dots, n$. For brevity, set $I = \{i \mid x_i' \neq x_i''\}$. Since G is a P_0-mapping, there exists an index $k \in I$ such that

$$[G_k(x') - G_k(x'')](x_k' - x_k'') = \max_{1 \leq i \leq n} [G_i(x') - G_i(x'')](x_i' - x_i''). \tag{A.8}$$

Then, by definition, $[G_k(x') - G_k(x'')](x_k' - x_k'') \geq 0$. Due to (A.7) we now obtain

$$(g_k' - g_k'')(x_k'' - x_k') \geq 0, \tag{A.9}$$

which is a contradiction, since f_k is strictly convex, that is, ∂f_k is strictly monotone because of Lemma 1.3(b). The proof is complete. □

Proof of Theorem 4.6. By Proposition 3.1, the initial problem is equivalent to the following VI: find $x^* \in K$ such that

$$\begin{aligned} \exists g_i^* \in \partial f_i(x_i^*) : (G_i(x^*) + \varepsilon x_i^*)(x_i - x_i^*) \\ + (g_i^* - \varepsilon x_i^*)(x_i - x_i^*) \geq 0 \quad \forall x_i \in K_i,\ i = 1, \dots, n; \end{aligned} \tag{A.10}$$

which can be rewritten equivalently as

$$\begin{aligned} \exists t_i^* \in \partial \varphi_i(x_i^*) : F_i^{(\varepsilon)}(x^*)(x_i - x_i^*) \\ + t_i^*(x_i - x_i^*) \geq 0 \quad \forall x_i \in K_i,\ i = 1, \dots, n; \end{aligned} \tag{A.11}$$

where $F_i^{(\varepsilon)}(x) = G_i(x) + \varepsilon x_i$ and $\varphi_i(\sigma) = f_i(\sigma) - \varepsilon \sigma^2/2$. Again, on account of Proposition 3.1, problem (A.11) is equivalent to the MVI: find $x^* \in K$ such that

$$\langle F^{(\varepsilon)}(x^*), x - x^* \rangle + \sum_{i=1}^{n} [\varphi_i(x_i) - \varphi_i(x_i^*)] \geq 0 \quad \forall x \in K. \tag{A.12}$$

From Lemma 3.6 it follows that $F^{(\varepsilon)}$ is a strict P-mapping for every $\varepsilon > 0$. We will show that each φ_i is a convex function for some $\varepsilon > 0$. Since f_i is strongly convex, we see that for all x_i', x_i'' and $t_i' \in \partial \varphi_i(x_i')$, $t_i'' \in \partial \varphi_i(x_i'')$, we have

$$(t_i' - t_i'')(x_i' - x_i'') = (g_i' - g_i'')(x_i' - x_i'') - \varepsilon (x_i' - x_i'')(x_i' - x_i'') \tag{A.13}$$

for some $g_i' \in \partial f_i(x_i')$, $g_i'' \in \partial f_i(x_i'')$, hence

$$(t_i' - t_i'')(x_i' - x_i'') \geq \tau(x_i' - x_i'')^2 - \varepsilon(x_i' - x_i'')^2 \geq 0 \tag{A.14}$$

if $\varepsilon < \tau$, where τ is the smallest constant of strong monotonicity of ∂f_k (strong convexity of f_k). So, ψ_i is a convex function if $0 < \varepsilon < \tau$.

By Proposition 4.1(ii), problem (A.12) has a unique solution. However, problem (A.12) is equivalent to (A.11), that is, it is equivalent to (A.10). This completes the proof. □

Proof of Theorem 4.7. First we note that

$$\nabla G(x) = \begin{pmatrix} A_k(x) & B_k' \\ B_k'' & C_k \end{pmatrix}, \tag{A.15}$$

where B_k' is a rectangular matrix which has k rows and $n-k$ columns, B_k'' is a rectangular matrix which has $n-k$ rows and k columns, and C_k is an $(n-k)\times(n-k)$ matrix. By assumption, there exists $\varepsilon' > 0$ such that $A_k(x) - \varepsilon' I_k$ is an M-matrix. Without loss of generality, we suppose that $\varepsilon' \leq \tau$, where τ is the smallest constant of strong monotonicity of ∂f_i (strong convexity of f_i). Let us consider the mapping $\tilde{G} : V \to \mathbb{R}^n$, whose components are defined by

$$\tilde{G}_i(x) = \begin{cases} G_i(x), & \text{if } 1 \leq i \leq k; \\ G_i(x) + \varepsilon'' x_i, & \text{if } k < i \leq n; \end{cases} \tag{A.16}$$

with $0 < \varepsilon'' < \varepsilon'$. Clearly, its Jacobian

$$\nabla \tilde{G}(x) = \begin{pmatrix} A_k(x) & B_k' \\ B_k'' & C_k + \varepsilon'' I_{n-k} \end{pmatrix} \tag{A.17}$$

is an M-matrix (see [8]). Moreover, $\nabla\tilde{G}(x) - \gamma I_n$ is an M-matrix for any $\gamma > 0$ such that $0 < \gamma < \varepsilon''$. By definition, $\tilde{G}$ is a strict P-mapping. Next, consider the functions

$$\tilde{f}_i(x_i) = \begin{cases} f_i(x_i), & \text{if } 1 \leq i \leq k; \\ f_i(x_i) - \varepsilon'' \dfrac{x_i^2}{2}, & \text{if } k < i \leq n. \end{cases} \tag{A.18}$$

Since $f_i(x_i)$, $i > k$ are strongly convex, for all x_i', x_i'' and $g_i' \in \partial f_i(x_i')$, $g_i'' \in \partial f_i(x_i'')$, we have

$$(g_i' - g_i'')(x_i' - x_i'') - \varepsilon''(x_i' - x_i'')(x_i' - x_i'') \geq (\tau - \varepsilon'')(x_i' - x_i'')^2 \geq 0. \tag{A.19}$$

Hence, $\partial f_i - \varepsilon'' I_1$ is nonempty and monotone for each $i > k$, that is, $\tilde{f}_i$ is convex for every $i = 1, \ldots, n$. Due to Proposition 4.1(ii), it follows that the problem: find $x^* \in K$ such that

$$\langle \tilde{G}(x^*), x - x^* \rangle + \tilde{f}(x) - \tilde{f}(x^*) \geq 0 \quad \forall x \in K \tag{A.20}$$

has a unique solution. However, this problem is clearly equivalent to MVI (1.1) and the result follows. □

Along the same lines, we obtain the result of Theorem 4.8 if we use Corollary 4.3 instead of Proposition 4.1.

We now turn to the assertions of Section 6. By assumption, the Jacobian of the cost mapping G in (6.1) can be rewritten as follows:

$$\nabla G(q) = \begin{pmatrix} \beta + \alpha_1 & \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \alpha_2 & \beta + \alpha_2 & \alpha_2 & \cdots & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \alpha_n & \alpha_n & \cdots & \beta + \alpha_n \end{pmatrix}, \tag{A.21}$$

where β denotes $-p'(\sigma_q)$ and α_i denotes $-p'(\sigma_q) - q_i p''(\sigma_q)$. We recall that $p(\sigma)$ is nonincreasing and $\mu(\sigma) = \sigma p(\sigma)$ is concave for all $\sigma \geq 0$, hence $p'(\sigma) \leq 0$ and $\mu''(\sigma) \leq 0$.

Proof of Lemma 6.2. By definition,

$$\det A_k(q) = \begin{vmatrix} \beta + \alpha_1 & \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \alpha_2 & \beta + \alpha_2 & \alpha_2 & \cdots & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_k & \alpha_k & \alpha_k & \ldots & \beta + \alpha_k \end{vmatrix}. \tag{A.22}$$

Adding all the rows to the first one, and subtracting the first column from others yields

$$\det A_k(q) = \begin{vmatrix} \beta + \sum_{i=1}^{k} \alpha_i & 0 & 0 & \cdots & 0 \\ \alpha_2 & \beta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_k & 0 & 0 & \ldots & \beta \end{vmatrix} = \beta^{k-1} \left(\beta + \sum_{i=1}^{k} \alpha_i \right). \tag{A.23}$$

Hence,

$$\begin{aligned}\det A_k(q) &= \left(-p'(\sigma_q)\right)^{k-1}\left[-(k+1)p'(\sigma_q)-\sigma_q p''(\sigma_q)\right]\\ &= \left(-p'(\sigma_q)\right)^{k-1}\left[-(k-1)p'(\sigma_q)-\mu''(\sigma_q)\right].\end{aligned} \tag{A.24}$$

□

Proof of Proposition 6.6. By assumption, $-p'(\sigma)\geq\delta>0$ for all $\sigma\geq 0$. Fix $\varepsilon\in(0,\delta)$. Then,

$$\begin{aligned}\det\left(A_k(q)-\varepsilon I_k\right) &= \begin{vmatrix}\beta_\varepsilon+\alpha_1 & \alpha_1 & \cdots & \alpha_1\\ \alpha_2 & \beta_\varepsilon+\alpha_2 & \cdots & \alpha_2\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_k & \alpha_k & \cdots & \beta_\varepsilon+\alpha_k\end{vmatrix}\\ &= \left(\beta_\varepsilon\right)^{k-1}\left[\beta_\varepsilon+\sum_{i=1}^{k}\alpha_i\right],\end{aligned} \tag{A.25}$$

where $\beta_\varepsilon=\beta-\varepsilon$, that is,

$$\det\left(A_k(q)-\varepsilon I_k\right)=\left(-p'(\sigma_q)-\varepsilon\right)^{k-1}\left[-(k-1)p'(\sigma_q)-\varepsilon-\mu''(\sigma_q)\right]>0 \tag{A.26}$$

if either $-\mu''(\sigma)\geq\delta$ or $p''(\sigma)\leq 0$. Therefore, G is now a strict P-mapping. Using now Proposition 4.1(ii), we conclude that problem (6.1) has a unique solution, as desired. □

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