MULTIPLE SOLUTIONS FOR A PROBLEM WITH RESONANCE INVOLVING THE p-LAPLACIAN

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ABSTRACT. In this paper we will investigate the existence of multiple solutions for the problem

$$(P) \qquad -\Delta_p u + g(x,u) = \lambda_1 h(x)\,|u|^{p-2}\,u, \quad \text{in} \quad \Omega, \quad u \in H_0^{1,p}(\Omega)$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$ is the p-Laplacian operator, $\Omega \subseteq I\!R^N$ is a bounded domain with smooth boundary, h and g are bounded functions, $N \geq 1$ and $1 < p < \infty$. Using the Mountain Pass Theorem and the Ekeland Variational Principle, we will show the existence of at least three solutions for (P).

1. Introduction

In this paper, we will investigate the existence of multiple solutions for the problem

$$(\text{P}) \qquad \begin{cases} -\Delta_p u + g(x,u) = \lambda_1 h(x)|u|^{p-2}u, & \text{in } \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Delta_p u$ =div$(|\nabla u|^{p-2}\,\nabla u)$ is the p-Laplacian operator, $1 < p < \infty$, $N \geq 1$, Ω is a bounded domain with smooth boundary,

$$(\text{G}_1) \qquad g : \Omega \times I\!R \to I\!R \text{ is bounded continuous function satisfying } \quad g(x,0) = 0,$$

and its primitive denoted by

1991 *Mathematics Subject Classification.* Primary: 35A05, 35A15, 35J20.

Key words and phrases. Radial solutions, Critical Sobolev exponents, Palais-Smale condition, Mountain Pass Theorem.

* Supported in part by Fapemig/ Brazil.

** Supported in part by Fapemig and CNPq/Brazil.

*** Supported in part by Fapemig and CNPq/ Brazil.

Received: March 18, 1998.

$$(\mathrm{G}_2) \qquad G(x,s)=\int_0^s g(x,t)dt \quad \text{is assumed to be bounded,}$$

λ_1 is the first eigenvalue of the following eigenvalue problem with weight

$$(\mathrm{P}_A) \qquad \begin{cases} -\Delta_p u = \lambda_1 h(x)|u|^{p-2}u, & \text{in } \Omega, \\ u=0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where

(h) $0 \leq h \in L^{\infty}(\Omega)$ with $h>0$ on subset of Ω with positive measure.

We recall that λ_1 is simple, isolated and it is the unique eigenvalue with positive eigenfunction Φ_1 (see [1] or [2]). There are many papers treating problem (P) with $h=1$, among others, we would like to mention Lazer & Landesman [3], Ahmad, Lazer & Paul [4], De Figueiredo & Gossez [5], Amann, Ambrosetti & Mancini [6], Ambrosetti & Mancini [7], Thews [8], Bartolo, Benci & Fortunato [9], Ward [10], Arcoya & Cañada [11], Costa & Silva [12], Fu [13], Gonçalves & Miyagaki [14] when $p=2$, and Boccardo, Drábek & Kučera [15], Anane & Gossez [16], Ambrosetti & Arcoya [17], Arcoya & Orsina [18], Fu & Sanches [19] when $p \neq 2$.

We shall show in this paper, the existence of multiple solutions for problem (P), by using similar arguments explored in [14] and [19]. Combining a version of the Mountain Pass Theorem due to Ambrosetti & Rabinowitz (see [20] and [25]) and the Ekeland variational principle (see [21, Theorem 4.1]), we will find nontrivial critical points of Euler- Lagrange functional associated to (P) given by

$$(1) \qquad I(u)=\frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega h\,|u|^p + \int_\Omega G(x,u) \ \ , \ u \in H_0^{1,p}(\Omega),$$

which are weak solutions of (P).

Hereafter, we will denoted by $\| \ \|$ and $| \ |_p$ the usual norms on the spaces $H_0^{1,p}(\Omega)$ and $L^p(\Omega)$ respectively, and by W the closed subspace

$$W=\left\{ u \in H_0^{1,p}(\Omega) \ / \ \int_\Omega h\,u\,|\Phi_1|^{p-2}\,\Phi_1 = 0 \right\}.$$

We can easily prove that W is a complementary subspace of $\langle \Phi_1 \rangle$. Therefore we have the following direct sum (see e.g. Brézis [22])

$$H_0^{1,p}(\Omega)=\langle \Phi_1 \rangle \oplus W.$$

We will be denoted by λ_2, the following real number

$$\lambda_2=\inf_{u\in W}\left\{ \int_\Omega |\nabla u|^p \ ; \int_\Omega h\,|u|^p = 1 \right\},$$

and we remind that this value is the second eigenvalue of the p-Laplacian (see [23] or [24]).

From simplicity and isolation of λ_1 (see [1] or [2]), we have $0<\lambda_1<\lambda_2$ and by definition of λ_2 it follows that

$$\int_\Omega h\,|w|^p \leq \frac{1}{\lambda_2}\int_\Omega |\nabla w|^p \ \ , \ \ \forall w \in W.$$

In this work, we will impose the following condition

$$(\mathrm{G}_3) \qquad g(x,t)t \to 0, \quad \text{as} \quad |t| \to \infty, \quad \forall x \in \Omega,$$

which appeared in [7] for $p = 2$ and [17] for the general case $p > 1$. This condition together with the assumptions on the limits

$$T(x) = \liminf_{|t|\to\infty} G(x,t) \quad \text{and} \quad S(x) = \limsup_{|t|\to\infty} G(x,t), \quad \forall x \in \Omega,$$

imply that problem (P) is in the class of the strongly resonance problem in the sense of Bartolo-Benci & Fortunato [9].
The following condition means a nonresonance with higher eigenvalues

$$(\mathrm{G}_4) \qquad G(x,t) \geq \left(\frac{\lambda_1 - \lambda_2}{p}\right) h(x)\,|t|^p\,, \quad \forall x \in \Omega, \quad \forall t \in I\!R.$$

In addition to (G_3) which is a behaviour of g at infinity, we assume a condition of the behaviour of G at origin

$$(\mathrm{G}_5) \qquad \begin{array}{c} \text{there exist } 0 < \delta \quad \text{and} \quad 0 < m < \lambda_1 \quad \text{such that} \\ G(x,t) \geq \frac{m}{p} h(x)\,|t|^p\,, \quad \text{for } |t| < \delta,\ \forall x \in \Omega. \end{array}$$

Our main result is the following:

Theorem 1. *Assume conditions (h), (G_1)-(G_5). Then, problem (P) has at least three solutions u_1, u_2 and u_3, with*

$$I(u_1),\ I(u_2) < 0 \textit{ and } I(u_3) > 0,$$

provided that the following conditions hold

$$(\mathrm{G}_6) \qquad \begin{array}{c} \textit{there exist } t^-,\ t^+ \in I\!R \quad \textit{with} \quad t^- < 0 < t^+ \textit{ such that} \\ \int_\Omega G(x, t^\pm \Phi_1) \leq \int_\Omega T(x)dx < 0, \end{array}$$

and

$$(\mathrm{G}_7) \qquad \int_\Omega S(x)dx \leq 0.$$

Remark 1. Theorem 1 improves in some sense the main result proved in [14], since the proof given in [14] works only in Hilbert space framework.

2. Preliminary Results

In this section, we will state and prove some results required in the proof of Theorem 1. We recall that $I : H_0^{1,p}(\Omega) \to I\!R$ is said to satisfy Palais-Smale condition at the level $c \in I\!R$ ($(\mathrm{PS})_c$ in short), if any sequence $\{u_n\} \subset H_0^{1,p}(\Omega)$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0,$$

possesses a convergent subsequence in $H_0^{1,p}(\Omega)$.

Lemma 1. *Assume (h), (G1) and (G2). Then I satisfies the $(PS)_c$ condition $\forall\, c < \int_\Omega T(x)dx$.*

Proof. We are going to adapt some arguments used in [16, p.1148]. First of all, we shall show that $\{u_n\}$ is bounded. Assume that $\{u_n\}$ is unbounded, therefore, up to subsequence, we have

$$\|u_n\| \to \infty.$$

Letting

$$(*_n) \qquad v_n = \frac{u_n}{\|u_n\|},$$

we can assume that there exists $v \in H_0^{1,p}(\Omega)$ such that

$$v_n \rightharpoonup v \text{ in } H_0^{1,p}(\Omega)$$

and

$$v_n \to v \ \text{ in } \ L^s(\Omega), \text{ for } 1 \leq s < p^* = \frac{Np}{N-p}.$$

Now, we will show that $v \neq 0$ and that there exists $\gamma \in I\!R$ such that

$$v(x) = \gamma\Phi_1(x), \ \ \forall x \in \Omega.$$

From (1) and choosing $t_n = \|u_n\|$, we obtain

$$(2) \qquad \frac{I'(u_n)u_n}{t_n^p} = \int_\Omega |\nabla v_n|^p - \lambda_1 \int_\Omega h\,|v_n|^p + \frac{1}{t_n^p}\int_\Omega g(x,u_n)u_n.$$

Using (G1) together with the fact that

$$\lim_{n\to\infty} \frac{I'(u_n)u_n}{t_n^p} = 0,$$

we get

$$(3) \qquad \int_\Omega h\,|v|^p = \frac{1}{\lambda_1}$$

and therefore $v \neq 0$.
Using the weak convergence $v_n \rightharpoonup v$, we know that

$$(4) \qquad \|v\| \leq 1.$$

By (3) and (4), it follows that v is an eigenfunction for λ_1. Then there exists $\gamma \in I\!R$ such that

$$(5) \qquad v(x) = \gamma\Phi_1(x), \ \ \forall x \in \Omega.$$

In particular,

$$\frac{u_n}{\|u_n\|} \to \gamma\Phi_1, \ \forall x \in \Omega,$$

which implies

$$|u_n(x)| \to \infty, \ \ \forall x \in \Omega,$$

and by (G2) and Fatou's lemma, we have

$$(6) \qquad \liminf_{n\to\infty} \int_\Omega G(x,u_n(x))dx \geq \int_\Omega \liminf_{n\to\infty} G(x,u_n(x))dx \geq \int_\Omega T(x)dx.$$

Now, using the inequality

$$(7) \qquad c + o_n(1) = I(u_n) \geq \int_\Omega G(x,u_n(x))dx,$$

we have by (6) that

$$c \geq \int_{\Omega} T(x)dx,$$

which contradicts the hypothesis on the level c, then $\{u_n\}$ is bounded.
Let $u \in H_0^{1,p}(\Omega)$ be a function such that $u_n \rightharpoonup u$, using a similar arguments explored in [18], we can conclude that

$$u_n \to u \text{ in } H_0^{1,p}(\Omega),$$

and Lemma 1 follows. ∎

3. Existence of two solutions (Ekeland's principle)

We will denote by $Q^{\pm}$ the following sets

$$Q^+ = \{t\Phi_1 + w,\ t \geq 0 \text{ and } w \in W\}$$

and

$$Q^- = \{t\Phi_1 + w,\ t \leq 0 \text{ and } w \in W\}.$$

It is easy to see that

$$\partial Q^+ = \partial Q^- = W.$$

Lemma 2. *If conditions (h),(G2) and (G6) hold, then functional I is bounded from below on $H_0^{1,p}(\Omega)$. Moreover, the infimum is negative on Q^+ and Q^-.*

Proof. From condition (G2), its easy to see that I is bounded from below on $H_0^{1,p}(\Omega)$.
Using condition (G6), we have

$$I(t^{\pm}\Phi_1) = \int_{\Omega} G(x, t^{\pm}\Phi_1) \leq \int_{\Omega} T(x)dx < 0,$$

therefore

$$\inf_{u \in Q^{\pm}} I(u) < 0. \ ∎$$

Remark 2. Using condition (G_4) and the definition of the number λ_2, we remark that

$$I(w) \geq \frac{1}{p}\int_{\Omega} |\nabla w|^p - \frac{\lambda_2}{p}\int_{\Omega} h(x)\,|w|^p \geq 0, \ \ \forall w \in W.$$

Therefore Lemma 2 implies that if the infimum of I on $Q^{\pm}$ is achieved by, for example, $u_0^{\pm} \in Q^{\pm}$, we can assume that

(8) $$u_0^{\pm} \in Q^{\pm} \setminus W.$$

This fact is very important when we are working with Ekeland's variational principle.

Theorem 2. *If conditions (h),(G1),(G2),(G4) and (G6) hold, then there exist $u_1 \in Q^+$ and $u_2 \in Q^-$ solutions of (P), such that*

$$I(u_1),\ I(u_2) < 0.$$

Proof. From the proof of Lemma 2 we can conclude that

$$\inf_{u\in Q^{\pm}} I(u) \leq \int_\Omega G(x,t^{\pm}\Phi_1) \leq \int_\Omega T(x)dx < 0.$$

If

$$\inf_{u\in Q^{\pm}} I(u) = \int_\Omega G(x,t^{\pm}\Phi_1) = I(t^{\pm}\Phi_1) \leq \int_\Omega T(x)dx < 0,$$

occurs we can take $u_1 = t^+\Phi_1$ and $u_2 = t^-\Phi_1$. Otherwise if

$$\inf_{u\in Q^{\pm}} I(u) < \int_\Omega G(x,t^{\pm}\Phi_1) \leq \int_\Omega T(x)dx,$$

holds using the Ekeland's variational principle and the same argument explored in [14], we can show that there exist sequences $\{u_n\} \subset Q^+$ and $\{v_n\} \subset Q^-$ satisfying

$$I(u_n) \to \inf_{u\in Q^+} I(u) \ \text{ and } \ I'(u_n) \to 0,$$

and

$$I(v_n) \to \inf_{u\in Q^-} I(u) \ \text{ and } \ I'(v_n) \to 0.$$

By Lemma 1, there exist u_1 and u_2 such that

$$u_n \to u_1 \ \text{ and } \ v_n \to u_2 \text{ in } H_0^{1,p}(\Omega).$$

Therefore, u_1 and u_2 are solutions of (P) verifying

$$I(u_1) = \inf_{u\in Q^+} I(u) < 0 \ \text{ and } \ I(u_2) = \inf_{u\in Q^-} I(u) < 0,$$

which implies from Remark 2 that $u_1 \in Q^+$ *and* $u_2 \in Q^-$. This completes the proof of Theorem 2. ▪

4. Existence of a third solution (Mountain Pass)

Using condition (G5) and arguing as in [14], we can easily show that

$$G(x,t) \geq \frac{m}{p} h(x)\,|t|^p - C\,|t|^\sigma\,, \quad \forall x\in\Omega, \quad t \in I\!R \tag{9}$$

where $p < \sigma < p^*$ and C is a constant independent of x.
By (9), we have that

$$I(u) \geq \frac{m}{p\lambda_1}\int_\Omega |\nabla u|^p - C\int_\Omega |u|^\sigma\,,$$

and then

$$I(u) \geq \frac{m}{p\lambda_1}\,\|u\|^p + o\left(\|u\|\right), \ \text{ as } \ \|u\| \to 0. \tag{10}$$

Using (G6), we obtain

$$I(t^{\pm}\Phi_1) < 0,$$

which together with (10) imply that there exist $r, \rho > 0$ and $e = t^+\Phi_1$ such that

$$I(u) \geq r > 0, \ \text{ for } \ \|u\| \leq \rho \ \text{ and } \ I(e) < 0.$$

Therefore, using a version of the Mountain Pass Theorem without a sort of Palais-Smale condition [25, Theorem 6], there exists a sequence $\{u_n\} \subset H_0^{1,p}(\Omega)$ satisfying

$$I(u_n) \to c \geq r > 0 \text{ and } \|I'(u_n)\|_{H_0^{1,p}(\Omega)^*}(1 + \|u_n\|) \to 0. \tag{11}$$

Remark 3. We recall the sequence obtained in (11) was introduced by Cerami in [26].

Theorem 3. *If conditions (h),(G1) -(G3) and (G5)-(G7) hold, then problem (P) has a solution* u_3, *with*

$$I(u_3) > 0.$$

Proof. Let $\{u_n\} \subset H_0^{1,p}(\Omega)$ be the sequence obtained in (11); then arguing as in Lemma 1, if $\{u_n\}$ is unbounded, we can assume that

$$|u_n(x)| \to \infty, \quad \forall x \in \Omega. \tag{12}$$

Using (11), we have

$$o_n(1) = I'(u_n)(u_n) = \|u_n\|^p - \lambda_1 |u_n|_p^p + \int_\Omega g(x, u_n)u_n dx,$$

and then

$$0 \leq \|u_n\|^p - \lambda_1 |u_n|_p^p \leq -\int_\Omega |g(x, u_n)u_n| + o_n(1).$$

Combining (12), (G3) with the inequality above, we conclude that

$$\|u_n\|^p - \lambda_1 |u_n|_p^p \to 0.$$

Now, using the equality

$$c + o_n(1) = I(u_n) = \frac{1}{p}\left[\|u_n\|^p - \lambda_1 |u_n|_p^p\right] + \int_\Omega G(x, u_n(x))dx,$$

together with Fatou Lemma and (G7) we obtain

$$c \leq \limsup_{n\to\infty} \int_\Omega G(x, u_n(x))dx \leq \int_\Omega S(x)dx \leq 0,$$

which is a contradiction , because $c > 0$ by (11). Then $\{u_n\}$ is bounded. Let $u_3 \in H_0^{1,p}(\Omega)$ be such that

$$u_n \rightharpoonup u_3. \tag{13}$$

By a similar argument explored in [18], we have that

$$u_n \to u_3 \text{ in } H_0^{1,p}(\Omega), \tag{14}$$

and consequently

$$I(u_3) = c \geq r > 0 \text{ and } I'(u_3) = 0,$$

which shows that u_3 is a solution of problem (P). ∎

5. Proof of Theorem 1

Theorem 1 is an immediate consequence of Theorems 2 and 3. ∎

6. Example

Making $\Omega = (0,1),\ p = 2$, and $h = 1$, we shall give an elementary example of a nonlinearity g verifying the set of assumptions.
We recall that $\lambda_n = n^2$, $n = 1, 2, ...$ are eigenvalues of (P_A) and $\Phi_1 = \sin \pi x$ is the first eigenvalue of (P_A).
Let $g : \Omega \times I\!R \to I\!R$ defined by

$$g(x,s) = R(x) g_1(s),$$

where $g_1 : I\!R \to I\!R$ is given by

$$g_1(s) = \begin{cases} s, & \text{for} \quad 0 \le s \le 1, \\ 2 - s, & \text{for} \quad 1 < s \le 5, \\ s - 8 & \text{for} \quad 5 < s \le 8 + \frac{\sqrt{30}}{2}, \\ 8 + \sqrt{30} - s, & \text{for} \quad 8 + \frac{\sqrt{30}}{2} < s \le 8 + \sqrt{30}, \\ 0 & \text{for} \quad s \ge 8 + \sqrt{30}, \\ -g(-s) & \text{for} \quad s \le 0, \end{cases}$$

and $R : \Omega \to I\!R$ is defined by

$$R(x) = \begin{cases} 4x + 1, & \text{for} \quad 0 \le x \le \frac{1}{2}, \\ -4x + 5, & \text{for} \quad \frac{1}{2} \le x \le 1. \end{cases}$$

Then

$$G_1(s) = \int_0^s g_1(t)dt \quad \text{and} \quad G(x,s) = \int_0^s g(x,t)dt = R(x)G_1(s)$$

and

$$S(x) = T(x) = -\frac{R(x)}{2}.$$

By the definition of g, it is easy to see that it verifies the conditions (Gi) for i$\neq 6$.
Thus, we shall prove that G satisfies (G6), for $t^+ = 8$.
Indeed, observe that

$$G(x, 8\Phi_1(x)) = G(1 - x, 8\Phi_1(1 - x)),\ \ x \in \Omega,$$

that is, the function above is symmetric with respect to $x = \frac{1}{2}$.
Then,

$$\begin{aligned} \int_0^1 G(x, 8\Phi_1(x))\, dx &= 2\int_0^{\frac{1}{2}} R(x) G_1(8\Phi_1(x)) dx \\ &= 2\int_0^{\frac{1}{2}} 4x\, G_1(8\Phi_1(x)) dx + 2\int_0^{\frac{1}{2}} G_1(8\Phi_1(x)) dx \\ &\equiv I_1 + I_2. \end{aligned}$$

Now, we shall estimate each integrals $I_j, (j = 1, 2)$. Since $G_1(2+\sqrt{2}) = 0$, choosing $\alpha_0 \in I\!R$ such that $8\Phi_1(\alpha_0) = 2+\sqrt{2}$, which satisfies $0 < \alpha_0 < \frac{1}{6}$, we obtain

$$I_1 \leq 2\int_0^{\alpha_0} 4x\, G_1(8\Phi_1(x))dx \leq 8\alpha_0 < \frac{4}{3}.$$

On the other hand,

$$\begin{aligned} I_2 &= 2\left(\int_0^{\frac{1}{6}} + \int_{\frac{1}{6}}^{\frac{1}{3}} + \int_{\frac{1}{3}}^{\frac{1}{2}}\right)G_1(8\Phi_1(x))dx \\ &\leq 2\left(\int_0^{\frac{1}{6}} G_1(2)dx + \int_{\frac{1}{6}}^{\frac{1}{3}} G_1(4)dx + \int_{\frac{1}{3}}^{\frac{1}{2}} G_1(6)dx\right) \\ &\leq -2. \end{aligned}$$

Therefore,

$$\int_0^1 G(x, 8\Phi_1(x))\, dx = I_1 + I_2 < -\frac{2}{3} < \int_0^1 T(x)dx.$$

Analogously for $t^- = -8$. This proves that G satisfies (G6). ∎

Acknowledgments. The authors are grateful to the referee for his helpful remarks. Also, the first and the third author wish to thank the Departamento de Matemática, Universidade Federal de Minas Gerais, for the warm hospitality.

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