MODELLING OF THE CZOCHRALSKI FLOW

JAN FRANCŮ

ABSTRACT. The Czochralski method of the industrial production of a silicon single crystal consists of pulling up the single crystal from the silicon melt. The flow of the melt during the production is called the Czochralski flow. The mathematical description of the flow consists of a coupled system of six P.D.E. in cylindrical coordinates containing Navier-Stokes equations (with the stream function), heat convection-conduction equations, convection-diffusion equation for oxygen impurity and an equation describing magnetic field effect.

This paper deals with the analysis of the system in the form used for numerical simulation. The weak formulation is derived and the existence of the weak solution to the stationary and the evolution problem is proved.

INTRODUCTION

Single crystal (monocrystal) silicon is an important raw material for electronic semiconductor parts. It is produced from polycrystalline silicon. The most important methods for producing silicon single crystals are floating-zone method and the Czochralski method. The latter consists in pulling up the single crystal from silicon melt in a device called Czochralski puller. Since impurities in the melt (mostly oxygen atoms from the silica (SiO_2) walls of the pot) build in the single crystal, the producers are interested in the character of the melt flow. The flow is not visible, it is very hard to measure during the procedure, therefore producers are interested in the mathematical modelling of the flow on computers.

We shall call this flow of the melt in the Czochralski puller during the single crystal growth *Czochralski flow.* The mathematical model of the flow

---

1991 *Mathematics Subject Classification.* 76D05, 76Rxx, 35Q10, 46E35.

Key words and phrases. mathematical modelling, viscous flow, Czochralski method, single crystal growth, weak solution, operator equation, existence theorem, weighted Sobolev spaces, Rothe method.

This research was partially supported by grants No. 201/95/1557 and 201/97/0153 of the Grant Agency of the Czech Republic.

Received: August 3, 1996.

used for numerical simulation is represented in the following system of six coupled partial differential equations

$$
\begin{gathered}
\frac{\partial S}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ruS)+\frac{\partial}{\partial z}(wS)+\frac{\partial}{\partial z}\left(\frac{\Omega^2}{r^4}\right) \\
=\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 S\right)\right]+\frac{\partial^2 S}{\partial z^2}\right]+\alpha_T\frac{1}{r}\frac{\partial T}{\partial r}+\alpha_C\frac{1}{r}\frac{\partial C}{\partial r}+\alpha_m\frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2}\,, \\
\frac{\partial \Omega}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ru\Omega)+\frac{\partial}{\partial z}(w\Omega)=\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{\Omega}{r^2}\right)\right]+\frac{\partial^2\Omega}{\partial z^2}\right]-\alpha_m\frac{\partial\chi}{\partial z}\,, \\
\frac{\partial T}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}(ruT)+\frac{\partial}{\partial z}(wT)=\nu_T\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right], \\
\frac{\partial C}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}(ruC)+\frac{\partial}{\partial z}(wC)=\nu_C\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)+\frac{\partial^2 C}{\partial z^2}\right], \\
\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\;=-rS\,, \\
\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\chi}{\partial z^2}\;=\frac{1}{r}\frac{\partial\Omega}{\partial z}
\end{gathered}
$$

for unknown functions S,Ω,T,C,ψ,χ ($u=u(\psi), w=w(\psi)$). For the meaning of the other variables and constants see List of symbols preceeding Section 1. The system is accompanied by boundary conditions, see Section 2. A brief derivation of the system and comments can be found in Section 2.

There are many papers dealing with modelling of the Czochralski flow, e. g. [6], [15], [8], [2], [7], [11]. Usually the above introduced system (often without unknowns C and χ and their equations) is studied from the physical point of view, several discretization schemes and numerical experiments are introduced.

On the other hand there is an extensive bibliography dealing with the Navier-Stokes system and its analysis, e. g. [3], [21], [1], [14]. But the Navier-Stokes system is usually uncoupled, formulated in terms of velocity vector (not in terms of flow function) in Cartesian coordinates (not in cylindrical coordinates) and mostly with homogeneous Dirichlet boundary conditions.

The aim of this paper is to give a precise weak formulation of the problem and to prove existence of the weak solution. We shall investigate the system in the form which is used for numerical simulation. In contrast to the pure mathematics which often solves *what can be done as it ought to be done* and the applied mathematics which solves *what ought to be done as it can be done* the paper is an attempt to solve *what ought to be done as it ought to be done.* Thus we use cylindrical coordinates, Navier-Stokes equations with the flow function and derived variables Svanberg vorticity S, swirl Ω etc.

The problem is rather complicated. Special difficulties arise from the so-called “wet axis”, in the cylindrical coordinates the coefficients have singularities, which involves the use of weighted Sobolev spaces. The Navier-Stokes equations are formulated here in terms of stream function, vorticity and swirl. They are coupled with heat convection-conduction equation and oxygen concentration convection-diffusion equation. The last equation in the system describes the effect of the axial magnetic field. The system is evolution but not in all unknowns, it is elliptic in χ.

In the paper we derive the weak formulation, justify it and prove the existence of the weak solution to both the stationary and evolution problem. After setting the problem in Section 1, the mathematical model is briefly derived from its physical grounds in Section 2. The integral identities derived in Section 3 form the base for the weak formulation of the problem. Section 4 contains weighted function spaces and inequalities that are used in Section 5 and 6. The stationary problem is studied in Section 5. The problem is reformulated into an operator equation for a vector of unknowns. The existence of the solution is proved by means of an abstract existence theorem for weakly continuous operators, see [5]. The evolution problem with time dependent data is studied in Section 6. The weak formulation is derived and justified. The existence of the solution is proved by means of the Rothe method.

List of symbols.

V — the melt volume in Cartesian coordinates (x_1, x_2, x_3)
r, φ, z — cylindrical coordinates, t — time
G, Γ — the melt "volume" in the r–z plane and its boundary
$\Gamma_p, \Gamma_s, \Gamma_c, \Gamma_a$ — parts of the boundary (pot, free surface, crystal, axis)
$n \equiv (n_r, n_z)$ — the unit vector of outer normal to Γ
$s \equiv (-n_z, n_r)$ — the unit tangent vector to Γ
$\partial/(\partial n), \partial/(\partial s)$ the normal and tangent derivatives, see (3.4)
u, v, w — the r, φ, z-components of velocity of the flow
Ω — swirl (angular momentum, $\Omega = rv$)
S, ψ — Svanberg vorticity and stream function, see (2.6), (2.8)
T — temperature of the melt
C — oxygen concentration in the melt
χ — stream function for induced electric current in the melt

A_k, B — generalized Laplace and convection operator, see (2.18)
ν, ν_T, ν_C, α_T, α_C, α_m, β_T, β_C, γ_T, γ_C, g_T, g_C, o_p, o_c, T_p, C_p, T_c
— constants and data functions, see Section 2, Summary of data

$\widetilde{\psi}, \widetilde{\Omega}, \widetilde{T}, \widetilde{C}, \widetilde{\chi}$ — test functions related to the unknowns ψ, Ω, T, C, χ
$a_1(u, v)$, $a_{-1}(u, v)$, $\mathcal{a}(u, v)$ — bilinear forms, see (3.5), (3.19)
$b(\psi, u; v)$ — convective trilinear form, see (3.6)
Ω_b, T_b, C_b — auxiliary functions having
prescribed boundary values for the unknowns Ω, T, C
$\mathcal{M}(G)$ — space of Lebesgue measurable functions on G
$L^p_r(G)$, $L^p_{1/r}(G)$ — Lebesgue spaces with weight r and $1/r$
$\|u\|_{p;r}$, $\|u\|_{p;1/r}$ — corresponding norms
(u, v) — scalar product in $L^2_r(G)$
$W^{1,2}_r(G)$, $W^{1,2}_{1/r}(G)$, $W^{2*,2}_{1/r}(G)$ — weighted Sobolev spaces

$\|u\|_{1,2;r}$, $\|u\|_{1,2;1/r}$, $\|u\|_{2*,2;1/r}$ — the corresponding norms, see Section 4
$|u|_{1,2;r}, |u|_{1,2;1/r}, |u|_{2*,2;1/r}$ — the corresponding seminorms, see Section 4
$V_\psi, V_\Omega, V_T, V_C, V_\chi$ — function spaces for unknowns ψ, Ω, T, C, χ
k_T, k_C, k_χ — positive multiplicative constants
$\mathrm{U} = (\psi, \Omega, T, C, \chi)$ — vector of the unknowns
$\mathrm{V} = (\widetilde{\psi}, \widetilde{\Omega}, \widetilde{T}, \widetilde{C}, \widetilde{\chi})$ — vector of the test functions
$\mathbb{W}, \mathbb{V}, \mathbb{H}, \mathbb{H}_0$ — spaces of vector functions, see (5.8), (5.9), (6.4)
$\|\mathrm{U}\|_{\mathbb{W}}$, $\|\mathrm{U}\|_{\mathbb{H}}$, $\|\mathrm{U}\|_{\mathbb{H}_0}$ — the corresponding norms, see (5.17), (6.7), (6.6)
$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}_0, \mathscr{F}_s, \mathscr{F}_e$
— operators and functionals, see (5.10) – (5.15), (6.1), (6.2)
$\mathscr{A}^i, \dots, \mathscr{T}^i, \mathrm{U}^i$ — semidiscretizated functions, see (6.26), (6.27)
$\overline{\mathrm{U}}_n, \widehat{\mathrm{U}}_n$ — Rothe stair function (6.34) and Rothe polygonal function (6.35)
$\overline{\mathscr{A}}_n, \dots, \overline{\mathscr{T}}_n$ — stair approximations, see (6.36).

1. Setting up the problem

We shall deal with modelling of melt flow during single crystal growth by the Czochralski method in a device called the crystal puller or Czochralski device.

Czochralski device. The apparatus is outlined in Fig. 1. The heart of the device consists of a melting pot (crucible) set on a turning base.

Polycrystalline silicon is put into the pot (crucible) and heated by a ring of electric carbon heaters around the pot. When the silicon is melted, a single crystal nucleus tightened in a turning hanger touches the surface of the melt. The single crystal starts “growing” as the silicon melt contacts the silicon solid. Both the pot and the hanger rotate around the common vertical axis (usually with the opposite orientation) to obtain the axially symmetric single crystal. The pot and the hanger are movable also in the vertical direction to set up a suitable position in the middle of the heaters as the melt level decreases and the single crystal grows.

The diameter of the single crystal is controlled by speed of pulling up the single crystal and also by changing the heat power. The single crystal grows in a protective inert atmosphere and in an axial magnetic field produced by an electromagnetic coil. Also other types of magnetic fields have been used but we will not deal with them in this paper. Czochralski crystals can also be grown with no magnetic field, this case is included by setting the constant $\alpha_m = 0$ and omitting the equation for χ.

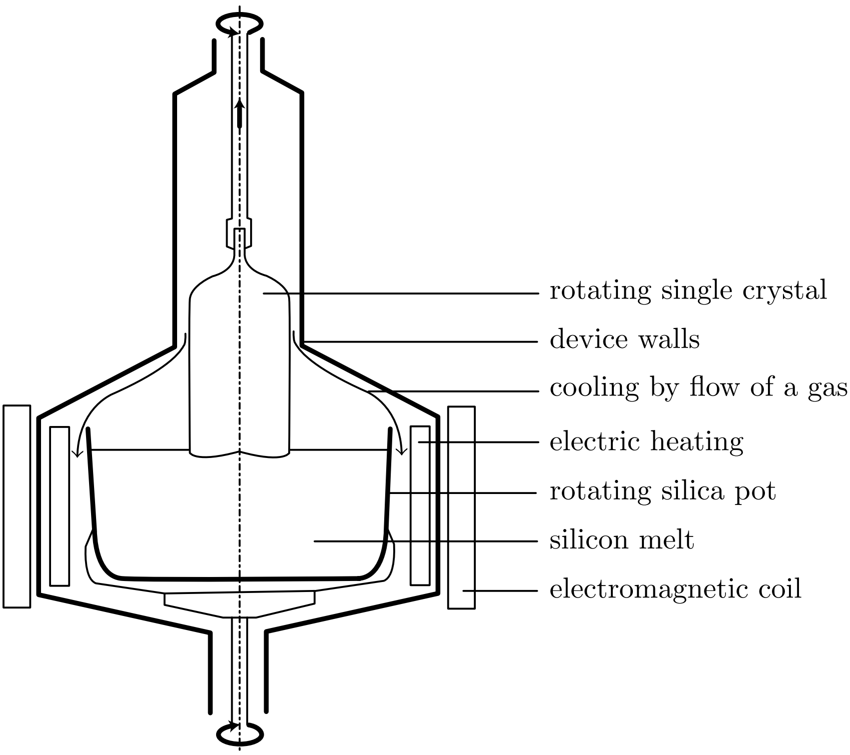

FIG. 1. CZOCHRALSKI DEVICE.

At the walls of the silica (SiO_2) melting pot the atoms of oxygen get into the melt, and on the free surface of the melt they escape into the atmosphere. Thus character of the flow influences oxygen concentration of the melt in the area of crystallization and successively oxygen concentration in the single crystal.

We shall deal with the model taking into account the following phenomena:

— incompressible viscous liquid
— axially symmetric flow in a cylindrical domain
— forced convection caused by rotation of the melting pot and the crystal
— natural convection driven by thermal expansion buoyance and oxygen concentration expansion buoyance
— Marangoni convection caused by surface tension variations in the free surface of the liquid
— thermal convection and conduction in the melt
— oxygen concentration convection and diffusion in the melt
— forces caused by the external magnetic field inducing electric current in the melt.

2. The mathematical model

We shall confine our modelling to the region V of the melt in the melting pot. We get use of axial symmetry of the problem. In this section we briefly derive the system of partial differential equations on domain G with corresponding boundary conditions.

Geometry of the problem. We shall assume that the region occupied by the melt is constant and known. This region is denoted by V in Cartesian coordinates $x = (x_1, x_2, x_3)$. In the cylindrical coordinates (r, φ, z) given by

$$x_1 = r\cos\varphi \qquad x_2 = r\sin\varphi \qquad x_3 = z$$

the region V corresponds (up to a zero measure set) to $G \times (0, 2\pi)$. The domain G represents a radial cross-section of V in the r, z–half plane ($r > 0$).

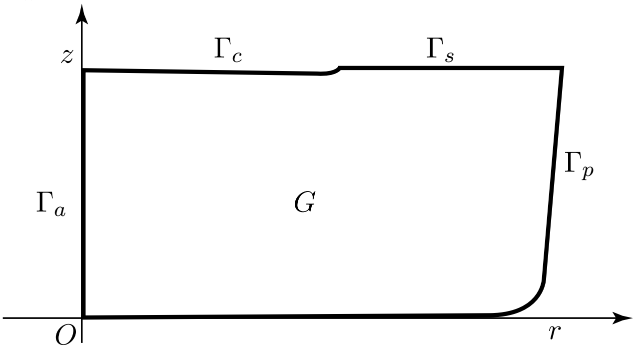

Fig. 2. Domain G and its boundary.

Due to symmetry of the device we shall assume *axial symmetry* of the problem, i. e. all variables are independent of φ. Thus the problem will be considered in the domain G. Boundary Γ of the domain G is divided into four parts, see Fig. 2:

Γ_p — contact with the bottom and wall of the melting <u>p</u>ot,
Γ_s — free <u>s</u>urface of the melt,
Γ_c — contact with the <u>c</u>rystal and
Γ_a — <u>a</u>xis of the symmetry.

We shall assume that the free surface of the melt has a plane shape, i. e. Γ_s is a subset of a line $z = \text{const}$.

We deduce the model for evolution (time-dependent) case with time variable t. Omitting the terms with time derivative we obtain the stationary problem.

Equations of motion. The flow of an incompressible viscous liquid is described by the Navier-Stokes system of equations. In the cylindrical coordinates for axially symmetric problem the system reads as follows

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} + w\,\frac{\partial u}{\partial z} - \frac{1}{r}\,v^2 = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(r\,u)\right) + \frac{\partial^2 u}{\partial z^2}\right] - \frac{\partial p}{\partial r} + f_r\,, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial r} + w\,\frac{\partial v}{\partial z} + \frac{1}{r}\,u\,v = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(r\,v)\right) + \frac{\partial^2 v}{\partial z^2}\right] + f_\varphi\,, \tag{2.2}$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial r} + w\,\frac{\partial w}{\partial z} = \nu\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right] - \frac{\partial p}{\partial z} + f_z\,, \tag{2.3}$$

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{1}{r}\,u = 0\,, \tag{2.4}$$

where u, v, w are the r, φ, z–components of velocity vector, ν the kinematic viscosity coefficient ($1/\nu$ corresponds to the Reynolds number), p the kinematic pressure i. e. the real pressure divided by the mean density ρ_0 from (2.12) and f_r, f_φ, f_z are components of the volume force vector $\mathbf{f}$. It consists of force $\mathbf{f}_V$ caused by gravity and volume expansion and of force $\mathbf{f}_m$ caused by outer magnetic field, $\mathbf{f} = \mathbf{f}_V + \mathbf{f}_m$.

In the literature on Czochralski flow the problem is often formulated in terms of Stokes stream function ψ, Svanberg vorticity S and swirl Ω (angular moment) instead of velocity components u, v, w.

Variable Ω — *swirl* defined by $\Omega = r\,v$ is used instead of φ-component v of velocity vector. Replacing v in (2.2) by Ω we obtain

$$\frac{\partial \Omega}{\partial t} + u\,\frac{\partial \Omega}{\partial r} + w\,\frac{\partial \Omega}{\partial z} = \nu\left[\frac{\partial^2 \Omega}{\partial r^2} + \frac{\partial^2 \Omega}{\partial z^2} - \frac{1}{r}\,\frac{\partial \Omega}{\partial r}\right] + r\,f_\varphi\,. \tag{2.5}$$

Variable S — *Svanberg vorticity* is defined as a negative $1/r$ multiple of the φ-component of vorticity ω

$$S = -\frac{1}{r}\,\omega_\varphi \equiv \frac{1}{r}\left[\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right]\,. \tag{2.6}$$

Subtracting equation (2.1) differentiated with respect to z and (2.3) differentiated with respect to r we obtain the equation for S

$$\begin{aligned} &\frac{\partial S}{\partial t} + \frac{1}{r}\,\frac{\partial}{\partial r}(ruS) + \frac{\partial}{\partial z}(wS) + \frac{\partial}{\partial z}\left(\frac{\Omega^2}{r^4}\right) \\ &= \nu\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(r^2 S)\right) + \frac{\partial^2 S}{\partial z^2}\right] + \frac{1}{r}\frac{\partial f_z}{\partial r} - \frac{1}{r}\frac{\partial f_r}{\partial z}\quad, \end{aligned} \tag{2.7}$$

where v was replaced by Ω/r.

Due to continuity equation (2.4) there exists a function ψ describing the r, z-components of velocity vector:

$$u = \frac{1}{r}\,\frac{\partial \psi}{\partial z}\,, \qquad w = -\,\frac{1}{r}\,\frac{\partial \psi}{\partial r}\,. \tag{2.8}$$

The equation (2.4) can be omitted since each functions u, w defined by (2.8) satisfy (2.4). On the other hand we have to add a relation between S and ψ. Replacing u and w by ψ in (2.6) we obtain the last equation:

$$r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} = -r^2\,S \tag{2.9}$$

Equation for temperature and oxygen concentration. Equation describing heat convection and conduction in cylindrical coordinates for axisymmetric problem admits the form

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial r} + w\,\frac{\partial T}{\partial z} = \nu_T \left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right], \tag{2.10}$$

where T is temperature, ν_T coefficient of thermal diffusivity. The equation describing diffusion and transport of oxygen in the melt is of the same form

$$\frac{\partial C}{\partial t} + u\,\frac{\partial C}{\partial r} + w\,\frac{\partial C}{\partial z} = \nu_C \left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial C}{\partial r} \right) + \frac{\partial^2 C}{\partial z^2} \right], \tag{2.11}$$

where C is oxygen concentration and ν_C diffusion coefficient.

Volume expansion. The temperature and oxygen concentration variations cause volume expansion and consequently density variations

$$\rho = \rho_0(1-\delta)\,, \qquad \delta \doteq \mathrm{const}_T(T-T_o) + \mathrm{const}_C(C-C_o)\,. \tag{2.12}$$

The linear dependence of ρ on T and C is sometimes called Boussinesq approximation. The gravity force $(0,0,-\rho g)$ acting on the melt can now be written as $\rho_0 \mathbf{f}_V$ with

$$\mathbf{f}_V = (0,0,f_V)\,, \qquad f_V = -g + \alpha_T(T-T_0) + \alpha_C(C-C_0)\,. \tag{2.13}$$

The constants α_T, α_C include both the volume expansion coefficients and the gravitational acceleration g.

Magnetic field. We suppose that the electromagnetic coil installed around the melting pot creates in the melt a known homogeneous axial magnetic field. The field is described by the magnetic induction vector $\mathbf{B} = (0,0,B_z)$ with single nonzero component. In the moving melt the magnetic field induces an electric field $\mathbf{E}$ and an electric current $\mathbf{j}$.

Since the electric current $\mathbf{j}$ satisfies continuity equation (the same equation as (2.4) for velocity vector), its components j_r, j_z can be expressed by means of an electric current stream function denoted by χ

$$j_r = \frac{1}{r}\,q\,B_z\,\frac{\partial \chi}{\partial z}\,, \qquad j_z = -\frac{1}{r}\,q\,B_z\,\frac{\partial \chi}{\partial r}\,.$$

where q is the constant of electrical conductivity. On the other hand $\mathbf{j}$ obeys Ohm's law $\mathbf{j} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Moreover the field $\mathbf{E}$ is potential, thus $\mathbf{E} = \nabla\Phi$. Comparing these four relations we obtain

$$j_r = \frac{1}{r}\,q\,B_z\,\frac{\partial \chi}{\partial z} = q\,\frac{\partial \Phi}{\partial r} + q\,v\,B_z\,, \qquad j_z = -\,\frac{1}{r}\,q\,B_z\,\frac{\partial \chi}{\partial r} = q\,\frac{\partial \Phi}{\partial z}\,.$$

Combining derivatives $\partial_z j_r - \partial_r j_z$ we obtain the equation for electric stream function

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial \chi}{\partial r} \right) + \frac{1}{r}\,\frac{\partial^2 \chi}{\partial z^2} = \frac{\partial v}{\partial z}\,. \tag{2.14}$$

Since the neighbourhood of the melt is electrically insulating the equation is completed by natural boundary conditions $\chi = 0$ on boundary Γ.

Finally the magnetic field acts on the moving melt by the force $\mathbf{f}_m = \mathbf{j} \times \mathbf{B}$. Inserting for $\mathbf{j}$ we obtain the second part of volume force $\mathbf{f}$

$$\mathbf{f}_m = (f_r, f_\varphi, 0)\,, \qquad f_r = -\,\frac{1}{r}\, q\, B_z^2\, \frac{\partial \psi}{\partial z}\,, \quad f_\varphi = -\,\frac{1}{r}\, q\, B_z^2\, \frac{\partial \chi}{\partial z}\,. \tag{2.15}$$

Geometric boundary conditions. The forced convection is caused by rotation of the melting pot and by rotation or counter-rotation of the crystal. Let us denote the angular velocity of the pot by o_p and of the crystal by o_c. Then due to assumption of viscous flow we have the following geometric boundary conditions

$$u = w = 0 \quad \text{on } \Gamma_p \cup \Gamma_c\,, \qquad v = r \cdot o_p \quad \text{on } \Gamma_p\,, \qquad v = r \cdot o_c \quad \text{on } \Gamma_c\,.$$

On the plane free surface and the axis of symmetry the normal component of the velocity vector equals to zero

$$u\, n_r + w\, n_z \equiv w = 0 \quad \text{on } \Gamma_s\,, \qquad u = 0 \quad \text{on } \Gamma_a\,.$$

We have to rewrite these conditions for variables ψ and Ω. The stream function ψ has zero tangent derivatives on Γ, thus we can put $\psi = 0$ on Γ. Moreover we have $\nabla\, \psi = 0$ on $\Gamma_p \cup \Gamma_c \cup \Gamma_a$. Conditions for v yields conditions for Ω:

$$\Omega = r^2 o_p \quad \text{on } \Gamma_p\,, \qquad \Omega = r^2 o_c \quad \text{on } \Gamma_c\,.$$

Conditions on the free surface. On the free surface of the melt the surface tension variations occur due to temperature and concentration gradients. This surface tension variations produce shear stress which generates a surface flow — the so-called Marangoni effect.

Let us suppose linear dependence of the surface tension A on T and C

$$A = A_o[1 - \mathrm{const}_T(T - T_o) - \mathrm{const}_C(C - C_o)]\,. \tag{2.16}$$

The shear stress is given by the surface gradient of A and it represents the only tangential surface force acting on the free surface. Denoting the stress tensor by τ we have

$$\mathbf{t} \cdot \nabla A = \mathbf{t} \cdot \tau \mathbf{n}$$

for any tangential $\mathbf{t}$ and the normal vector $\mathbf{n} = (n_r, 0, n_z)$ to the surface. Between the stress tensor τ and the stretching tensor $\varepsilon(\mathbf{v}) = (\nabla \mathbf{v} + (\nabla \mathbf{v})^\top)/2$ we assume linear dependence (Newton law) $\tau = 2\,\nu\rho\,\varepsilon(\mathbf{v})$. Combining these relations we obtain

$$\mathbf{t} \cdot \nabla A = \nu\rho\, \mathbf{t} \cdot [\nabla \mathbf{v} + (\nabla \mathbf{v})^\top]\, \mathbf{n} \tag{2.17}$$

In our case of plane free surface Γ_s we have $\mathbf{n} = (0, 0, 1)$. First in (2.17) we take the tangent vector $\mathbf{t} = (0, 1, 0)$. Since A, u, w are independent of φ on the plane surface we obtain $0 = -\nu\rho\,(\partial v)/(\partial z)$ which rewritten for Ω yields the condition

$$\frac{\partial \Omega}{\partial n} \equiv \frac{\partial \Omega}{\partial z} = 0 \quad \text{on } \Gamma_s\,.$$

Then in (2.17) we take the tangent vector in the r,z plane $\mathbf{t} = (1,0,0)$. We obtain

$$\frac{\partial A}{\partial r} = \nu\rho\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).$$

Since $w = 0$ on Γ_s we have $\partial w/\partial r = 0$ and using (2.6) and (2.16), we can rewrite the boundary condition for S

$$S = \beta_T\,\frac{1}{r}\,\frac{\partial T}{\partial r} + \beta_C\,\frac{1}{r}\,\frac{\partial C}{\partial r} \quad \text{on } \Gamma_s$$

with material constants β_T and β_C.

Boundary conditions for temperature and concentration. We shall assume that the temperature is known at the pot walls and crystal interface: $T = T_p$ on Γ_p and $T = T_c$ on Γ_c. At the free surface we consider heat flow caused by both the conduction to cooling inert gas and the radiation to device's walls. The linearized heat flow can be described by

$$\frac{\partial T}{\partial z} = g_T - \gamma_T T \quad \text{on } \Gamma_s\,,$$

with a function g_T and a constant γ_T. Symmetry of the problem yields $\partial T/\partial r = 0$ at axis Γ_a.

Similarly we assume that the concentration is known at the pot walls, it is symmetric at the axis and no segregation occurs at the crystal interface:

$$C = C_p \quad \text{on } \Gamma_p\,, \qquad \frac{\partial C}{\partial r} = 0 \quad \text{on } \Gamma_a\,, \qquad \frac{\partial C}{\partial n} = 0 \quad \text{on } \Gamma_c\,.$$

On the free surface we consider an oxygen flow due to evaporating. The linearized flow can be described by

$$\frac{\partial C}{\partial z} = g_C - \gamma_C C \quad \text{on } \Gamma_s$$

with a function g_C and a constant γ_C. The last condition is often replaced by $C = 0$, in that case also $\gamma_C = 0$.

Summary of differential equations. The mathematical model of the Czochralski flow consists of Navier-Stokes equations (2.5), (2.7), (2.9). We used (2.4), inserted coupling volume forces (2.13), (2.15) and set $\alpha_m = qB_z^2$. Adding equations (2.10), (2.11) and (2.14) and using (2.4) we obtain a system of six partial differential equations mentioned in Introduction, that are used in papers dealing with numerical simulations of Czochralski flow.

Now we substitute u and w from (2.8). We shall simplify the notation of the system. In the equations two types of operators appear: a generalized Laplace operator and a convection operator. Denoting the operators by A_k and B

$$A_k(f) = -\frac{k}{r}\frac{\partial f}{\partial r} - \frac{\partial^2 f}{\partial r^2} - \frac{\partial^2 f}{\partial z^2}\,, \quad B(\psi, f) = \frac{1}{r}\left(\frac{\partial \psi}{\partial z}\,\frac{\partial f}{\partial r} - \frac{\partial \psi}{\partial r}\,\frac{\partial f}{\partial z}\right) \tag{2.18}$$

we can rewrite the system as follows:

$$\frac{\partial S}{\partial t} + \nu\, A_3(S) + B(\psi, S) + \frac{1}{r^4}\,\frac{\partial}{\partial z}\left(\Omega^2\right) = \tag{2.19}$$

$$= \alpha_T \frac{1}{r}\frac{\partial T}{\partial r} + \alpha_C \frac{1}{r}\frac{\partial C}{\partial r} + \alpha_m \frac{1}{r^2}\frac{\partial^2 \psi}{\partial z^2},$$

$$\frac{\partial \Omega}{\partial t} + \nu\, A_{-1}(\Omega) + B(\psi, \Omega) = -\,\alpha_m \frac{\partial \chi}{\partial z}, \tag{2.20}$$

$$\frac{\partial T}{\partial t} + \nu_T A_1(T) + B(\psi, T) = 0, \tag{2.21}$$

$$\frac{\partial C}{\partial t} + \nu_C A_1(C) + B(\psi, C) = 0, \tag{2.22}$$

$$A_{-1}(\psi) = r^2 S, \tag{2.23}$$

$$A_{-1}(\chi) = -\frac{\partial \Omega}{\partial z}. \tag{2.24}$$

Summary of boundary conditions. The system of differential equations is completed with the following system of boundary conditions:

— at the melting pot wall Γ_p:

$$\Omega = r^2 o_p, \quad T = T_p, \quad C = C_p, \quad \psi = 0, \quad \nabla\psi = 0, \quad \chi = 0, \tag{2.25}$$

— at the crystal interface Γ_c:

$$\Omega = r^2 o_c, \quad T = T_c, \quad \frac{\partial C}{\partial n} = 0, \quad \psi = 0, \quad \nabla\psi = 0, \quad \chi = 0, \tag{2.26}$$

— at the free surface Γ_s:

$$\begin{gathered} S = \beta_T \frac{1}{r}\frac{\partial T}{\partial r} + \beta_C \frac{1}{r}\frac{\partial C}{\partial r}, \qquad \frac{\partial \Omega}{\partial z} = 0, \\ \frac{\partial T}{\partial z} = g_T - \gamma_T T, \qquad \frac{\partial C}{\partial z} = g_C - \gamma_C C, \qquad \psi = 0, \qquad \chi = 0, \end{gathered} \tag{2.27}$$

— and at the symmetry axis Γ_a:

$$\Omega = 0, \quad \frac{\partial T}{\partial r} = 0, \quad \frac{\partial C}{\partial r} = 0, \quad \psi = 0, \quad \nabla\psi = 0, \quad \chi = 0. \tag{2.28}$$

Summary of the data. The data describing the problem can be divided into two groups: constants of the constitutive relations i. e. material properties and operational data.

Coefficients of the constitutive relations

ν — silicon melt viscosity, $\nu > 0$,

ν_T — thermal diffusivity of the silicon melt, $\nu_T > 0$,

ν_C — oxygen diffusion coefficient in the silicon melt, $\nu_C > 0$,

α_T, α_C — coefficient of buoyance caused by thermal and oxygen volume expansion in the gravitation field. The constants determine natural convection,

β_T, β_C — coefficients of condition describing the surface flow in the free surface

γ_T, g_T, γ_C, g_C — data in conditions describing the linearized heat and oxygen flow on the free surface. They depend also on the surrounding walls and the flow of cooling gas. We assume $\gamma_T \geq 0$, $\gamma_C \geq 0$. Of course g_T, g_C would rather belong to the operational data.

Operational data

G — the cross-section of the volume of the melt, namely inner dimensions of the melting pot: its radius r_p and height of the melt, crystal diameter etc.,

o_p, o_c — angular velocity of the pot and the crystal rotation. The constants determine the forced convection,

α_m — constant describing effect of the applied magnetic field,

T_p, C_p, T_c — given temperature and oxygen concentration on the pot walls and temperature on the crystal surface.

The data of functional character g_T, g_C, T_p, C_p, T_c may be dependent on the space variables r, z.

In the evolution case the problem is completed by initial conditions giving the value of ψ, Ω, T, C in time $t = 0$. Moreover, the operational data may be time-dependent, namely o_p, o_c, T_p, C_p, T_c, α_m, g_T, g_C may vary in time. In Sections 5 and 6 operational data o_p, o_c, T_p, C_p, T_c will be included in functions Ω_b, T_b, C_b on $\overline{G}$ that have the prescribed values on the corresponding parts of the boundary.

Normalization

For computation the variables are often normalized. The space and time variables are rescaled such that the radius of the pot and the circumference velocity of the pot (or crystal) are of unit magnitude, the temperature T is shifted and rescaled to take its values in $[0, 1]$ and the oxygen concentration C is rescaled to maximum value 1. Then some constants of the system can be expressed by means of dimensionless criteria

$$\begin{array}{lcllcl}
\nu & = & 1/\mathrm{Re} & \alpha_T & = & \mathrm{Gr}/\mathrm{Re}^2 \\
\nu_T & = & 1/(\mathrm{Re}\,\mathrm{Pr}) & \alpha_C & = & \mathrm{Gr_d}/\mathrm{Re}^2 \\
\nu_C & = & 1/(\mathrm{Re}\,\mathrm{Sc}) & \beta_T & = & \mathrm{Mn}/(\mathrm{Re}\,\mathrm{Pr}) \\
\alpha_m & = & \mathrm{St} & \beta_C & = & \mathrm{Mn_d}/(\mathrm{Re}\,\mathrm{Sc})\,,
\end{array}$$

where the criteria are (their values for the real problem are introduced in the brackets — if they are known to the author)

Re	—	Reynolds number	$(10^4 - 10^6)$
Pr	—	Prandtl number	(0.024)
Sc	—	Schmidt number	
Gr	—	Grashof number	$(10^8 - 10^{10})$
Gr_d	—	Grashof diffusion number	
Mn	—	Marangoni number	$(10^4 - 10^5)$
Mn_d	—	Marangoni diffusion number	
St	—	Stuart number	$(0 - 10^3)$.

The criteria and their values are taken from the physical part of [17] written by O. Litzman.

3. INTEGRAL IDENTITIES

Integral identities derived in this section will be the base for generalized formulation of the problems.

Each equation of the system (2.19)–(2.24) will be multiplied by a weight r or $1/r$ and by a test function and integrated over G. Using Green's theorem ("integration by parts" in the plane) and taking into account the boundary conditions we obtain integral identities with lower order derivatives.

We shall suppose that all integrals exist and are finite. All computations are based on the following two formulas

$$\int_G \frac{\partial u}{\partial r}\, v \,\mathrm{d}G = \int_\Gamma u\, v\, n_r \,\mathrm{d}\Gamma - \int_G u\, \frac{\partial v}{\partial r} \,\mathrm{d}G\,, \tag{3.1}$$

$$\int_G \frac{\partial u}{\partial z}\, v \,\mathrm{d}G = \int_\Gamma u\, v\, n_z \,\mathrm{d}\Gamma - \int_G u\, \frac{\partial v}{\partial z} \,\mathrm{d}G\,. \tag{3.2}$$

To simplify the notation we denote the scalar product with weight r by

$$(u, v) = \int_G r\, u\, v \,\mathrm{d}G\,. \tag{3.3}$$

We suppose that the normal vector $n = (n_r, n_z)$ exists on the boundary Γ (except for a finite number of points) and we define normal and tangent derivatives by

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} n_r + \frac{\partial u}{\partial z} n_z \qquad \frac{\partial u}{\partial s} = -\frac{\partial u}{\partial r} n_z + \frac{\partial u}{\partial z} n_r\,. \tag{3.4}$$

Since in the equations some operators appear several times we start with couple of lemmas transforming the common integrals simultaneously.

Transformation of some integrals.

Lemma 3.1. *The integrals with operators A_k with $k = 1, -1$ can be transformed as follows*

$$\int_G r^k\, A_k(u) v \,\mathrm{d}G = a_k(u, v) - \int_\Gamma r^k\, \frac{\partial u}{\partial n}\, v \,\mathrm{d}\Gamma\,,$$

where

$$a_k(u, v) = \int_G r^k \left(\frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) \mathrm{d}G\,. \tag{3.5}$$

The proof consists in applying (3.1), (3.2) to the second order derivatives.

Lemma 3.2. *The trilinear form $b(u, v; w)$ born by the operator $B(u, v)$*

$$b\,(u, v; w) \equiv \int_G r\, B(u, v) w \,\mathrm{d}G = \int_G \left(\frac{\partial u}{\partial z}\, \frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\, \frac{\partial v}{\partial z} \right) w \,\mathrm{d}G \tag{3.6}$$

transforms as follows

$$b\,(u,v;w) = -b\,(w,v;u) - \int_\Gamma u\,\frac{\partial v}{\partial s}\,w\,\mathrm{d}\Gamma,$$
$$b\,(u,v;w) = -b\,(u,w;v) + \int_\Gamma \frac{\partial u}{\partial s}\,v\,w\,\mathrm{d}\Gamma.$$

Equation for concentration C **and temperature** T**.** The unknown function C has prescribed values at Γ_p, thus we choose the corresponding condition for its test function $\widetilde{C}$

$$\widetilde{C} = 0 \quad \text{on } \Gamma_p\,. \tag{3.7}$$

We multiply equation (2.22) by r and by test function $\widetilde{C}$ and integrate it over domain G

$$\int_G r\,\frac{\partial C}{\partial t}\,\widetilde{C}\,\mathrm{d}G + \nu_C \int_G r\,A_1(C)\,\widetilde{C}\,\mathrm{d}G + \int_G r\,B(\psi,C)\,\widetilde{C}\,\mathrm{d}G = 0\,.$$

We rewrite the first evolution term using the scalar product (3.3), the second diffusion term is transformed using Lemma 3.1; the boundary condition for C with (3.7) yields an integral over Γ_s, we put it to the right-hand side. The third convective term is rewritten using notation (3.6). Thus we obtain the integral identity corresponding to the equation for C

$$\left(\frac{\partial C}{\partial t}\,,\,\widetilde{C}\right) + \nu_C\,a_1(C,\widetilde{C}) + b\,(\psi,C;\widetilde{C}) = \nu_C \int_{\Gamma_s} r(g_C - \gamma_C C)\widetilde{C}\,\mathrm{d}\Gamma. \tag{3.8}$$

The values for T are prescribed on $\Gamma_p \cup \Gamma_c$ thus we choose its test function $\widetilde{T}$ satisfying

$$\widetilde{T} = 0 \quad \text{on } \Gamma_p \cup \Gamma_c\,. \tag{3.9}$$

We multiply equation (2.21) by $r\,\widetilde{T}$ and integrate it over G

$$\int_G r\,\frac{\partial T}{\partial t}\,\widetilde{T}\,\mathrm{d}G + \nu_T \int_G r\,A_1(T)\widetilde{T}\,\mathrm{d}G + \int_G r\,B(\psi,T)\widetilde{T}\,\mathrm{d}G = 0\,.$$

Again, like in the previous case, we transform the terms using Lemma 3.1 and Lemma 3.2; the boundary integral due to boundary condition yields an integral on the right-hand side. We obtain the integral identity corresponding to the equation for T:

$$\left(\frac{\partial T}{\partial t}\,,\,\widetilde{T}\right) + \nu_T\,a_1(T,\widetilde{T}) + b\,(\psi,T;\widetilde{T}) = \nu_T \int_{\Gamma_s} r(g_T - \gamma_T T)\widetilde{T}\,\mathrm{d}\Gamma. \tag{3.10}$$

Equation for electric flow function χ**.** Due to zero boundary condition for χ we choose the test function $\widetilde{\chi}$ satisfying

$$\widetilde{\chi} = 0 \quad \text{on } \Gamma\,. \tag{3.11}$$

In this case we multiply equation (2.24) by $\widetilde{\chi}/r$ and integrate over G:

$$\int_G \frac{1}{r}A_{-1}(\chi)\widetilde{\chi}\,\mathrm{d}G = -\int_G \frac{1}{r}\,\frac{\partial \Omega}{\partial z}\,\widetilde{\chi}\,\mathrm{d}G\,.$$

Using Lemma 3.1 we obtain the integral identity for χ:

$$a_{-1}(\chi,\widetilde{\chi}) = -\left(\frac{1}{r}\,\frac{\partial\Omega}{\partial z}\,,\,\frac{\widetilde{\chi}}{r}\right)\,. \tag{3.12}$$

Equation for swirl Ω. The boundary condition for Ω are prescribed on $\Gamma-\Gamma_s$. Thus we choose a test function $\widetilde{\Omega}$ satisfying

$$\widetilde{\Omega} = 0 \quad \text{ on } \Gamma-\Gamma_s\,. \tag{3.13}$$

We multiply equation (2.20) by $\widetilde{\Omega}/r$ and integrate it over G:

$$\begin{aligned}\int_G \frac{1}{r}\,\frac{\partial\Omega}{\partial t}\,\widetilde{\Omega}\,\mathrm{d}G + \nu\int_G \frac{1}{r}\,A_{-1}(\Omega)\,\widetilde{\Omega}\,\mathrm{d}G + \int_G \frac{1}{r}\,B(\psi,\Omega)\,\widetilde{\Omega}\,\mathrm{d}G&\\ = -\,\alpha_m\int_G \frac{1}{r}\,\frac{\partial\chi}{\partial z}\,\widetilde{\Omega}\,\mathrm{d}G.&\end{aligned}$$

We rewrite the second term using Lemma 3.1; the boundary integral vanishes due to boundary condition for $\widetilde{\Omega}$. The term on the right hand side is converted by (3.2), the boundary term vanishes due to $\chi=0$ on Γ. Thus we obtain the identity corresponding to the equation for Ω

$$\left(\frac{\partial}{\partial t}\frac{\Omega}{r}\,,\,\frac{\widetilde{\Omega}}{r}\right) + \nu\,a_{-1}(\Omega,\widetilde{\Omega}) + b\left(\psi,\Omega;\frac{\widetilde{\Omega}}{r^2}\right) = \alpha_m\left(\frac{\chi}{r},\frac{1}{r}\,\frac{\partial\widetilde{\Omega}}{\partial z}\right)\,. \tag{3.14}$$

Equation for vorticity S and stream function ψ. The remaining two equations present a problem. On the boundary $\Gamma-\Gamma_s$ the second order equation (2.23) for ψ has two boundary conditions $\psi=0$, $\partial\psi/\partial n=0$ (they are equivalent to $\psi=0, \nabla\psi=0$) while equation (2.19) for S has no boundary condition. If we express S by ψ using equation (2.23)

$$S \equiv S(\psi) = r^{-2}\,A_{-1}(\psi) \tag{3.15}$$

and insert it into equation (2.19) we obtain a fourth order equation for ψ which has two boundary conditions: $\psi=0$ on Γ and

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{ on } \Gamma-\Gamma_s\,, \qquad S(\psi) = \beta_T\,\frac{1}{r}\frac{\partial T}{\partial s} + \beta_C\,\frac{1}{r}\frac{\partial C}{\partial s} \quad \text{ on } \Gamma_s\,.$$

We choose a test function $\widetilde{\psi}$ satisfying

$$\widetilde{\psi} = 0 \quad \text{ on } \Gamma\,, \qquad \nabla\widetilde{\psi} = 0 \quad \text{ on } \Gamma-\Gamma_s\,. \tag{3.16}$$

We multiply equation (2.19) by $r\,\widetilde{\psi}$ and integrate it over domain G:

$$\begin{aligned}\int_G r\,\frac{\partial S(\psi)}{\partial t}\widetilde{\psi}\,\mathrm{d}G + \nu\int_G r\,A_3(S(\psi))\,\widetilde{\psi}\,\mathrm{d}G + \int_G r\,B(\psi,S(\psi))\,\widetilde{\psi}\,\mathrm{d}G&\\ + \int_G \frac{1}{r^3}\,\frac{\partial}{\partial z}(\Omega^2)\,\widetilde{\psi}\,\mathrm{d}G&\\ = \int_G\left[\alpha_T\,\frac{\partial T}{\partial r} + \alpha_C\,\frac{\partial C}{\partial r}\right]\widetilde{\psi}\,\mathrm{d}G + \alpha_m\int_G \frac{1}{r}\,\frac{\partial^2\psi}{\partial z^2}\widetilde{\psi}\,\mathrm{d}G\,.&\end{aligned} \tag{3.17}$$

The first integral can be rewritten using Lemma 3.1; due to boundary conditions the boundary integrals vanish

$$\int_G r\,\frac{\partial S(\psi)}{\partial t}\widetilde{\psi}\,\mathrm{d}G = \int_G \frac{1}{r}\frac{\partial}{\partial t}\left(A_{-1}(\psi)\right)\widetilde{\psi}\,\mathrm{d}G = a_{-1}\left(\frac{\partial\psi}{\partial t},\widetilde{\psi}\right).$$

In the second term we use the following forms of the operators A_3 and A_{-1}

$$\begin{aligned} A_3(S) &= -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 S\right)\right] - \frac{\partial^2 S}{\partial z^2}, \\ A_{-1}(\psi) &= -r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{\partial^2\psi}{\partial z^2} \end{aligned} \tag{3.18}$$

and the computation with double use of (3.1) and (3.2) yields

$$\int_G r\,A_3(S(\psi))\,\widetilde{\psi}\,\mathrm{d}G = -\int_\Gamma \frac{1}{r}\frac{\partial}{\partial n}\left(r^2\,S(\psi)\right)\widetilde{\psi}\,\mathrm{d}\Gamma + \int_\Gamma r\,S(\psi)\frac{\partial\widetilde{\psi}}{\partial n}\,\mathrm{d}\Gamma$$

$$+\int_G r\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\right]\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial z^2}\right]\mathrm{d}G\,.$$

Due to (3.16) the first integral over Γ is zero and the second one vanishes on $\Gamma - \Gamma_s$. The last integral over G has integrand of the form $(a+b)(\widetilde{a}+\widetilde{b})$. We use integration by parts (3.1) and (3.2) twice to the "mixed" term $a\widetilde{b}$ to obtain an integrand of form $c\,\widetilde{c}$:

$$\int_G \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\frac{\partial^2\widetilde{\psi}}{\partial z^2}\mathrm{d}G = \int_\Gamma \frac{1}{r}\frac{\partial\psi}{\partial r}\left[\frac{\partial^2\widetilde{\psi}}{\partial z^2}n_r - \frac{\partial^2\widetilde{\psi}}{\partial r\,\partial z}n_z\right]\mathrm{d}\Gamma + \int_G \frac{1}{r}\frac{\partial^2\psi}{\partial r\,\partial z}\frac{\partial^2\widetilde{\psi}}{\partial r\,\partial z}\mathrm{d}G.$$

The integrals over Γ vanish due to $\partial\psi/\partial r = 0$ on Γ. The second "mixed" term $\widetilde{a}b$ can be transformed in the same way. Thus we obtained

$$\int_G r\,A_3(S(\psi))\,\widetilde{\psi}\,\mathrm{d}G = \int_{\Gamma_s} r\,S(\psi)\frac{\partial\widetilde{\psi}}{\partial n}\,\mathrm{d}\Gamma + \mathfrak{a}(\psi,\widetilde{\psi})$$

where $\mathfrak{a}(\psi, v)$ is a bilinear form with second order derivatives

$$\mathfrak{a}(\psi,\widetilde{\psi}) = \tag{3.19}$$

$$= \int_G r\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial r}\right) + 2\,\frac{1}{r}\frac{\partial^2\psi}{\partial r\,\partial z}\,\frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\,\frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial z^2}\right]\mathrm{d}G$$

$$= \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right), \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial r}\right)\right) + 2\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\,\partial z}, \frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial r\,\partial z}\right) + \left(\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}, \frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial z^2}\right).$$

In the integral over Γ_s we use (2.27) and put it to the right-hand side.

We rewrite the third integral of identity (3.17) using Lemma 3.2 and (3.15), the fourth term remains unchanged. The last integral with α_m is rewritten using (3.2) and (3.16).

Thus we obtain the last integral identity

$$
\begin{aligned}
&a_{-1}\left(\frac{\partial\psi}{\partial t},\widetilde{\psi}\right)+\nu\,\boldsymbol{a}(\psi,\widetilde{\psi})-b\left(\psi,\widetilde{\psi};\frac{1}{r^2}A_{-1}(\psi)\right)\\
&\qquad+\int_G\frac{1}{r^3}\frac{\partial}{\partial z}\left(\Omega^2\right)\widetilde{\psi}\,\mathrm{d}G+\alpha_m\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\,,\,\frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial z}\right)\\
&\qquad=\left(\alpha_T\frac{\partial T}{\partial r}+\alpha_C\frac{\partial C}{\partial r}\,,\,\frac{\widetilde{\psi}}{r}\right)-\nu\left\langle\beta_T\frac{\partial T}{\partial r}+\beta_C\frac{\partial C}{\partial r}\,,\,\frac{\partial\widetilde{\psi}}{\partial z}\right\rangle_{\Gamma_s},
\end{aligned}
\tag{3.20}
$$

where $\langle\cdot\,,\,\cdot\rangle_{\Gamma_s}$ means the integral

$$
\langle u\,,\,v\rangle_{\Gamma_s}=\int_{\Gamma_s}u\,v\;\mathrm{d}\Gamma\,. \tag{3.21}
$$

We have obtained a system of five integral identities. The relation between them and the original system of pointwise equations can be stated in the following assertion:

Theorem 3.3. (Relation between pointwise equations and integral identities)

(i) *Let the functions* S,Ω,T,C,ψ,χ *satisfy the system of pointwise equations (2.19)–(2.24) with the boundary conditions (2.25)–(2.28).*

Then the integral identities (3.20), (3.14), (3.10), (3.8) and (3.12) hold for all sufficiently smooth test functions $\widetilde{\psi},\widetilde{\Omega},\widetilde{T},\widetilde{C},\widetilde{\chi}$ *satisfying the corresponding boundary conditions: (3.16), (3.13), (3.9), (3.7) and (3.11).*

(ii) *On the other hand let the functions* ψ,Ω,T,C,χ *satisfy the derived integral identities for all smooth test functions* $\widetilde{\psi},\widetilde{\Omega},\widetilde{T},\widetilde{C},\widetilde{\chi}$ *satisfying the corresponding boundary conditions and let the functions* ψ,Ω,T,C,χ *be sufficiently smooth. Then they also satisfy the system of pointwise differential equations with corresponding boundary conditions, where* S *is given by (3.15).*

4. Auxiliary results

In this section we introduce function spaces for the unknowns and the test functions and some inequalities. They are directed to Theorem 4.9 saying that all terms in the integral identities are “well defined”. For weighted Lebesgue and Sobolev spaces see e. g. [13].

Function spaces. Our domain G represents radial cross-section of the melt volume. We assume it is a bounded domain with Lipschitz boundary. Due to cylindrical coordinates the basic function space will be Lebesgue space of square integrable functions with weight r

$$
L^2_r(G)=\left\{u\in\mathcal{M}(G)\,|\;\int_G r\,u^2\,\mathrm{d}G<\infty\right\},
$$

where $\mathcal{M}(G)$ is the space of classes of a. e. equal measurable functions on G. It is a Hilbert space with scalar product and norm defined by

$$(u,v) \equiv (u,v)_r = \int_G r\,u\,v\,\mathrm{d}G\,, \qquad \|u\|_{2;r} = \left[\int_G r\,|u|^2\,\mathrm{d}G\right]^{1/2}.$$

In case of $(\cdot,\cdot)_r$ we often omit the subscript r and write only $(\cdot,\cdot)$, see (3.3). We can obtain the same space by completion of smooth functions $C^\infty(G)$ in the norm $\|\cdot\|_{2;r}$. In the same way we introduce weighted spaces $L^p_r(G)$ of functions integrable with p-th power $(1 \le p < \infty)$ and norm $\|\cdot\|_{p;r}$

$$L^p_r(G) = \left\{u \in \mathcal{M}(G) \mid \|u\|_{p;r} \equiv \left[\int_G r\,|u|^p\,\mathrm{d}G\right]^{1/p} < \infty\right\}.$$

Spaces $L^p_{1/r}(G)$ with weight $1/r$ and norm $\|\cdot\|_{r;1/r}$ are introduced similarly

$$L^p_{1/r}(G) = \left\{u \in \mathcal{M}(G) \mid \|u\|_{p;1/r} \equiv \left[\int_G \frac{1}{r}\,|u|^p\,\mathrm{d}G\right]^{1/p} < \infty\right\}.$$

In the same way using convenient norms generated by the bilinear forms $a_k(\cdot,\cdot)$ we introduce weighted Sobolev spaces for the unknowns: $W^1_r(G)$ space with weight r for the unknowns T, C is defined by completion of $C^\infty(G)$ in the norm

$$\|u\|_{1,2;r} = \Big[\|u\|^2_{2;r} + a_1(u,u)\Big]^{1/2} \equiv \left[\|u\|^2_{2;r} + \left\|\frac{\partial u}{\partial r}\right\|^2_{2;r} + \left\|\frac{\partial u}{\partial z}\right\|^2_{2;r}\right]^{1/2}.$$

Similarly, space $W^1_{1/r}(G)$ with weight $1/r$ for the unknowns Ω, χ is defined as completion of C^∞ functions in the norm

$$\|u\|_{1,2;1/r} = \Big[\|u\|^2_{2;1/r} + a_{-1}(u,u)\Big]^{\frac{1}{2}} \equiv \left[\|u\|^2_{2;1/r} + \left\|\frac{\partial u}{\partial r}\right\|^2_{2;1/r} + \left\|\frac{\partial u}{\partial z}\right\|^2_{2;1/r}\right]^{\frac{1}{2}}.$$

For the last unknown ψ we introduce a special weighted second order derivative Sobolev space $W^{2*}_{1/r}(G)$ by completion of $C^\infty(G)$ in the norm containing the bilinear form $\mathcal{a}(u,v)$:

$$\begin{aligned}\|u\|_{2*,2;1/r} &= \Big[\|u\|^2_{1,2;1/r} + \mathcal{a}(u,u)\Big]^{1/2}\\ &\equiv \Bigg[\|u\|^2_{2;1/r} + \left\|\frac{\partial u}{\partial r}\right\|^2_{2;1/r} + \left\|\frac{\partial u}{\partial z}\right\|^2_{2;1/r} + \left\|r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right)\right\|^2_{2;1/r}\\ &\qquad + 2\left\|\frac{\partial^2 u}{\partial r\,\partial z}\right\|^2_{2;1/r} + \left\|\frac{\partial^2 u}{\partial z^2}\right\|^2_{2;1/r}\Bigg]^{1/2}.\end{aligned}$$

Remarks. (i) The weighted space $L^2_r(G)$ is a natural counterpart to the space $L^2(V)$, where V is the cylindrical domain in $\mathbb{R}^3$ having the cross-section G. Indeed, the following equality holds

$$2\pi \int_G r\,u(r,z)\,\mathrm{d}G = \int_V u^*(x_1,x_2,x_3)\mathrm{d}x\,,$$

where $u^*(x_1, x_2, x_3) = u(\sqrt{x_1^2 + x_2^2}, x_3)$.

In the same way we see that the space $W_r^1(G)$ is a Hilbert space and it is connected to its natural counterpart $W^1(V)$ by equality

$$2\pi \|u\|_{1,2;r}^2 = \|u^*\|_{1,2}^2 \,.$$

The equality is used in several proofs.

(ii) For the unknown ψ we need a weighted Sobolev space of functions with second order derivatives. The standard Sobolev space $W_{1/r}^2(G)$ defined by completion of the smooth functions in the standard norm

$$\|u\|_{2,2;1/r}^2 = \sum_{|\alpha|\le 2} \|\mathrm{D}^\alpha u\|_{2;1/r}^2$$

is not convenient, since it is too narrow. Indeed, its norm requires condition $(\partial^2 u)/(\partial r^2) = 0$ a. e. along the axis Γ_a. The norm induced by a admits nonzero values of $\partial^2\psi/\partial r^2$ at the axis, e. g. $u(r,z) = r^2$ is in $W_{1/r}^{2*}(G)$ but not in $W_{1/r}^2(G)$.

(iii) As we shall see in Lemma 4.1 the weighted space $W_r^1(G)$ is imbedded into $L_r^q(G)$ for $q \le 6$ only. In case of the standard Sobolev spaces this is true for domains in $\mathbb{R}^3$ and for $G \subset \mathbb{R}^2$ the standard imbedding holds for $q < \infty$. It is caused by the correspondence mentioned in the preceeding Remark (i).

(iv) As we shall see in Lemma 4.2 functions of the weighted space $W_{1/r}^1(G)$ have zero traces on the axis Γ_a.

Imbedding, traces and inequalities.

Lemma 4.1. (On embedding) *For $q \le 6$ we have $W_r^1(G) \subset L_r^q(G)$ i. e.*

$$\|u\|_{q;r} \le \text{const} \cdot \|u\|_{1,2;r} \qquad \textit{for } q \le 6 \,.$$

For $q < 6$ the imbedding is compact, namely any sequence u_n converging weakly in $W_r^1(G)$ is converging strongly in $L_r^q(G)$.

The assertion is a direct consequence of the corresponding imbedding in three dimensional domains if we rewrite the integrals over G to integrals over the corresponding volume V in $\mathbb{R}^3$.

Lemma 4.2. (On traces)

(i) *The functions of $W_r^1(G)$ have traces on Γ_s in $L^2(\Gamma_s)$, i. e.*

$$\|u\|_{L^2(\Gamma_s)} \le c_s \|u\|_{1,2;r} \,,$$

where the constant c_s depends on G and Γ_s only.

(ii) *The functions of $W_{1/r}^1(G)$ has zero trace on Γ_a and the functions u of $W_{1/r}^{2*}(G)$ satisfy $u = 0$ and $\nabla u = 0$ on Γ_a in sense of traces.*

Since $\Gamma_s \subset \Gamma$ has positive distance from the axis $r = 0$ the first part of the lemma is a direct consequence of Theorem on traces in $\mathbb{R}^3$, see e. g. [12]. The second part follows from the fact that any continuous function $u \in W_{1/r}^1(G)$

cannot have nonzero value and nonzero gradient on Γ_a. For functions of $W^{2*}_{1/r}(G)$ the same reasoning can be used.

Lemma 4.3. (On bilinear forms) *Let us introduce seminorms generated by the bilinear forms* $a_1(u,v)$, $a_{-1}(u,v)$, $\mathcal{A}(u,v)$*:*

$$|u|_{1,2;r} = [a_1(u,u)]^{1/2}\,, \quad |u|_{1,2;1/r} = [a_{-1}(u,u)]^{1/2}\,, \quad |u|_{2*,2;1/r} = [\mathcal{A}(u,u)]^{1/2}\,.$$

The first two seminorms represent equivalent norms on subspaces of functions with zero trace on a part of the boundary $M \subset \Gamma - \Gamma_a$ *of positive measure, the third semimorm* $|u|_{2*,2:1/r}$ *is an equivalent norm on the subspace of functions with traces of both* u *and* ∇u *vanishing on* M*. Thus on these subspaces the bilinear forms satisfy*

$$a_1(u,u) \geq c_1\|u\|^2_{1,2;r}\,, \quad a_{-1}(u,u) \geq c_{-1}\|u\|^2_{1,2;1/r}\,, \quad \mathcal{A}(u,u) \geq c_3\|u\|^2_{2*,2;1/r}\,.$$

On the other hand, on the whole spaces $W^1_r(G), W^1_{1/r}(G), W^{2*}_{1/r}(G)$ *we have*

$$\begin{gathered} a_1(u,v) \leq \|u\|_{1,2;r}\,\|v\|_{1,2;r}\,, \quad a_{-1}(u,v) \leq \|u\|_{1,2;1/r}\,\|v\|_{1,2;1/r}\,, \\ \mathcal{A}(u,v) \leq \|u\|_{2*,2;1/r}\,\|v\|_{2*,2;1/r}\,. \end{gathered}$$

The results can be proved by transforming the integral over G into integral over the volume V in $\mathbb{R}^3$ and using corresponding results for standard Sobolev spaces.

Lemma 4.4. *The following inequalities hold*

$$\begin{aligned} |(u,v)| &\leq \|u\|_{2;r}\,\|v\|_{2;r}\,, \\ \left|\left(\frac{u}{r},\frac{v}{r}\right)\right| &\leq \left\|\frac{u}{r}\right\|_{2;r}\,\left\|\frac{v}{r}\right\|_{2;r} = \|u\|_{2;1/r}\,\|v\|_{2;1/r}\,, \\ \left|\int_G u\,v\,\mathrm{d}G\right| &\leq \|u\|_{2;r}\,\left\|\frac{v}{r}\right\|_{2;r} = \|u\|_{2;r}\|v\|_{2;1/r}\,, \\ \left|\int_G u\,v\,w\,\mathrm{d}G\right| &\leq \left\|\frac{u}{r}\right\|_{4;r}\,\|v\|_{2;r}\,\|w\|_{4;r}\,, \\ \left|\int_G \frac{1}{r^2}u\,v\,w\,\mathrm{d}G\right| &\leq \left\|\frac{u}{r}\right\|_{4;r}\,\|v\|_{2;1/r}\,\left\|\frac{w}{r}\right\|_{4;r}\,. \end{aligned}$$

The proofs are based on Cauchy inequality $|\int f\,g\,\mathrm{d}G| \leq \|f\|_2 \cdot \|g\|_2$ and on Hölder inequality $|\int f\,g\,h\,\mathrm{d}G| \leq \|f\|_4 \cdot \|g\|_2 \cdot \|h\|_4$.

Lemma 4.5. *Functions with zero trace* $u = 0$ *on* Γ_a *satisfy*

$$\left\|\frac{u}{r^2}\right\|_{2;r} \leq \left\|\frac{\partial u}{\partial r}\right\|_{2;1/r}\,, \qquad \left\|\frac{u}{r^2}\right\|_{4;r} \leq \frac{2}{3}\,\left\|\frac{1}{r}\,\frac{\partial u}{\partial r}\right\|_{4;r}\,.$$

Adding inequalities of the previous lemma we obtain

$$\left\| \frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right\|_{2;r} \le \mathrm{const}\cdot \|u\|_{1,2;1/r} \quad \textit{for } u \in W^1_{1/r}(G)\,,$$

$$\left\|\frac{u}{r}\right\|_{q;r} \le \mathrm{const}\cdot \|u\|_{1,2;1/r} \quad \textit{for } u \in W^1_{1/r}(G) \quad q \le 6\,,$$

$$\left\|\frac{1}{r}\frac{\partial u}{\partial r}\right\|_{4;r}, \ \left\|\frac{1}{r}\frac{\partial u}{\partial z}\right\|_{4;r} \le \mathrm{const}\cdot \|u\|_{2*,2;1/r} \quad \textit{for } u \in W^{2*}_{1/r}(G)\,.$$

The proof is based on the following two cases of Hardy's inequality [13]

$$\int_0^\delta r\left|\frac{F(r)}{r^2}\right|^2 \mathrm{d}r \le \int_0^\delta \frac{1}{r}|f(r)|^2 \mathrm{d}r, \quad \int_0^\delta r\left|\frac{F(r)}{r^2}\right|^4 \mathrm{d}r \le \left(\frac{2}{3}\right)^4 \int_0^\delta \frac{1}{r^3}|f(r)|^4 \mathrm{d}r\,,$$

where $F(r) = \int_0^r f(s)\mathrm{d}s$ and Lemma 4.2 (ii).

It remains to estimate the boundary integrals $\langle\cdot\,,\,\cdot\rangle_{\Gamma_s}$ in identity (3.20). The standard trace theorem (Lemma 4.2) is not sufficient, the boundary integral exists only for smooth functions. Using a more delicate estimate we can extend it to the desired case:

Lemma 4.6. *The bilinear form $\langle\cdot\,,\,\cdot\rangle_{\Gamma_s}$ on a smooth part of boundary Γ_s defined for smooth functions by (3.21) can be extended by continuity to the case*

$$\left\langle \frac{\partial u}{\partial r}\,, \frac{\partial v}{\partial z}\right\rangle_{\Gamma_s}, \qquad u \in W^1_r(G)\,,\ v \in W^{2*}_{1/r}(G)$$

satisfying

$$\left|\left\langle \frac{\partial u}{\partial r}\,, \frac{\partial v}{\partial z}\right\rangle_{\Gamma_s}\right| \le \mathrm{const}\cdot \|u\|_{1,2;r}\|v\|_{2*,2;1/r}\,.$$

Sketch of the proof. Since both weights r and $1/r$ are bounded on Γ_s we need not take care of them. We shall use results saying that functions of Sobolev spaces have their traces in fraction order spaces. A function $v \in W^{2*}_{1/r}(G)$ has its derivative in $W^1_{1/r}(G)$ and the trace of the derivative is in $W^{1/2}(\Gamma_s)$. The second function $u \in W^1_r(G)$ has its trace in $W^{1/2}(\Gamma_s)$ and its tangent derivatives in the space $W^{-1/2}(\Gamma_s)$ dual to $W^{1/2}(\Gamma_s)$. The pairing inequality on $W^{-1/2}(\Gamma_s)\times W^{1/2}(\Gamma_s)$ yields the inequality and justifies the extension.

Estimates of the nonlinear terms and the conclusion.

Lemma 4.7. *We have the following estimates of the nonlinear convective terms*

$$\begin{aligned}
|b\,(u, v_1; w_1)| &\le \mathrm{const}\cdot \|u\|_{2*,2;1/r}\,\|v_1\|_{1,2;r}\,\|w_1\|_{1,2;r}\,,\\
\left|\int_G \frac{u}{r^3}\,\frac{\partial}{\partial z}(v_2^2)\;\mathrm{d}G\right| &\le \mathrm{const}\cdot \|u\|_{2*,2;1/r}\,\|v_2\|^2_{1,2;1/r}\,,\\
\left|b\left(u, v_2; \frac{w_2}{r^2}\right)\right| &\le \mathrm{const}\cdot \|u\|_{2*,2;1/r}\,\|v_2\|_{1,2;1/r}\,\|w_2\|_{1,2;1/r}\,,\\
\left|b\left(u, v_3; \frac{1}{r^2}A_{-1}(w_3)\right)\right| &\le \mathrm{const}\cdot \|u\|_{2*,2;1/r}\,\|v_3\|_{2*,2;1/r}\,\|w_3\|_{2*,2;1/r}
\end{aligned}$$

provided $v_1, w_1 \in W^1_r(G)\,,\;\; v_2, w_2 \in W^1_{1/r}(G)$ *and* $u, v_3, w_3 \in W^{2*}_{1/r}(G)$.

The inequalities follows from Lemmas 4.4, 4.5 and (3.18). For the evolution case in Section 6 we need more delicate estimates of the nonlinear terms:

Lemma 4.8. *Under the assumptions of Lemma 4.7 we have*

$$|b\,(u, v_1; w_1)| \le \mathrm{const}\cdot \|u\|^{1/4}_{1,2;1/r}\, \|u\|^{3/4}_{2*,2;1/r}\, \|v_1\|_{1,2;r}\, \|w_1\|^{1/4}_{2;r}\, \|w_1\|^{3/4}_{1,2;r}\,,$$

$$\left|\int_G \frac{u}{r^3}\frac{\partial}{\partial z}(v_2^2)\,\mathrm{d}G\right| \le \mathrm{const}\cdot \|u\|^{1/4}_{1,2;1/r}\, \|u\|^{3/4}_{2*,2;1/r} \|v_2\|^{1/4}_{2;1/r}\, \|v_2\|^{7/4}_{1,2;1/r}\,,$$

$$\left|b\left(u, v_2; \frac{w_2}{r^2}\right)\right| \le \mathrm{const}\cdot \|u\|^{1/4}_{1,2;1/r}\, \|u\|^{3/4}_{2*,2;1/r}\, \|v_2\|_{1,2;1/r}\, \|w_2\|^{1/4}_{2;1/r}\, \|w_2\|^{3/4}_{1,2;1/r}$$

and

$$\begin{aligned}&\left|b\left(u, v_3; \frac{1}{r^2}A_{-1}(w_3)\right)\right|\\ &\le \mathrm{const}\cdot \|u\|^{1/4}_{1,2;1/r}\, \|u\|^{3/4}_{2*,2;1/r}\, \|v_3\|_{2*,2;1/r}\, \|w_3\|^{1/4}_{1,2;1/r}\, \|w_3\|^{3/4}_{2*,2;1/r}\,,\end{aligned}$$

provided $v_1, w_1 \in W^1_r(G)\,,\;\; v_2, w_2 \in W^1_{1/r}(G)$ *and* $u, v_3, w_3 \in W^{2*}_{1/r}(G)$.

The estimates follow from Lemmas 4.4 and 4.5 if we estimate the first and the third factor using the following inequality which follows from Hölder inequality and Lemma 4.1

$$\|u\|_{4;r} \le \mathrm{const}\cdot \|u\|^{1/4}_{2;r}\, \|u\|^{3/4}_{6;r} \le \mathrm{const}\cdot \|u\|^{1/4}_{2;r}\, \|u\|^{3/4}_{1,2;r}$$

or using a similar estimate for $\|u/r\|_{4;r}$.

Applying all previous lemmas estimating individual terms we can conclude the section with the following statement:

Theorem 4.9. *Let the unknown functions* ψ, Ω, T, C, χ *and the test functions* $\widetilde{\psi}, \widetilde{\Omega}, \widetilde{T}, \widetilde{C}, \widetilde{\chi}$ *be in corresponding spaces i. e.*

$$\psi, \widetilde{\psi} \in W^{2*}_{1/r}(G)\,, \quad \Omega, \widetilde{\Omega}, \chi, \widetilde{\chi} \in W^1_{1/r}(G)\,, \quad T, \widetilde{T}, C, \widetilde{C} \in W^1_r(G)$$

and the data functions $g_T, \gamma_C \in L^2(\Gamma_s)$. *In the evolution case in addition we assume*

$$\frac{\partial\psi}{\partial t} \in W^1_{1/r}(G)\,, \qquad \frac{\partial\Omega}{\partial t} \in L^2_{1/r}(G)\,, \qquad \frac{\partial T}{\partial t}\,,\, \frac{\partial C}{\partial t} \in L^2_r(G)\,.$$

Then all integrals in identities (3.8), (3.10), (3.12), (3.14), (3.20) are well defined and finite.

5. Stationary problem

In this section we introduce the weak and operator formulation of the stationary problem and prove existence of the solution — Theorem 5.2. To obtain the stationary problem we drop evolution terms in the integral identities (3.20), (3.14), (3.10), (3.8), (3.12) of Section 3 with the time independent unknowns ψ, Ω, T, C, χ and the test functions $\widetilde{\psi}, \widetilde{\Omega}, \widetilde{T}, \widetilde{C}, \widetilde{\chi}$:

$$\begin{aligned} \nu\, \boldsymbol{a}(\psi, \widetilde{\psi}) - b(\psi, \widetilde{\psi}; \frac{1}{r^2} A_{-1}(\psi)) + \int_G \frac{1}{r^3} \frac{\partial}{\partial z} \left(\Omega^2\right) \widetilde{\psi}\, \mathrm{d}G \\ + \alpha_m \left(\frac{1}{r} \frac{\partial \psi}{\partial z} \,,\, \frac{1}{r} \frac{\partial \widetilde{\psi}}{\partial z} \right) \\ = \left(\alpha_T \frac{\partial T}{\partial r} + \alpha_C \frac{\partial C}{\partial r} \,,\, \frac{\widetilde{\psi}}{r} \right) - \nu \left\langle \beta_T \frac{\partial T}{\partial r} + \beta_C \frac{\partial C}{\partial r} \,,\, \frac{\partial \widetilde{\psi}}{\partial z} \right\rangle_{\Gamma_s} , \end{aligned} \tag{5.1}$$

$$\nu\, a_{-1}(\Omega, \widetilde{\Omega}) + b\left(\psi, \Omega; \frac{\widetilde{\Omega}}{r^2} \right) = \alpha_m \left(\frac{\chi}{r}, \frac{1}{r} \frac{\partial \widetilde{\Omega}}{\partial z} \right) , \tag{5.2}$$

$$\nu_T\, a_1(T, \widetilde{T}) + b\,(\psi, T; \widetilde{T}) = \nu_T \int_{\Gamma_s} r(g_T - \gamma_T T)\, \widetilde{T}\, \mathrm{d}\Gamma\,, \tag{5.3}$$

$$\nu_C\, a_1(C, \widetilde{C}) + b\,(\psi, C; \widetilde{C}) = \nu_C \int_{\Gamma_s} r(g_C - \gamma_C C)\, \widetilde{C}\, \mathrm{d}\Gamma\,, \tag{5.4}$$

$$a_{-1}(\chi, \widetilde{\chi}) = - \left(\frac{1}{r} \frac{\partial \Omega}{\partial z} \,,\, \frac{\widetilde{\chi}}{r} \right) , \tag{5.5}$$

where the forms $a_i(\cdot, \cdot)$, $\boldsymbol{a}(\cdot, \cdot)$, $b\,(\cdot, \cdot\,; \cdot)$ are defined by (3.5), (3.19), (3.6) and $\langle \cdot \,,\, \cdot \rangle_{\Gamma_s}$ is justified in Lemma 4.6.

Function spaces. According to various boundary conditions for the test functions we introduce function spaces. The spaces will be denoted by subscript of the unknown. The equalities on the boundary are taken in sense of traces:

$$\begin{aligned} V_\psi &= \left\{ \psi \in W^{2*}_{1/r}(G) \,\middle|\, \psi = 0 \text{ on } \Gamma, \quad \nabla \psi = 0 \text{ on } \Gamma - \Gamma_s \right\} , \\ V_\Omega &= \left\{ \Omega \in W^1_{1/r}(G) \,\middle|\, \Omega = 0 \text{ on } \Gamma - \Gamma_s \right\} , \\ V_T &= \left\{ T \in W^1_r(G) \,\middle|\, T = 0 \text{ on } \Gamma_p \cup \Gamma_c \right\} , \\ V_C &= \left\{ C \in W^1_r(G) \,\middle|\, C = 0 \text{ on } \Gamma_p \right\} , \\ V_\chi &= \left\{ \chi \in W^1_{1/r}(G) \,\middle|\, \chi = 0 \text{ on } \Gamma \right\} . \end{aligned}$$

On a part of the boundary the unknowns Ω, T, C have prescribed nonhomogeneous boundary values. Therefore for the weak formulation we introduce auxiliary functions having prescribed boundary conditions: Let Ω_b, T_b and C_b be arbitrary functions

$$\Omega_b \in W^1_{1/r}(G)\,, \qquad T_b \in W^1_r(G)\,, \qquad C_b \in W^1_r(G) \tag{5.6}$$

chosen such that they satisfy the corresponding nonhomogeneous boundary conditions prescribed for Ω, T, C by (2.25), (2.26). In Lemma 5.8 the functions Ω_b, T_b, C_b will be modified.

Remark. The physical substance of the problem enables to choose “better” (e. g. bounded) auxiliary functions Ω_b, T_b, C_b, nevertheless the assumption (5.6) is sufficient for proving the existence result.

Weak formulation. Let us summarize the assumptions. We assumed that the bounded domain G of shape sketched in Fig. 2 has Lipschitz boundary. Let the data be such that there exist auxiliary functions Ω_b, T_b, C_b satisfying (5.6). Further, we assume

$$\nu,\, \nu_T,\, \nu_C > 0\,, \qquad \gamma_T,\, \gamma_C,\, \alpha_m \geq 0\,, \qquad g_T,\, g_C \in L^2(\Gamma_s)\,. \tag{5.7}$$

The other constants $\alpha_T, \alpha_C, \beta_T, \beta_C$ need not be positive.

Problem 5.1. *We say that the functions ψ, Ω, T, C, χ are the* **weak solution** *to the stationary problem iff*

$$\psi \in V_\psi\,, \quad \Omega - \Omega_b \in V_\Omega\,, \quad T - T_b \in V_T\,, \quad C - C_b \in \mathrm{V}_C\,, \quad \chi \in V_\chi,$$

and the integral identities (5.1) – (5.5) hold for each test functions

$$\widetilde{\psi} \in V_\psi\,, \quad \widetilde{\Omega} \in V_\Omega\,, \quad \widetilde{T} \in V_T\,, \quad \widetilde{C} \in V_C\,, \quad \widetilde{\chi} \in V_\chi\,.$$

Vector formulation. To simplify the notation we gather all the unknowns into a vector U of unknown functions and all the test functions into a vector V of test functions

$$\mathrm{U} = (\psi, \Omega, T, C, \chi)\,, \qquad \mathrm{V} = (\widetilde{\psi}, \widetilde{\Omega}, \widetilde{T}, \widetilde{C}, \widetilde{\chi})\,.$$

We introduce a basic space $\mathbb{W}$ for the vector functions U and V

$$\mathbb{W} = W^{2*}_{1/r}(G) \times W^1_{1/r}(G) \times W^1_r(G) \times W^1_r(G) \times W^1_{1/r}(G)\,, \tag{5.8}$$

its subspace $\mathbb{V}$ of functions with prescribed zero traces for the test vector V

$$\mathbb{V} = V_\psi \times V_\Omega \times V_T \times V_C \times V_\chi, \tag{5.9}$$

and a vector for nonhomogenous boundary conditions $\mathrm{U}_b = (0, \Omega_b, T_b, C_b, 0)$.

Operators and functionals. We multiply the equations (5.1) – (5.5) with positive constants $k_\psi, k_\Omega, k_T,\ k_C, k_\chi$, respectively and sum them up. We choose $k_\psi = k_\Omega = 1$ since both ψ and Ω correspond to the same physical quantity. The constants k_T, k_C, k_χ will be chosen such that they ensure coercivity of the operator. They will be specified later in Lemma 5.7.

Summing the identities we obtain an identity containing 17 terms, each linear in its test function. We associate the terms into four groups, the first three represent defining formulae for operators $\mathscr{A}, \mathscr{B}, \mathscr{C} : \mathbb{W} \to \mathbb{V}^*$:

The first *principal* linear operator $\mathscr{A}$ contains all scalar product like terms:

$$\begin{aligned}
\langle \mathscr{A}(\mathrm{U})\,, \mathrm{V}\,\rangle &= \nu\, \boldsymbol{a}(\psi, \widetilde{\psi}) + \nu\, a_{-1}(\Omega, \widetilde{\Omega}) \\
&\quad + k_T \nu_T \Big[a_1(T, \widetilde{T}) + \gamma_T \int_{\Gamma_s} r\, T\, \widetilde{T}\, \mathrm{d}\Gamma \Big] \\
&\quad + k_C \nu_C \Big[a_1(C, \widetilde{C}) + \gamma_C \int_{\Gamma_s} r\, C\, \widetilde{C}\, \mathrm{d}\Gamma \Big] + k_\chi\, a_{-1}(\chi, \widetilde{\chi}) \\
&\quad + \alpha_m \left(\frac{1}{r} \frac{\partial \psi}{\partial z}, \frac{1}{r} \frac{\partial \widetilde{\psi}}{\partial z} \right)\,.
\end{aligned} \tag{5.10}$$

The second *convective* nonlinear operator $\mathscr{B}$ consists of all trilinear forms:

$$\begin{aligned}\langle \mathscr{B}(\mathrm{U})\,,\mathrm{V}\,\rangle = -b\left(\psi,\widetilde{\psi};\frac{1}{r^2}A_{-1}(\psi)\right) + \int_G \frac{1}{r^3}\frac{\partial}{\partial z}(\Omega^2)\,\widetilde{\psi}\,\mathrm{d}G \\ + b\left(\psi,\Omega;\frac{\widetilde{\Omega}}{r^2}\right) + k_T\,b\,(\psi,T;\widetilde{T}) + k_C\,b\,(\psi,C;\widetilde{C})\,.\end{aligned} \tag{5.11}$$

The third *coupling* operator $\mathscr{C}$ contains remaining bilinear terms that couple the equations together:

$$\begin{aligned}\langle \mathscr{C}(\mathrm{U})\,,\mathrm{V}\,\rangle = -\left(\alpha_T\frac{\partial T}{\partial r} + \alpha_C\frac{\partial C}{\partial r}\,,\,\frac{1}{r}\,\widetilde{\psi}\right) \\ + \nu\left\langle \beta_T\frac{\partial T}{\partial r} + \beta_C\frac{\partial C}{\partial r}\,,\,\frac{\partial\widetilde{\psi}}{\partial z}\right\rangle_{\Gamma_s} - \alpha_m\left(\frac{\chi}{r},\frac{1}{r}\,\frac{\partial\widetilde{\Omega}}{\partial z}\right) \\ + k_\chi\left(\frac{1}{r}\,\frac{\partial\Omega}{\partial z},\frac{\widetilde{\chi}}{r}\right)\,.\end{aligned} \tag{5.12}$$

The remaining terms form a functional $\mathscr{F}_0$ on $\mathbb{V}$:

$$\langle \mathscr{F}_0\,,\mathrm{V}\,\rangle = k_T\nu_T\int_{\Gamma_s} r\,g_T\,\widetilde{T}\,\mathrm{d}\Gamma + k_C\nu_C\int_{\Gamma_s} r\,g_C\,\widetilde{C}\,\mathrm{d}\Gamma\,. \tag{5.13}$$

Using the operators we can reformulate the stationary problem:

Problem 5.1a. *The functions* $(\psi,\Omega,T,C,\chi)\equiv\mathrm{U}$ *are the* **weak solution** *to the stationary problem iff* $\mathrm{U}\in\mathbb{W}$ *such that* $\mathrm{U}-\mathrm{U}_b\in\mathbb{V}$ *and the following operator equation on* $\mathbb{V}^*$ *is satisfied*

$$\mathscr{A}(\mathrm{U}) + \mathscr{B}(\mathrm{U}) + \mathscr{C}(\mathrm{U}) = \mathscr{F}_0\,.$$

Conversion to homogeneous boundary conditions. In order to get rid of nonhomogeneous boundary conditions we replace the unknown U by $\mathrm{U}+\mathrm{U}_b$ with $\mathrm{U}\in\mathbb{V}$. Then the operator equation reads

$$\mathscr{A}(\mathrm{U}+\mathrm{U}_b) + \mathscr{B}(\mathrm{U}+\mathrm{U}_b) + \mathscr{C}(\mathrm{U}+\mathrm{U}_b) = \mathscr{F}_0\,.$$

We shall rewrite the equation to the form

$$\mathscr{A}(\mathrm{U}) + \mathscr{B}(\mathrm{U}) + \mathscr{C}(\mathrm{U}) + \mathscr{D}(\mathrm{U}) = \mathscr{F}_s\,,$$

where the operator $\mathscr{D}$ contains all new terms linear in the unknown U:

$$\begin{aligned}\langle \mathscr{D}(\mathrm{U})\,,\mathrm{V}\,\rangle = 2\int_G \frac{1}{r^3}\,\frac{\partial}{\partial z}(\Omega\,\Omega_b)\widetilde{\psi}\,\mathrm{d}G + b\left(\psi,\Omega_b;\frac{\widetilde{\Omega}}{r^2}\right) \\ + k_T\,b\,(\psi,T_b;\widetilde{T}) + k_C\,b\,(\psi,C_b;\widetilde{C})\end{aligned} \tag{5.14}$$

and the functional $\mathscr{F}_s$ consists of all terms without the unknown U:

$$\begin{aligned}\langle \mathscr{F}_s\,,\mathrm{V}\,\rangle = k_T\nu_T\int_{\Gamma_s} r\,g_T\,\widetilde{T}\,\mathrm{d}\Gamma + k_C\nu_C\int_{\Gamma_s} r\,g_C\,\widetilde{C}\,\mathrm{d}\Gamma \\ - \langle \mathscr{A}(\mathrm{U}_b)\,,\mathrm{V}\,\rangle - \int_G \frac{1}{r^3}\,\frac{\partial}{\partial z}\left(\Omega_b^2\right)\widetilde{\psi}\,\mathrm{d}G - \langle \mathscr{C}(\mathrm{U}_b)\,,\mathrm{V}\,\rangle\,.\end{aligned} \tag{5.15}$$

Thus we obtained the final formulation of the stationary problem:

Problem 5.1b. *The function* $\mathrm{U}+\mathrm{U}_b$ *is* **weak solution** *to the stationary problem iff* $\mathrm{U}\in\mathbb{V}$ *and the following operator equation holds on* $\mathbb{V}^*$

$$\mathscr{A}(\mathrm{U})+\mathscr{B}(\mathrm{U})+\mathscr{C}(\mathrm{U})+\mathscr{D}(\mathrm{U})=\mathscr{F}_s\,. \tag{5.16}$$

Main result – Existence theorem.

Theorem 5.2. *Let the assumption (5.6) and (5.7) be satisfied. Then the stationary problem 5.1b is well defined and admits a weak solution.*

Well possedness of the problem is a consequence of the Theorem 4.9. The proof of existence of the solution is based on the following abstract existence theorem:

Theorem 5.3. ([5]). *Let V be a reflexive separable Banach space and T an operator $T:V\to V^*$ which is*

— *coercive i. e.*

$$\lim_{\|u\|\to\infty}\frac{\langle T(u)\,,u\,\rangle}{\|u\|}=\infty\,,$$

— *weakly continuous i. e. it preserves weakly continuous sequences*

$$u_n\rightharpoonup u\quad\Longrightarrow\quad T(u_n)\rightharpoonup T(u).$$

Then for any $b\in V^*$ the equation $T(u)=b$ admits a solution.

Sketch of the proof of Theorem 5.3. We consider a sequence of finite dimensional subspaces V_n of the space V and corresponding sequence of Galerkin approximation of the problem $T(u)=b$ from V to subspaces V_n:

Find $u_n\in V_n$ *such that* $\langle T(u_n)\,,\,v\rangle=\langle b\,,\,v\rangle$ *holds for all* $v\in V_n$.

Since T is coercive on V it is coercive on V_n, weak continuity on V yields continuity on finite dimensional subspace V_n and thus the approximative solution u_n exists. Moreover u_n is bounded by a constant independent of V_n.

Due to reflexivity of V there is a subsequence $\{u_{n'}\}$ weakly converging to an element $u\in V$. It remains to prove that the limit u is a solution of $T(u)=b$, i. e. for each $v\in V$ we have $\langle T(u)\,,\,v\rangle=\langle b\,,\,v\rangle$.

Since V is separable the sequence of its subspaces V_n can be chosen such that the distance of any $v\in V$ from V_n tends to zero. Let us take a $v\in V$. There exists a sequence $v_n\in V_n$ such that $v_n\to v$. Now putting this v_n into the equality for u_n we can pass to the limit in $\langle T(u_{n'})\,,\,v_{n'}\rangle=\langle b\,,\,v_{n'}\rangle$.

Indeed, since T is weakly continuous $u_{n'}\rightharpoonup u$ implies $T(u_{n'})\rightharpoonup T(u)$. Further $T(u_{n'})$ is bounded, $v_n\to v$ and the result follows. For details see [5]. ∎

at the end. The proof of Theorem 5.2 consists of verifying the assumptions of Theorem 5.3, namely weak continuity and coercivity of the operator.

Weak continuity.

Lemma 5.4. *The operator* $\mathscr{A}+\mathscr{B}+\mathscr{C}+\mathscr{D}$ *is weakly continuous.*

Proof. Since weakly continuous operators form a linear space we can study their weak continuity separately. Linear continuous operators are also weakly continuous (see e. g. [5]), thus $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{D}$ are weakly continuous.

The remaining nonlinear operator $\mathscr{B}$ is weakly continuous, too. The proof is based on compactness of the following imbedding $W^1_r(G) \subset L^4_r(G)$, see Lemma 4.1.

Let $\mathrm{U}_n \rightharpoonup \mathrm{U}$ in $\mathbb{W}$ weakly, where U_n, U are vector functions with components $(\psi, \Omega, T, c, \chi)$. Then the following strong convergences in $L^4_r(G)$ and also in $L^2_r(G)$ hold:

$$\frac{1}{r}\frac{\partial \psi_n}{\partial r} \to \frac{1}{r}\frac{\partial \psi}{\partial r}\,, \quad \frac{1}{r}\frac{\partial \psi_n}{\partial z} \to \frac{1}{r}\frac{\partial \psi}{\partial z}\,, \quad \frac{1}{r}\Omega_n \to \frac{1}{r}\Omega\,, \quad T_n \to T\,, \quad C_n \to C\,.$$

Expression $\langle \mathscr{B}(\mathrm{U}_n)\,,\mathrm{V}\,\rangle$ consists of trilinear forms. Each term represents an integral of a product of three functions $\int_G f_n\, g_n\, h\, \mathrm{d}G$: the first one is the unknown in the highest derivative — thus converging weakly, the second one is the unknown in lower order derivative — thus converging strongly and the third one is the stationary test function. Thus their product converges to product of their limits and the weak continuity of operator $\mathscr{B}$ follows. ∎

Coercivity. To prove the coercivity we choose a special norm on $\mathbb{W}$ and the vector U_b such that the operator $\mathscr{A}$ is coercive and the other operators $\mathscr{B}, \mathscr{C}, \mathscr{D}$ are sufficiently small not to violate the coercivity.

We start with introducing a convenient *norm* on the product space $\mathbb{W}$ with vectors $\mathrm{U} = (\psi, \Omega, T, C, \chi)$. The definition contains constants k_T, k_C, k_χ:

$$\begin{aligned} \|\mathrm{U}\|_{\mathbb{W}} = \Big[&\nu\|\psi\|^2_{2*,2;1/r} + \nu\|\Omega\|^2_{1,2;1/r} \\ &+ k_T\nu_T\|T\|^2_{1,2;r} + k_C\nu_C\|C\|^2_{1,2;r} + k_\chi\|\chi\|^2_{1,2;1/r}\Big]^{1/2}\,. \end{aligned} \tag{5.17}$$

Let us remark that the introduced norm is equivalent to the standard product norm on the space $\mathbb{W}$.

Lemma 5.5. *The principal linear operator $\mathscr{A}$ satisfies estimates:*

$$\langle \mathscr{A}(\mathrm{U})\,,\mathrm{U}\,\rangle \geq K_0\ \|\mathrm{U}\|^2_{\mathbb{W}} \qquad \textit{for}\ \ \mathrm{U} \in \mathbb{V}$$

$$\langle \mathscr{A}(\mathrm{U})\,,\mathrm{V}\,\rangle \leq K_{\mathscr{A}}\ \|\mathrm{U}\|_{\mathbb{W}} \cdot \|\mathrm{V}\|_{\mathbb{W}} \qquad \textit{for}\ \ \mathrm{U},\mathrm{V} \in \mathbb{W}\,,$$

where the constant K_0 equals to minimum of the equivalent norm constants of Lemma 4.3 and both $K_0, K_{\mathscr{A}}$ are independent of the choice of k_T, k_C, k_χ.

Lemma 5.6. *The convective nonlinear operator $\mathscr{B}$ satisfies*

$$\langle \mathscr{B}(\mathrm{U})\,,\mathrm{U}\,\rangle = 0 \qquad \textit{for}\ \ \mathrm{U} \in \mathbb{W}\,.$$

Proof. Proof According to Lemma 3.2 the first, fourth and fifth term in $\langle \mathscr{B}(\mathrm{U})\,,\mathrm{U}\,\rangle$ equals zero. We transform the third term using Lemma 3.2 and carrying out differentiation $\frac{\partial}{\partial r}(\Omega/r^2)$ we find

$$b\Big(\psi, \Omega; \frac{\Omega}{r^2}\Big) = -b\Big(\frac{\Omega}{r^2}, \Omega; \psi\Big) = -b\Big(\Omega, \Omega; \frac{\psi}{r^2}\Big) - 2\int_G \frac{1}{r^3}\frac{\partial \Omega}{\partial z}\Omega\,\psi\,\mathrm{d}G\,.$$

Again the convection form b is zero and the integral is cancelled by the second term in $\langle \mathscr{B}(\mathrm{U}), \mathrm{U}\rangle$ and the equality is proved. ∎

Lemma 5.7. *The coupling linear operator $\mathscr{C}$ is a continuous operator satisfying*

$$\langle \mathscr{C}(\mathrm{U}), \mathrm{U}\rangle \le K_{\mathscr{C}}\, \|\mathrm{U}\|^2_{\mathbb{W}} \qquad \textit{for}\ \ \mathrm{U} \in \mathbb{V}\,.$$

By proper choice of constants k_T, k_C, k_χ the positive constant $K_{\mathscr{C}}$ can be arbitrary small, particularly there exist k_T, k_C, k_χ such that $K_{\mathscr{C}} \le K_0/4$.

Proof. In case $\alpha_m > 0$ choosing $k_\chi = \alpha_m$ the last two terms in $\langle \mathscr{C}(\mathrm{U}), \mathrm{V}\rangle$ cancel for $\mathrm{U} = \mathrm{V}$. The remaining four integrals contain product of two unknowns ψ and T or C. They can be estimated by means of inequality $ab \le a^2\,\varepsilon/2 + b^2/(2\varepsilon)$. Let us deal with e. g. the first integral

$$\alpha_T \left(\frac{\partial T}{\partial r}\,,\,\frac{\psi}{r}\right) \le \alpha_T \|\psi\|_{2*,2;1/r} \cdot \|T\|_{1,2;r} \le \frac{\varepsilon\,\alpha_T}{2}\,\|\psi\|^2_{2*,2;1/r} + \frac{\alpha_T}{2\varepsilon}\|T\|^2_{1,2;r}\,.$$

We choose $\varepsilon > 0$ such that the constant by $\|\psi\|^2$ is less than $K_0\,\nu/16$. Let k_{TG} be a constant dominating the constant standing by $\|T\|^2$ and let $k_{T\Gamma}$ be an analogous constant obtained during the estimate of the second integral containing T. Then we choose k_T such that $(k_{TG} + k_{T\Gamma}) = K_0 k_T \nu_T/8$. In the same way we choose the constant k_C to dominate the integrals with C and the desired estimate follows.

In case $\alpha_m = 0$ we cannot put $k_\chi = 0$ and thus the term with multiplier k_χ remains. Nevertheless it can be estimated by the same trick. Indeed, the term is estimated by $k_\chi\,\|\Omega\| \cdot \|\chi\|$ which is less or equal to $\|\Omega\|^2\,\varepsilon/2 + \|\chi\|^2\,k_\chi^2/(2\varepsilon)$. With $\varepsilon = K_0/2$ we can choose k_χ such that also the constant by $\|\chi\|^2$ is dominated by $K_0/4$ and the estimate is true even in this case. ∎

Lemma 5.8. *The forced convection operator $\mathscr{D}$ satisfies*

$$\langle \mathscr{D}(\mathrm{U}), \mathrm{U}\rangle \le K_{\mathscr{D}} \|\mathrm{U}\|^2_{\mathbb{W}} \qquad \textit{for}\ \ \mathrm{U} \in \mathbb{V}\,.$$

By proper choice of auxiliary functions Ω_b, T_b, C_b (satisfying (5.6) and the corresponding nonhomogeneous boundary conditions (2.25), (2.26)) the positive constant $K_{\mathscr{D}}$ can be made arbitrary small, particularly, there exist functions Ω_b^*, T_b^*, C_b^* such that $K_{\mathscr{D}} = K_0/4$.

The proof is based on the following lemma, see e. g. [3], [18]:

Lemma 5.9. (Lemma on cut off function) *For each $\varepsilon > 0$ there exists a smooth function $\zeta \in C^\infty(\overline{G})$, $\ 0 \le \zeta \le 1\ $ with $\zeta = 1$ on Γ and satisfying* $\mathrm{meas}(\mathrm{supp}(\zeta)) \le \varepsilon$.

Proof of Lemma 5.8. Taking a cut off function ζ of the previous lemma we replace the functions Ω_b, T_b, C_b by functions

$$\Omega_b^* = \Omega_b\zeta\,, \qquad T_b^* = T_b\zeta\,, \qquad C_b^* = C_b\zeta\,.$$

Let us remark that we replace U_b by U_b^* not only in $\mathscr{D}$ but also in the functional $\mathscr{F}_s$ denoted again by $\mathscr{F}_s$.

Since $\zeta = 1$ at the boundary the new functions Ω_b^*, T_b^*, C_b^* coincide with Ω_b, T_b, C_b on Γ and satisfy the required boundary conditions. Since ζ is smooth (5.6) remains valid. Since ζ can have support of arbitrarily small measure we get the desired estimate.

The expression $\langle \mathscr{D}(\mathrm{U})\,, \mathrm{U}\,\rangle$ contains one integral and three forms b. The form with T_b is transformed by Lemma 3.2 and estimated using Lemmas 4.4, 4.5

$$|b(\psi, T_b^*; T)| = |b(\psi, T; T_b^*)| \leq \mathrm{const} \cdot \|\psi\|_{2*,2;1/r} \cdot \|T\|_{1,2;r} \cdot \|T_b^*\|_{4;r}\,.$$

Due to imbedding $W_r^1(G) \subset L^6(G)$ (Lemma 4.1) and Hölder inequality we obtain

$$\|T_b^*\|_{4;r} = \|T_b\zeta\|_{4;r} \leq \|T_b\|_{6;r} \cdot \|\zeta\|_{12;r} \leq \mathrm{const} \cdot \|T_b\|_{1,2;r} \cdot \|\zeta\|_{12;r}\,,$$

where $\|\zeta\|_{12;r}$ can be made arbitrary small. Indeed, following Lemma on cut off function we can write

$$\|\zeta\|_{12;r}^{12} = \int_G r\,\zeta^{12}\,\mathrm{d}G \leq \max_G r \cdot \mathrm{meas}(\mathrm{supp}(\zeta)) \leq \mathrm{const} \cdot \varepsilon\,.$$

In the same way we can handle the form b with C_b.

The term b with Ω_b can be estimated similarly. Using Lemmas 3.2, 4.4 and 4.5 we get

$$\left| b\left(\psi, \Omega_b^*; \frac{\widetilde{\Omega}}{r^2}\right)\right| = \left| b\left(\psi, \frac{\widetilde{\Omega}}{r^2}; \Omega_b^*\right)\right| \leq \mathrm{const}\|\psi\|_{2*,2;1/r}\,\|\widetilde{\Omega}\|_{1,2;1/r}\,\left\|\frac{\Omega_b^*}{r}\right\|_{4;r}\,.$$

Due to the imbedding $\Omega_b/r \in W_r^1(G) \subset L^6(G)$ we have $\|\Omega_b\zeta/r\|_{4;r} \leq \|\Omega_b/r\|_{6;r} \cdot \|\zeta\|_{12;r}$ and we can proceed in the same way as above. Applying the integration by parts (3.2) to obtain an expression with undifferentiated Ω_b^* we can handle the last term and the proof is complete. ▪

Proof of Theorem 5.2. Due to estimates in the previous lemmas all terms are integrable i. e. the problem is well defined. Moreover all operators and functionals are bounded and continuous.

Let us verify the assumptions of the abstract existence Theorem 5.3. The space $\mathbb{V}$ is a separable reflexive Banach space since it is a product of closed subspaces of separable reflexive Banach spaces.

Weak continuity of the operator $\mathscr{A} + \mathscr{B} + \mathscr{C} + \mathscr{D}$ was proved in Lemma 5.4. The coercivity is a consequence of Lemmas 5.5–5.8: the coercivity of operator $\mathscr{A}$ is not violated if we use the indicated choice of the constants k_T, k_C, k_χ and the auxiliary functions Ω_b^*, T_b^*, C_b^* with the cut off function ζ:

$$\langle (\mathscr{A} + \mathscr{B} + \mathscr{C} + \mathscr{D})(\mathrm{U})\,, \mathrm{U}\,\rangle \geq \frac{1}{2}\,K_0\|\mathrm{U}\|_{\mathbb{W}}^2 \qquad \text{for} \quad \mathrm{U} \in \mathbb{V}. \tag{5.18}$$

Thus the solution exists and the proof is complete. ▪

Remark. For small data we can prove even the uniqueness of the solution.

6. Evolution problem

In this section we introduce the weak formulation of the evolution problem with some time dependent data and prove existence of its weak solution by means of the Rothe method.

Evolution problem. In the evolution problem all the unknowns are time dependent and the equations contain terms with time derivatives. The functions Ω_b, T_b, C_b and the data $\gamma_T, \gamma_C, g_T, g_C$ and α_m may be time dependent. The boundary value problem is completed by initial conditions. The basis for the formulation are integral identities (3.20), (3.14), (3.10), (3.8), (3.12).

Operator equation. Like in the stationary case we gather the unknowns into a vector function

$$\mathrm{U}(t) \equiv (\psi, \Omega, T, c, \chi)(t)$$

and test functions into a vector $\mathrm{V} \equiv (\widetilde{\psi}, \widetilde{\Omega}, \widetilde{T}, \widetilde{C}, \widetilde{\chi})$. We sum up the integral identities multiplied with the same constants $1, 1, k_T, k_C, k_\chi$ as in the stationary problem of Section 5. Further we split the terms into operators $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ defined by (5.10)–(5.12), (5.14). Since we admit time dependent data $\Omega_b(t)$, $T_b(t)$, $C_b(t)$, $\gamma_T(t)$, $\gamma_C(t)$, $g_T(t)$, $g_C(t)$, $\alpha_m(t)$ the operators (except for $\mathscr{B}$) are time dependent $\mathscr{A}(t)$, $\mathscr{C}(t)$, $\mathscr{D}(t)$.

The remaining evolution terms with time derivatives will be gathered into an operator $\mathscr{E}$ defined by

$$\langle \mathscr{E}(\mathrm{U})\,,\mathrm{V}\,\rangle = \,a_{-1}\Big(\psi, \widetilde{\psi}\Big) + \left(\frac{\Omega}{r}, \frac{\widetilde{\Omega}}{r}\right) + k_T\left(T, \widetilde{T}\right) + k_C\left(C, \widetilde{C}\right). \tag{6.1}$$

Also the right-hand side functional is time dependent

$$\begin{aligned}\langle \mathscr{F}_e(t)\,,\mathrm{V}\,\rangle = k_T\,\nu_T \int_{\Gamma_s} r\, g_T(t)\,\widetilde{T}\,\mathrm{d}\Gamma + k_C\,\nu_C \int_{\Gamma_s} r\, g_C(t)\,\widetilde{C}\,\mathrm{d}\Gamma \\ - \langle \mathscr{A}(\mathrm{U}_b(t))\,,\mathrm{V}\,\rangle - \int_G \frac{1}{r^3}\frac{\partial}{\partial z}\Big(\Omega_b^2(t)\Big)\widetilde{\psi}\,\mathrm{d}G \\ - \langle \mathscr{C}(\mathrm{U}_b(t))\,,\mathrm{V}\,\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\,\langle \mathscr{E}(\mathrm{U}_b(t))\,,\mathrm{V}\,\rangle\,.\end{aligned} \tag{6.2}$$

Thus we obtain an operator equation — evolution equivalent of (5.16):

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U}(t)) + (\mathscr{A}(t) + \mathscr{B} + \mathscr{C}(t) + \mathscr{D}(t))\,(\mathrm{U}(t)) = \mathscr{F}_e(t)\,. \tag{6.3}$$

Vector function spaces. In Section 5 by (5.8) we introduced the basic Sobolev space $\mathbb{W}$ and by (5.9) its subspace $\mathbb{V}$ of vectors satisfying homogeneous boundary conditions. Both spaces are equipped with the norm $\|\cdot\|_{\mathbb{W}}$ defined by (5.17). The dual space to $\mathbb{V}$ is denoted by $\mathbb{V}^*$ and equipped with the natural norm $\|\cdot\|_{\mathbb{V}^*}$.

We introduce another space $\mathbb{H}$ for time derivatives of the unknowns

$$\mathbb{H} = W^1_{1/r} \times L^2_{1/r}(G) \times L^2_r(G) \times L^2_r(G) \times L^2_{1/r}(G)\,, \tag{6.4}$$

and its subspace $\mathbb{H}_0$ of vectors with the last zero component

$$\mathbb{H}_0 = W^1_{1/r} \times L^2_{1/r}(G) \times L^2_r(G) \times L^2_r(G) \times \{0\} \tag{6.5}$$

since $\mathscr{E}$ does not contain the component χ. Using the operator $\mathscr{E}$ we endow the space $\mathbb{H}_0$ with a scalar product $(\cdot,\cdot)_{\mathbb{H}_0}$ and the corresponding norm

$$(\mathrm{U}\,,\mathrm{V})_{\mathbb{H}_0} = \langle \mathscr{E}(\mathrm{U})\,,\mathrm{V}\,\rangle\;, \qquad \|\mathrm{U}\|_{\mathbb{H}_0} = (\mathrm{U},\mathrm{U})^{1/2}_{\mathbb{H}_0} \equiv \langle \mathscr{E}(\mathrm{U})\,,\mathrm{U}\,\rangle^{1/2}\;. \tag{6.6}$$

To cover the whole space $\mathbb{H}$ we complete the scalar product and the norm with the last component

$$(\mathrm{U}\,,\mathrm{V})_{\mathbb{H}} = \langle \mathscr{E}(\mathrm{U})\,,\mathrm{V}\,\rangle + k_\chi \left(\frac{\chi}{r}, \frac{\widetilde{\chi}}{r}\right)\,, \qquad \|\mathrm{U}\|_{\mathbb{H}} = (\mathrm{U}\,,\mathrm{U})_{\mathbb{H}}^{1/2} \tag{6.7}$$

Clearly on $\mathbb{H}_0$ both norms coincide. In case of vectors $\mathrm{U} = \{\psi, \Omega, T, C, \chi\}$ with components Ω, χ satisfying equation (2.24) i. e. in the weak formulation

$$\text{Find } \chi \in V_\chi \ \text{ such that } \quad a_{-1}(\chi, \widetilde{\chi}) = -\left(\frac{1}{r}\frac{\partial \Omega}{\partial z}\,, \frac{\widetilde{\chi}}{r}\right) \quad \forall\, \widetilde{\chi} \in V_\chi\,. \tag{6.8}$$

We see $\|\chi\|_{1,2;1/r} \le \mathrm{const}\|\Omega\|_{1,2;1/r}$ and thus both norms are equivalent i. e. there exists a constant $c_\chi > 0$ such that

$$c_\chi\, \|\mathrm{U}\|_{\mathbb{H}} \le \|\mathrm{U}\|_{\mathbb{H}_0} \le \|\mathrm{U}\|_{\mathbb{H}} \tag{6.9}$$

for all $\mathrm{U} = (\psi, \Omega, T, C, \chi) \in \mathbb{V}$ satisfying (6.8).

We shall deal with the space $\mathbb{H}^*$ dual to the space $\mathbb{H}$. Using Riesz representation theorem we identify both spaces and identify the duality map $\langle\cdot\,,\cdot\,\rangle_{\mathbb{H}}$ on $\mathbb{H}^* \times \mathbb{H}$ with the scalar product $(\cdot\,,\cdot)_{\mathbb{H}}$ on $\mathbb{H} \times \mathbb{H}$. The evolution problem will be considered in a time interval $I = [0, \Theta]$. The vector function $\mathrm{U}(t)$ takes its values in the space $\mathbb{V}$. The time derivative $(\mathrm{d}/\mathrm{d}t)\mathscr{E}(\mathrm{U})$ is considered in generalized sense — in the sense of scalar distribution in t. Thus the operator equation (6.3) with space test function V and time test function φ is taken in the sense

$$\begin{aligned} &\int_I [\, -\langle \mathscr{E}(\mathrm{U}(t))\,,\mathrm{V}\,\rangle\, \varphi'(t) \\ &\qquad + \langle (\mathscr{A}(t) + \mathscr{B} + \mathscr{C}(t) + \mathscr{D}(t))(\mathrm{U}(t))\,,\mathrm{V}\,\rangle\, \varphi(t)]\ \mathrm{d}t \\ &= \int_I \langle \mathscr{F}_e(t)\,,\mathrm{V}\,\rangle\, \varphi(t)\,\mathrm{d}t\,, \quad \forall \mathrm{V} \in \mathbb{V}\,, \forall \varphi \in C_0^\infty(I)\,. \end{aligned} \tag{6.10}$$

In order to formulate the evolution problem we need to specify time dependence. We adopt standard notation for spaces of abstract functions. The space of abstract functions with values in $\mathbb{X}$ Bochner integrable on I with $p-$th power are denoted by $L^p(I, \mathbb{X})$ and essentially bounded functions by $L^\infty(I, \mathbb{X})$. Further the space of continuous functions with values in $\mathbb{X}$ will be denoted by $C(I, \mathbb{X})$.

We shall work with a sequence of four imbedded Banach spaces:

$$\mathbb{V} \subset \mathbb{H} \cong \mathbb{H}^* \subset \mathbb{V}^*\,. \tag{6.11}$$

All spaces are reflexive separable and the imbedding $\mathbb{V} \subset \mathbb{H}$ is compact. Following Lions theorem (see e. g. [3], section 8.6) we have the following result:

Lemma 6.1. *Let $\mathbb{V}, \mathbb{H}, \mathbb{V}^*$ be spaces introduced above and let us denote*

$$\mathscr{W} = \{\mathrm{U} \in L^2(I, \mathbb{V}) \mid \tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U}) \in L^{4/3}(I, \mathbb{V}^*)\}\,, \tag{6.12}$$

where the time derivative is taken in the sense of distribution. Then

(i) *$\mathscr{W} \subset L^2(I, \mathbb{H})$ is compact imbedding,*
(ii) *$\mathscr{W} \subset C(I, \mathbb{V}^*)$ is continuous imbedding.*

Weak formulation of the problem.

Assumptions. (i) *The constitutive relation constants are positive*

$$\nu,\ \nu_T,\ \nu_C > 0\,, \tag{6.13}$$

constants α_T, α_C, β_T, β_C *may be any real numbers.*

(ii) *Functions describing heat conduction, radiation and oxygen evaporation on* Γ_s *satisfy*

$$\gamma_T(t),\ \gamma_C(t) \ge 0\,, \quad \gamma_T,\ \gamma_C \in L^\infty(I)\,, \quad g_T, g_C \in L^2(I, L^2(\Gamma_s))\,. \tag{6.14}$$

(iii) *Forced convection data* o_p, o_c, T_p, C_p, T_c *are such that there exist auxiliary functions* $\mathrm{U}_b = (0, \Omega_b, T_b, C_b, 0)$ *satisfying*

$$\mathrm{U}_b \ \equiv\ (0, \Omega_b, T_b, C_b, 0) \in L^\infty(I, \mathbb{W})\,, \tag{6.15}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{U}_b \ \equiv\ \left(0, \tfrac{\mathrm{d}}{\mathrm{d}t}\Omega_b, \tfrac{\mathrm{d}}{\mathrm{d}t}T_b, \tfrac{\mathrm{d}}{\mathrm{d}t}C_b, 0\right) \in L^2(I, \mathbb{V}^*)\,. \tag{6.16}$$

(iv) *Parameter* $\alpha_m(t)$ *characterizing intensity of the applied magnetic field satisfies*

$$\alpha_m(t) \ge 0\,, \qquad \alpha_m \in L^\infty(I)\,. \tag{6.17}$$

(v) *The initial value vector* U_0 *satisfies*

$$\mathrm{U}_0 \equiv (\psi_0, \Omega_0, T_0, c_0, 0) \in \mathbb{H}_0\,. \tag{6.18}$$

Since no initial condition is prescribed for χ we introduce a projection operator

$$\mathscr{P} : \mathrm{U} = (\psi, \Omega, T, C, \chi) \longmapsto \mathscr{P}(\mathrm{U}) = (\psi, \Omega, T, C, 0)\,. \tag{6.19}$$

Problem 6.2. *A function* $\mathrm{U} + \mathrm{U}_b : I \to \mathbb{W}$ *is called* **weak solution** *to the evolution problem iff*

$$\mathrm{U} \in L^2(I, \mathbb{V}) \cap L^\infty(I, \mathbb{H})\,,$$
$$\mathscr{P}(\mathrm{U}(0) + \mathrm{U}_b(0)) = \mathrm{U}_0 \qquad \text{in } \mathbb{V}^*\,,$$

and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U}(t)) + (\mathscr{A}(t) + \mathscr{B} + \mathscr{C}(t) + \mathscr{D}(t))(\mathrm{U}(t)) = \mathscr{F}_e(t)\,, \tag{6.20}$$

where the operator equation on $\mathbb{V}^*$ *with time derivative is taken in the sense of distributions (6.10).*

Justification of the weak formulation. Taking into account Assumptions 6.1 we can justify the weak formulation:

Lemma 6.3. *Let the assumptions (6.13) – (6.18) be satisfied. If* $\mathrm{U} \in L^2(I, \mathbb{V})$ *then*

$$\mathscr{A}(\mathrm{U}) \in L^2(I, \mathbb{V}^*)\,, \quad \mathscr{C}(\mathrm{U}) \in L^2(I, \mathbb{V}^*)\,, \quad \mathscr{D}(\mathrm{U}) \in L^2(I, \mathbb{V}^*)\,, \tag{6.21}$$

$$\mathscr{F}_e \in L^2(I, \mathbb{V}^*)\,. \tag{6.22}$$

If moreover $\mathrm{U} \in L^\infty(I, \mathbb{H})$ *then we obtain*

$$\mathscr{B}(\mathrm{U}) \in L^{4/3}(I, \mathbb{V}^*) \tag{6.23}$$

If further U *solves the equation (6.20), then its distributional derivative satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U}) \in L^{4/3}(I,\mathbb{V}^*) \tag{6.24}$$

and $\mathrm{U} \in C(I,\mathbb{V}^*)$ *which gives sense to* $\mathscr{P}(\mathrm{U}(0)) + \mathrm{U}_b(0) = \mathrm{U}_0$ *in* $\mathbb{V}^*$.

Sketch of the proof. The properties of the operators $\mathscr{A}(t), \mathscr{B}, \mathscr{C}(t), \mathscr{D}(t)$ are described in lemmas of Section 5. Since the parameters $\gamma_T, \gamma_C, \alpha_m$ are bounded there exists a constant $K_{\mathscr{A}}$ of the estimate in Lemma 5.5 valid for a. e. $t \in I$ and we can estimate the linear operator $\mathscr{A}(t)$

$$\int_I |\langle \mathscr{A}(t)(\mathrm{U}(t)) \,, \mathrm{V}\,\rangle|^2 \,\mathrm{d}t \le \int_I K_{\mathscr{A}}^2 \|\mathrm{U}(t)\|^2 \cdot \|\mathrm{V}\|^2 \,\mathrm{d}t = K_{\mathscr{A}}^2 \cdot \|U\|^2_{L^2(I,\mathbb{W})} \cdot \|\mathrm{V}\|^2_{\mathbb{W}}\,.$$

In the same way we estimate also linear operators $\mathscr{C}(t)$ and $\mathscr{D}(t)$. In this place we need only $\mathrm{U}_b \in L^2(I,\mathbb{W})$. Thus (6.21) takes place.

Similar estimates are used in proof of (6.22). In terms with linear data functions $g_T(t), g_C(t)$ we use (6.14), in terms with time derivative we need (6.16). To obtain (6.23) we need a more delicate estimate of operator $\mathscr{B}$

$$|\langle \mathscr{B}(\mathrm{U}) \,, \mathrm{V}\,\rangle| \le K^*_{\mathscr{B}} \|\mathrm{U}\|^{3/2}_{\mathbb{W}} \cdot \|\mathrm{U}\|^{1/2}_{\mathbb{H}_0} \cdot \|\mathrm{V}\|_{\mathbb{W}}\,. \tag{6.25}$$

The inequality (6.25) is based on estimates of Lemma 4.8. We can write

$$\begin{aligned}\int_I |\langle \mathscr{B}(\mathrm{U}(t)) \,, \mathrm{V}\,\rangle|^{4/3} \,\mathrm{d}t &\le (K^*_{\mathscr{B}})^{4/3} \int_I \|\mathrm{U}(t)\|^2_{\mathbb{W}} \cdot \|\mathrm{U}(t)\|^{2/3}_{\mathbb{H}_0} \cdot \|\mathrm{V}\|^{4/3}_{\mathbb{W}} \,\mathrm{d}t \\ &\le (K^*_{\mathscr{B}})^{4/3} \|\mathrm{U}\|^2_{L^2(I,\mathbb{W})} \cdot \|\mathrm{U}\|^{2/3}_{L^\infty(I,\mathbb{H})} \cdot \|\mathrm{V}\|^{4/3}_{\mathbb{W}}\end{aligned}$$

which yield (6.24).

Let U satisfy the operator equality (6.20). Since the other terms are at least in $L^{4/3}(I,\mathbb{V}^*)$ the sixth assertion follows. Since the derivative is integrable in t Lemma 6.1 yields $\mathrm{U} \in C(I,\mathbb{V}^*)$ and the initial condition makes sense. ∎

Main result – Existence theorem.

Theorem 6.4. *Let the assumptions (6.13) – (6.18) be satisfied. Then there exists a solution to the evolution problem 6.2.*

The proof will be carried out by means of the Rothe method. First we deal with time dependent data. We use the existence result of Section 5. Due to the assumptions (6.13) – (6.17) the data $\gamma_T(t), \gamma_C(t), \alpha_m(t)$ are bounded and we can choose the constants k_T, k_C, k_χ such that the estimates of Lemmas 5.5, 5.7 are valid with the constants $K_0, K_{\mathscr{A}}, K_{\mathscr{C}}$ independent of t. Further, since Ω_b, T_b, C_b are bounded in time (6.15) we can replace them by Ω^*_b, T^*_b, C^*_b with a time independent cut off function ζ such that the estimate of Lemma 5.8 is valid for a. e. $t \in I$. Thus for a. e. $t \in I$ the operator $(\mathscr{A} + \mathscr{B} + \mathscr{C} + \mathscr{D})$ is weakly continuous and coercive.

We look for the solution in the form $\mathrm{U}(t) + \mathrm{U}^*_b(t)$, with $\mathrm{U}^*_b(t) = \zeta \mathrm{U}_b(t)$. Replacing $\mathrm{U}_b(t)$ by $\mathrm{U}^*_b(t)$ in $\mathscr{D}(t)$ we replace it also in $\mathscr{F}_e$ but the functional is again denoted by $\mathscr{F}_e(t)$. This modified U^*_b will be also denoted by U_b.

The proof can be divided into several steps. Let us characterize the individual steps:

- — By semidiscretization in time we obtain a sequence of stationary problems for "time level" solutions $\mathrm{U}^i \approx \mathrm{U}(t_i)$.
- — We prove existence of the level solutions U^i.
- — We find apriori estimates of the solutions U^i.
- — We introduce two sequences of Rothe functions $\overline{\mathrm{U}}_n(t)$, $\widehat{\mathrm{U}}_n(t)$.
- — Using the apriori estimates we extract weakly converging subsequences and prove that their limits coincide.
- — The operator equation (6.20) is satisfied in sense (6.37) if we replace U by Rothe function $\widehat{\mathrm{U}}_n$ in the term with time derivative and by $\overline{\mathrm{U}}_n$ in all other terms. We justify passing to the limit $n \to \infty$ in this equality and find that the limit U is the solution.

Semidiscretization. Let n be a positive integer and let us divide the time interval $I = [0, \Theta]$ into n parts by $t_i = i \cdot h\,,\ i = 0, \dots, n$ with time step $h = \Theta/n$. The vector U^i will correspond to the value $\mathrm{U}(t_i)$.

The first vector U^0 is given by

$$\mathrm{U}^0 = (\psi_0, \Omega_0, T_0, C_0, \chi_0) - \mathrm{U}_b(0)\,,$$

where $(\psi_0, \Omega_0, T_0, C_0, 0) \equiv \mathrm{U}_0$ is from the initial condition. Since we have no initial condition for χ we set $\chi_0 = 0$.

The other vectors $\mathrm{U}^i \in \mathbb{V}\quad (i = 1, 2, \dots, n)$ will be defined using the operator equation (6.20). Since some operators are time dependent we introduce the *average discretization.* Let $\mathscr{T}$ be an abstract function $\mathscr{T} : I \to \mathbb{X}$. Then by $\mathscr{T}^i$, $i = 1, \dots, n$ we denote a family of integral averages

$$\mathscr{T}^i = \frac{1}{h}\int_{t_{i-1}}^{t_i} \mathscr{T}(t)\,\mathrm{d}t\,. \tag{6.26}$$

Replacing the time derivative with time difference in the operator equation we obtain

$$\tfrac{1}{h}\Big[\mathscr{E}(\mathrm{U}^i - \mathrm{U}^{i-1})\Big] + \mathscr{A}^i(\mathrm{U}^i) + \mathscr{B}(\mathrm{U}^i) + \mathscr{C}^i(\mathrm{U}^i) + \mathscr{D}^i(\mathrm{U}^i) = \mathscr{F}^i\,, \tag{6.27}$$

where $\mathscr{A}^i$, $\mathscr{C}^i$, $\mathscr{D}^i$, $\mathscr{F}^i$ are operators $\mathscr{A}(t)$, $\mathscr{C}(t)$, $\mathscr{D}(t)$ and functional $\mathscr{F}_e(t)$ discretizated by (6.26).

Lemma 6.5. *The sequence of semidiscretized problems (6.27) admits solutions* $\mathrm{U}^1, \dots, \mathrm{U}^n$ *.*

Proof. The existence is obtained by induction. Having the vector U^{i-1} the next vector U^i is given as a solution of the following equation

$$\Big(\tfrac{1}{h}\mathscr{E} + \mathscr{A}^i + \mathscr{B} + \mathscr{C}^i + \mathscr{D}^i\Big)(\mathrm{U}^i) = \tfrac{1}{h}\mathscr{E}(\mathrm{U}^{i-1}) + \mathscr{F}^i\,. \tag{6.28}$$

The problem is a stationary equation of type studied in Section 5. Having chosen constants k_T, k_C, k_χ and auxiliary function $\mathrm{U}_b = (0, \Omega_b, T_b, C_b, 0)$ such that (5.18) holds for a. e. $t \in I$ we obtain the same estimate for the integral averages defined by (6.26). Thus the left hand side operator is coercive and weakly continuous and existence of the sequence $\mathrm{U}^1, \dots, \mathrm{U}^n$ follows. ∎

Apriori estimates.

Lemma 6.6. *The sequence of vectors* U^i , $i = 1, 2, \dots, n$ *satisfies the following estimates with a constants* $\mathbf{C}_1, \mathbf{C}_2$ *independent of* n

$$\max_{k \in \{1,\dots,n\}} \left\| \mathrm{U}^k \right\|_{\mathbb{H}} \leq \mathbf{C}_1 \,, \tag{6.29}$$

$$\sum_{i=1}^{n} \left\| \mathrm{U}^i - \mathrm{U}^{i-1} \right\|_{\mathbb{H}}^2 \leq \mathbf{C}_1 \,, \tag{6.30}$$

$$h \sum_{i=1}^{n} \left\| \mathrm{U}^i \right\|_{\mathbb{W}}^2 \leq \mathbf{C}_2 \,. \tag{6.31}$$

Proof. Let us apply equation (6.26) to the test function U^i and use (6.6)

$$\tfrac{1}{h} \left(\mathrm{U}^i - \mathrm{U}^{i-1} \,, \mathrm{U}^i \right)_{\mathbb{H}_0} + \left\langle (\mathscr{A}^i + \mathscr{B} + \mathscr{C}^i + \mathscr{D}^i)(\mathrm{U}^i) \,, \mathrm{U}^i \right\rangle = \left\langle \mathscr{F}^i \,, \mathrm{U}^i \right\rangle \tag{6.32}$$

Using inequality $ab \leq a^2/(4\varepsilon) + \varepsilon\, b^2$ the right hand side can be estimated

$$\left| \left\langle \mathscr{F}^i \,, \mathrm{U}^i \right\rangle \right| \leq \left\| \mathscr{F}^i \right\|_{\mathbb{V}^*} \cdot \left\| \mathrm{U}^i \right\|_{\mathbb{W}} \leq \frac{1}{4\varepsilon} \| \mathscr{F}^i \|_{\mathbb{V}^*}^2 + \varepsilon \| \mathrm{U}^i \|_{\mathbb{W}}^2 \,.$$

We multiply inequality (6.32) by $2h$. We estimate the second term of the left hand side using (5.18); the constant $K_0/2$ is diminished by ε from the estimate of $\mathscr{F}^i$ in the right hand side. We estimate the first term in the left hand side using equality $2(\|a\|^2 - (a, b)) = \|a\|^2 - \|b\|^2 + \|a - b\|^2$ with the scalar product on the space $\mathbb{H}_0$ defined by (6.6). Thus we obtain

$$\| \mathrm{U}^i \|_{\mathbb{H}_0}^2 - \| \mathrm{U}^{i-1} \|_{\mathbb{H}_0}^2 + \| \mathrm{U}^i - \mathrm{U}^{i-1} \|_{\mathbb{H}_0}^2 + 2\, h (K_0/2 - \varepsilon) \| \mathrm{U}^i \|_{\mathbb{W}}^2 \leq \frac{h}{2\varepsilon} \| \mathscr{F}^i \|_{\mathbb{V}^*}^2 \,.$$

Now the result follows. Summing the inequalities for $i = 1$ up to k we obtain the first inequality (6.29); summing the inequalities up to n we obtain (6.30), (6.31) since the right hand side is bounded by a constant independent of n

$$h \sum_{i=1}^{k} \| \mathscr{F}^i \|_{\mathbb{V}^*}^2 \leq h \sum_{i=1}^{n} \| \mathscr{F}^i \|_{\mathbb{V}^*}^2 \leq \int_I \| \mathscr{F}_e(t) \|_{\mathbb{V}^*}^2 \,\mathrm{d}t = \| \mathscr{F}_e \|_{L^2(I, \mathbb{V}^*)}^2 \,.$$

The components of vectors U^i satisfy (6.8) thus in the estimates we can replace the norm $\| \cdot \|_{\mathbb{H}_0}$ by $c_\chi \| \cdot \|_{\mathbb{H}}$, see (6.9). ∎

Lemma 6.7. *The sequence of vectors* U^i , $i = 1, 2, \dots, n$ *satisfies the following estimate with constant* $\mathbf{C}_3$ *independent of* n

$$h \sum_{i=1}^{n} \left\| \tfrac{1}{h}\, \mathscr{E}(\mathrm{U}^i - \mathrm{U}^{i-1}) \right\|_{\mathbb{V}^*}^{\frac{4}{3}} \leq \mathbf{C}_3. \tag{6.33}$$

Proof. We express $(\mathscr{E}(\mathrm{U}^i - \mathrm{U}^{i-1}))/h$ from the equation (6.27). Using estimates of Lemmas 5.5 − 5.8 and inequality (6.25) we obtain

$$\begin{aligned}&\left\|\tfrac{1}{h}\mathscr{E}(\mathrm{U}^i - \mathrm{U}^{i-1})\right\|_{\mathbb{V}^*}\\ &\qquad\le (K_{\mathscr{A}} + K_{\mathscr{C}} + K_{\mathscr{D}})\,\|\mathrm{U}^i\|_{\mathbb{W}} + K^*_{\mathscr{B}}\|\mathrm{U}^i\|^{3/2}_{\mathbb{W}}\|\mathrm{U}^i\|^{1/2}_{\mathbb{H}} + \|\mathscr{F}^i\|_{\mathbb{V}^*}\,.\end{aligned}$$

We put both sides to power 4/3, use inequality $(a+b+c)^p \le c_p(a^p+b^p+c^p)$ with $p = 4/3$. Multiplying the equations by h, summing them from 1 to n and using estimates (6.29), (6.30) of the preceding lemma we obtain the desired estimate. ∎

Rothe functions and their properties. We shall pass from semidiscretized problem to time dependent problem. For each n we introduce two Rothe functions: a "stair function" i. e. piecewise constant function $\overline{\mathrm{U}}_n(t)$

$$\overline{\mathrm{U}}_n(t) = \begin{cases} \mathrm{U}^0 & \text{for } t = 0\,, \\ \mathrm{U}^i & \text{for } t \in (t_{i-1}, t_i],\ i = 1, 2, \dots, n \end{cases} \tag{6.34}$$

and a "polygonal" i. e. continuous piecewise linear function $\widehat{\mathrm{U}}_n(t)$

$$\widehat{\mathrm{U}}_n(t) = \mathrm{U}^i - \frac{t_i - t}{h}\left(\mathrm{U}^i - \mathrm{U}^{i-1}\right) \quad \text{for } t \in [t_{i-1}, t_i]\,,\ i = 1, 2, \dots, n\,. \tag{6.35}$$

Since on $(0, t_1)$ the function $\widehat{\mathrm{U}}_n$ has values that need not be in $\mathbb{V}$ we introduce a modified Rothe polygonal function $\widehat{\mathrm{U}}^*_n(t)$ setting $\widehat{\mathrm{U}}^*_n(t) = \mathrm{U}^1$ for $t \in [0, h]$. The Rothe functions pass through values U^i i. e. we have $\overline{\mathrm{U}}_n(t_i) = \widehat{\mathrm{U}}_n(t_i) = \widehat{\mathrm{U}}^*_n(t_i) = \mathrm{U}^i$ for $i = 1, \dots, n$.

For a function $\mathscr{T} : I \to \mathbb{X}$ we defined the family $\mathscr{T}^1, \dots, \mathscr{T}^n$ by (6.26). Now using $\mathscr{T}^i$ we introduce a stair approximation of the function $\mathscr{T}$ setting

$$\overline{\mathscr{T}}_n(t) = \mathscr{T}^i \quad \text{for } t \in (t_{i-1}, t_i]\,, \quad i = 1, 2, \dots, n\,. \tag{6.36}$$

Since $\frac{1}{h}(\mathrm{U}^i - \mathrm{U}^{i-1}) = \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathrm{U}}(t)$ for $t \in (t_{i-1}, t_i)$ we observe that these functions satisfy the operator equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\widehat{\mathrm{U}}_n) + \overline{\mathscr{A}}_n(\overline{\mathrm{U}}_n) + \mathscr{B}(\overline{\mathrm{U}}_n) + \overline{\mathscr{C}}_n(\overline{\mathrm{U}}_n) + \overline{\mathscr{D}}_n(\overline{\mathrm{U}}_n) = \overline{\mathscr{F}}_n \quad \forall t \ne t_i\,, \tag{6.37}$$

where $\overline{\mathscr{A}}_n(t), \overline{\mathscr{C}}_n(t), \overline{\mathscr{D}}_n(t), \overline{\mathscr{F}}_n(t)$ are stair approximations of $\mathscr{A}^i, \mathscr{C}^i, \mathscr{D}^i, \mathscr{F}^i$ defined by (6.36).

The estimates for $\{\mathrm{U}^i\}$ of Lemmas 6.6, 6.7 yield similar estimates for the Rothe functions. Since $\widehat{\mathrm{U}}_n(t)$ need not be in $\mathbb{V}$ for $t < h$ the corresponding estimate is valid on $I_h = (h, \Theta)$ only:

Lemma 6.8. *The Rothe functions* $\overline{\mathrm{U}}_n(t)$ *and* $\widehat{\mathrm{U}}_n(t)$ *satisfy the following estimates with constants independent of* n*:*

$$\left\|\overline{\mathrm{U}}_n\right\|_{L^\infty(I,\mathbb{H})} \le \mathbf{C}_1\,, \qquad \left\|\widehat{\mathrm{U}}_n\right\|_{C(I,\mathbb{H})} \le \mathbf{C}_1\,, \tag{6.38}$$

$$\left\|\overline{\mathrm{U}}_n\right\|_{L^2(I,\mathbb{V})} \le \mathbf{C}_2\,, \quad \left\|\widehat{\mathrm{U}}_n\right\|_{L^2(I_h,\mathbb{V})} \le \mathbf{C}_2\,, \quad \left\|\widehat{\mathrm{U}}^*_n\right\|_{L^2(I,\mathbb{V})} \le \mathbf{C}_2\,, \tag{6.39}$$

$$\left\|\overline{\mathrm{U}}_n - \widehat{\mathrm{U}}_n\right\|^2_{L^2(I,\mathbb{H})} \le \frac{h}{3}\,\mathbf{C}_1\,, \tag{6.40}$$

$$\left\|\tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\widehat{\mathrm{U}}_n)\right\|_{L^{4/3}(I,\mathbb{V}^*)} \le \mathbf{C}_3^{3/4}\,, \tag{6.41}$$

where $I_h = (h,\Theta)$*.*

Compactness and limit procedure.

Lemma 6.9. *There exists a function* $\mathrm{U} : I \to \mathbb{V}\,,\ \mathrm{U} \in \mathscr{W}$ *and*

$$\mathrm{U} \in L^2(I,\mathbb{V}) \cap L^\infty(I,\mathbb{H})\,, \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U}) \in L^{4/3}(I,\mathbb{V}^*)$$

and a subsequence n' *such that* ($\varepsilon > 0$)

$$\overline{\mathrm{U}}_{n'},\, \widehat{\mathrm{U}}_{n'} \to \mathrm{U} \ \ \textit{weak-}\star \textit{ in } \ L^\infty(I,\mathbb{H}) \quad \textit{and strongly in } \ L^2(I,\mathbb{H})\,, \tag{6.42}$$

$$\overline{\mathrm{U}}_{n'}, \widehat{\mathrm{U}}^*_{n'} \rightharpoonup \mathrm{U} \textit{ weakly in } L^2(I,\mathbb{V})\,, \quad \widehat{\mathrm{U}}_{n'} \rightharpoonup \mathrm{U} \textit{ weakly in } L^2(I_\varepsilon,\mathbb{V})\,, \tag{6.43}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\widehat{\mathrm{U}}_{n'}) \rightharpoonup \tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U}) \ \ \textit{weakly in } \ L^{4/3}(I,\mathbb{V}^*)\,, \tag{6.44}$$

Moreover the limit U *satisfies the initial condition* $\mathscr{P}(\mathrm{U}+\mathrm{U}_b)(0) = \mathrm{U}_0$.

Sketch of the proof. We shall use the following properties of reflexive spaces $\mathbb{X}$:

— a sequence bounded in $L^p(I,\mathbb{X})$ $(1 < p < \infty)$ contains a weakly converging subsequence,

— a sequence bounded in $L^\infty(I,\mathbb{X})$ contains a weak-$\star$ converging subsequence.

The estimates of Lemma 6.8 ensure existence of a subsequence n' such that

— $\overline{\mathrm{U}}_{n'}\,,\ \widehat{\mathrm{U}}_{n'}$ converges weak-$\star$ in $L^\infty(I,\mathbb{H})$,

— $\overline{\mathrm{U}}_{n'}\ ,\ \widehat{\mathrm{U}}^*_{n'}$ converge weakly in $L^2(I,\mathbb{V})$ and $\widehat{\mathrm{U}}_{n'}$ converges weakly in $L^2(I_\varepsilon,\mathbb{V})$ for any $\varepsilon > 0$,

— $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\widehat{\mathrm{U}}_{n'})$ converges weakly in $L^{4/3}(I,\mathbb{V}^*)$.

Compactness of Lemma 6.1 yields strong convergence $\overline{\mathrm{U}}_{n'}, \widehat{\mathrm{U}}_{n'} \to \mathrm{U}$ in $L^2(I,\mathbb{H})$. Due to (6.40) the limits of $\overline{\mathrm{U}}_{n'}$ and $\widehat{\mathrm{U}}_{n'}$ coincide, let us denote the limit by U. Since differentiation is a linear operator we can prove that the limit of derivatives coincide with derivative of the limit function $\lim \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\widehat{\mathrm{U}}_{n'}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\mathrm{U})$. Finally, since $\widehat{\mathrm{U}}_n \in C(I,\mathbb{V}^*)$ and all $\widehat{\mathrm{U}}_n$ satisfy the initial condition then also their limit U in $L^\infty(I,\mathbb{V}^*)$ is continuous and satisfies the initial condition. ∎

Lemma 6.10. *Passing to the limit in (6.37) we obtain (6.10).*

Sketch of the proof. We have to prove convergence of terms of the following form:

$$\int_I \Big\langle \overline{\mathscr{T}}_{n'}(t)(\overline{\mathrm{U}}_{n'}(t)) - \mathscr{T}(t)(\mathrm{U}(t)\,,\mathrm{V} \Big\rangle\, \varphi(t)\,\mathrm{d}t \to 0 \quad \forall\,\mathrm{V}\in\mathbb{V}\,,\ \varphi\in C^\infty(I)\,.$$

In the proof we shall use the following property: Let $\mathscr{T}\in L^p(I,\mathbb{X})$ and $\overline{\mathscr{T}}_n$ be its stair (piecewise constant) approximation defined by (6.26) and (6.36). Then

(i) $\overline{\mathscr{T}}_n \to \mathscr{T}$ strongly in $L^p(I,\mathbb{X})$ if $p<\infty$,
(ii) $\overline{\mathscr{T}}_n \to \mathscr{T}$ weak-$\star$ in $L^p(I,\mathbb{X})$ if $p=\infty$.

Let us sketch the proof. Any continuous function $\mathscr{F}$ on a compact interval I is uniformly continuous with continuity modulus $\delta(\varepsilon)$ which yields the result for continuous functions. Indeed, $\|\overline{\mathscr{T}}_n(t)-\mathscr{T}(t)\| = \|\mathscr{T}(\xi)-\mathscr{T}(t)\| \le \delta(\Theta/n)$ with $|\xi-t|\le h=\Theta/n$. Further continuous functions are dense in L^p-space ($p<\infty$) which yields (i). For $p=\infty$ the density in not true and we obtain only weak-$\star$ convergence (ii).

Let us return to proof of Lemma 6.10. For each n the equality (6.37) is equivalent to

$$\begin{aligned}
&\int_I \Big[-\Big\langle \mathscr{E}(\widehat{\mathrm{U}}_n(t))\,,\mathrm{V}\Big\rangle\,\varphi'(t)\\
&\quad + \Big\langle (\overline{\mathscr{A}}_n+\mathscr{B}+\overline{\mathscr{C}}_n+\overline{\mathscr{D}}_n)(\overline{\mathrm{U}}_n(t))\,,\mathrm{V}\Big\rangle\,\varphi(t)\Big]\,\mathrm{d}t\\
&= \int_I \Big\langle \overline{\mathscr{F}}_n(t)\,,\mathrm{V}\Big\rangle\,\varphi(t)\,\mathrm{d}t\,,\quad \forall \mathrm{V}\in\mathbb{V}\,,\forall\varphi\in C_0^\infty(I)\,.
\end{aligned}\tag{6.45}$$

Taking the extracted subsequence n' and using convergences of Lemma 6.9 we examine all terms of the equality by passing $n'\to\infty$. We see that each term represents an integral of product of one or two sequences (and constant functions respectively) but at most one sequence converges weakly or weak-$\star$. Thus we can pass to the limit. By this argument we can handle all terms of the equality and the lemma is proved. ∎

Summary of the proof of Theorem 6.4. We constructed sequences of Rothe functions $\widehat{\mathrm{U}}_n, \overline{\mathrm{U}}_n$ that satisfy the equation (6.37) and initial condition. We extracted a subsequence n' such that Rothe functions converge to a function U (Lemma 6.9). Lemma 6.10 justify passing to the limit in (6.37). Thus $\mathrm{U}+\mathrm{U}_b$ is a solution to the evolution problem and the proof is complete. ∎

Remarks. (i) We did not admit all data to be time dependent. With respect to physical substance of the problem we supposed the constitutive relation constants are time independent. Nevertheless the proof can be carried out even in the case of bounded time dependent α_T,α_C and bounded continuously time dependent β_T,β_C.

(ii) Further we assumed bounded data U_b in order to find time independent cut off function ζ. Generalization to unbounded U_b would bring time dependent cut off function and other technical difficulties. Nevertheless to obtain $\mathscr{F}_e\in L^2(I,\mathbb{V}^*)$ we need at least $\Omega_b\in L^4(I,W^1_{1/r}(G))$. With some technical difficulties the proof can be extended to the case of e. g. $\mathscr{F}_e \in L^{4/3}(I, \mathbb{V}^*)$.

(iii) Under some additional assumptions the uniqueness of the weak solution can be proved using the method of proof of Theorem 8.7.76 in [3].

Acknowldegment. The author wishes to express his thanks to Professor Otto Litzman (a physicist at Masaryk University in Brno) who introduced him to the problem (they were developing numerical simulation of the flow for the Tesla company in Rožnov pod Radhoštem (Czech Republic) producing Silicon single crystals), and Professors Jindřich Nečas and Miloslav Feistauer for useful discussions on the analysis of Navier-Stokes equations. The existence results were first proved for small material constants $\alpha_T, \alpha_C, \beta_T, \beta_C$ only in 1992 [4]. A discussion with Dr. Knobloch [9] inspired the author to find a way of removing these restrictions.

Finally, the author expresses his thanks to the referee for his careful reading of the text and his helpful suggestions.

References

[1] P. Constantin and C. Foias: *Navier-Stokes equations,* Chicago Lect. Math., Univ. Chicago Press, Chicago, 1988.

[2] M. J. Crochet, P. J. Wouters, F. T. Geyling and A. S. Jordan, *Finite-element simulation of Czochralski bulk flow,* J. Cryst. Growth, **65** (1983), 153–165.

[3] M. Feistauer, *Mathematical methods in fluid dynamics,* Longman Sci. Tech., 1993.

[4] J. Franců, *On modelling of Czochralski flow* (Czech), Ph.D. Thesis, Masaryk Univ., Brno, 1992.

[5] J. Franců, *Weakly continuous operators. Applications to differential equations,* Appl. Math. **39** (1994), 45-56.

[6] G. H. Golub and W. E. Langlois, *Direct solution of the equation for the Stokes stream function,* Comput. Methods Appl. Mech. Engrg. **19** (1979), 391–399.

[7] L. N. Hjellming and J. S. Walker, *Melt motion in a Czochralski crystal puller with an axial magnetic field,* Isothermal Motion, J. Fluid Mech. **164** (1986), 237–273.

[8] K. M. Kim and W. E. Langlois, *Computer simulation of boron transport in magnetic Czochralski growth of silicon,* J. Electrochem. Soc. **133** (1986), 2586-90.

[9] P. Knobloch, *Weak solvability of a model related to crystal growth processes,* Otto-von-Guericke-Universität Magdeburg, Preprint MBI-94-2, April 1994.

[10] P. Knobloch, *Solvability and finite element discretization of a mathematical model related to Czochralski crystal growth,* Otto-von-Guericke-Universität Magdeburg, Preprint MBI-96-5, November 1996.

[11] S. Kobayashi, *Numerical analysis of oxygen transport in magnetic Czochralski crystal growth of silicon,* J. Crystal Growth, **85** (1987), 69–74.

[12] A. Kufner, O. John and S. Fučík, *Function spaces,* Academia, Prague, 1977.

[13] A. Kufner, *Weighted Sobolev spaces,* Teubner-Texte Math. #31, Teubner, Leipzig, 1980.

[14] O. A. Ladyzhenskaja, *Mathematical problems of dynamics of viscous incompressible liquid* (Russian), Gos. Izd. Fiz. Mat. Lit. Moscow, 1961.

[15] W. E. Langlois, *Conservative differencing procedures for rotationally symmetric flow with swirl,* Comput. Methods Appl. Mech. Engrg. **25** (1981), 315–333.

[16] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaries,* Gauthier-Villars, Paris, 1969.

[17] O. Litzman and J. Franců, *Modelling of Czochralski flow,* (Czech), Research report, Dept. Physics, Fac. Sciences, Masaryk University, 1988.

[18] J. Nečas, *Les méthodes directes en théorie des équations elliptiques,* Academia, Prague, 1967.
[19] V. M. Paskonov, V. I. Polezhajev and L. A. Tschudov: *Numerical modeling of heat and mass conduction processes,* (Russian), Nauka, Moscow, 1984.
[20] P. J. Roach, *Computational fluid dynamics,* Hermosa Publ., Albuquerque, 1976.
[21] R. Temam, *Navier-Stokes equations, theory and numerical analysis,* North - Holland, Amsterdam, 1979.

Department of Mathematics
Technical University Brno
Technická 2
616 69 Brno, CZECH REPUBLIC

E-mail address: francu@fme.vutbr.cz