NONANALYTIC SOLUTIONS OF THE KdV EQUATION

PETER BYERS AND A. ALEXANDROU HIMONAS

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To Professor E. K. Ifantis on the occasion of his 67th birthday

We construct nonanalytic solutions to the initial value problem for the KdV equation with analytic initial data in both the periodic and the nonperiodic cases.

1. Introduction

It is well known that the solution to the Cauchy problem of the KdV equation with an analytic initial profile is analytic in the space variable for a fixed time (see Trubowitz [11] and Kato [7]). However, analyticity in the time variable fails. Here, we will present several examples demonstrating this phenomenon of the KdV equation. More precisely, we will show that the initial value problem

$$\begin{gathered}\partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R} \text{ or } \mathbb{T},\ t \in \mathbb{R},\end{gathered} \tag{1.1}$$

where $\varphi(x)$ is an appropriate analytic function, cannot have an analytic solution in t for fixed x, say $x = 0$. By replacing x with $-x$, we see that it suffices to consider the equivalent problem

$$\partial_t u = \partial_x^3 u + u\partial_x u, \tag{1.2}$$

$$u(x,0) = \varphi(x). \tag{1.3}$$

If $u(x,t)$ was analytic in t at $t = 0$, then it could be written as a power series of the following form:

$$u(x,t) = \sum_{j=0}^{\infty} \frac{\partial_t^j u(x,0)}{j!} t^j \tag{1.4}$$

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with a nonzero radius of convergence. In particular, there must be some constant A such that

$$\partial_t^n u(x,0) \leq A^n n! \tag{1.5}$$

for every positive integer n.

In computing the values of $\partial_t^n u(x,0)$, we will rely on the fact (to be demonstrated shortly) that if $u(x,t)$ is a solution to (1.2), then $\partial_t^j u(x,t)$ can be written as a polynomial of $u(x,t), \partial_x u(x,t), \partial_x^2 u(x,t), \dots, \partial_x^{3j} u(x,t)$.

To motivate the discussion that will follow, we will first look at $\partial_t u(x,t)$ and $\partial_t^2 u(x,t)$. The initial value problem (1.2) and (1.3) gives

$$\partial_t u(x,0) = \varphi'''(x) + \varphi(x)\varphi'(x). \tag{1.6}$$

Then, differentiating (1.2), we obtain

$$\begin{aligned}
\partial_t^2 u &= \partial_t\left[\partial_x^3 u + u\partial_x u\right] = \partial_x^3 \partial_t u + u\partial_x \partial_t u + \partial_x u \partial_t u \\
&= \partial_x^3\left(\partial_x^3 u + u\partial_x u\right) + u\partial_x\left(\partial_x^3 u + u\partial_x u\right) + \partial_x u\left(\partial_x^3 u + u\partial_x u\right) \\
&= \partial_x^6 u + u\partial_x^4 u + 3\partial_x^3 u\partial_x u + 3\partial_x^2 u\partial_x^2 u + \partial_x^3 u\partial_x u \\
&\quad + u\partial_x^4 + u\partial_x u\partial_x u + uu\partial_x^2 u + \partial_x u\partial_x^3 u + u\partial_x u\partial_x u \\
&= \partial_x^6 u + 2u\partial_x^4 u + 5\partial_x u\partial_x^3 u + 3\partial_x^2 u\partial_x^2 u + uu\partial_x^2 u + 2u\partial_x u\partial_x u,
\end{aligned} \tag{1.7}$$

and hence

$$\partial_t^2 u(x,0) = \varphi^{(6)} + 2\varphi\varphi^{(4)} + 5\varphi^{(1)}\varphi^{(3)} + 3\varphi^{(2)}\varphi^{(2)} + \varphi\varphi\varphi^{(2)} + 2\varphi\varphi^{(1)}\varphi^{(1)}. \tag{1.8}$$

Now, suppose that the initial data is a function of the form

$$u(x,0) = \varphi(x) = (a-x)^{-d}, \tag{1.9}$$

where a is some (complex-valued) constant. Then we have

$$\partial_t u(x,0) = d(d+1)(d+2)(a-x)^{-(d+3)} + d(a-x)^{-(2d+1)}, \tag{1.10}$$

and we make our key observation: the exponent of $(a-x)$ will be the same for both terms if and only if

$$d = 2. \tag{1.11}$$

In a similar way, we compute for the terms of $\partial_t^2 u$ at $t = 0$,

$$\begin{aligned}
\partial_x^6 u &= d(d+1)(d+2)(d+3)(d+4)(d+5)(a-x)^{-(6+d)},\\
u\partial_x^4 u &= d(d+1)(d+2)(d+3)(a-x)^{-(4+2d)},\\
\partial_x u\partial_x^3 u &= d\cdot d(d+1)(d+2)(a-x)^{-(4+2d)},\\
\partial_x^2 u\partial_x^2 u &= d(d+1)\cdot d(d+1)(a-x)^{-(4+2d)},\\
uu\partial_x^2 u &= d(d+1)(a-x)^{-(2+3d)},\\
u\partial_x u\partial_x u &= d\cdot d(a-x)^{-(2+3d)},
\end{aligned} \tag{1.12}$$

and we again see that for all the terms in the expression of $\partial_t^2 u(x,0)$ to have equal exponents, we must have $d = 2$.

Next, we will show that with this choice of d, the "homogeneity" degree of all the terms in the expression of $\partial_t^j u(x,0)$ is the same number which is equal to $3j+2$. More precisely, if, for a term of the form

$$(\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u)\cdots(\partial_x^{\alpha_\ell} u), \quad \alpha_m \in \{0,1,2,\dots\}, \tag{1.13}$$

we assign the "homogeneity" degree

$$(\alpha_1+2)+(\alpha_2+2)+\cdots+(\alpha_\ell+2) = |\alpha|+2\ell, \tag{1.14}$$

then we have the following lemma.

Lemma 1.1. *If $u(x,t)$ is a solution to the initial value problem (1.2) and (1.3), then*

$$\partial_t^j u = \partial_x^{3j} u + \sum_{|\alpha|+2\ell=3j+2} C_\alpha (\partial_x^{\alpha_1} u)\cdots(\partial_x^{\alpha_\ell} u) \tag{1.15}$$

with $C_\alpha \geq 0$.

Here, for a multi-index $\alpha = (\alpha_1,\dots,\alpha_\ell)$, we use the notation $|\alpha| = \alpha_1+\cdots+\alpha_\ell$.

Proof. Our computations above show that Lemma 1.1 is true for $j = 0,1,2$. Next, we assume that it is true for j and we will show that it is true for $j+1$. We have

$$\partial_t^{j+1} u = \partial_x^{3j}(\partial_t u) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha \partial_t\big[(\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u)\cdots(\partial_x^{\alpha_\ell} u)\big]. \tag{1.16}$$

The first term is equal to

$$\partial_x^{3j}(\partial_x^3 u + u\partial_x u) = \partial_x^{3(j+1)} u + \partial_x^{3j}(u\partial_x u), \tag{1.17}$$

where $\partial_x^{3(j+1)} u$ is the leading term in the expression of $\partial_t^{j+1} u$ and $\partial_x^{3j}(u\partial_x u)$, and by using Leibniz rule, it can be written as a sum of terms of the form $(\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u)$ of homogeneity degree $3(j+1)+2$. Also, using the product rule for differentiation, each term of the sum in (1.15) gives

$$\begin{aligned}\partial_t\big[(\partial_x^{\alpha_1}u)(\partial_x^{\alpha_2}u)\cdots(\partial_x^{\alpha_\ell}u)\big]\\ =\partial_x^{\alpha_1}\partial_t u\cdot(\partial_x^{\alpha_2}u)\cdots(\partial_x^{\alpha_\ell}u)+\cdots+(\partial_x^{\alpha_1}u)(\partial_x^{\alpha_2}u)\cdots\partial_x^{\alpha_\ell}\partial_t u.\end{aligned}\tag{1.18}$$

Finally, replacing $\partial_t u$ by $\partial_x^3 u+u\partial_x u$ in each term of the last sum gives the desired result. For example, the first term is equal to

$$\partial_x^{\alpha_1}\big[\partial_x^3 u+u\partial_x u\big]\cdot(\partial_x^{\alpha_2}u)\cdots(\partial_x^{\alpha_\ell}u),\tag{1.19}$$

and each term that results by applying $\partial_x^{\alpha_1}$ on $\partial_x^3 u+u\partial_x u$ has nonnegative coefficients and homogeneity degree equal to

$$\alpha_1+5+(\alpha_2+2)+\cdots+(a_\ell+2)=|\alpha|+3=3j+2+3=3(j+1)+2.\tag{1.20}$$

□

2. Nonperiodic case

Using the initial condition

$$u(x,0)=(a-x)^{-2},\tag{2.1}$$

we find that

$$\partial_x^{\alpha_m}u(x,0)=(\alpha_m+1)!(a-x)^{-(\alpha_m+2)}.\tag{2.2}$$

Therefore, at $t=0$, relations (1.15) and (2.2) give that

$$\partial_t^j u(x,0)=\Bigg((3j+1)!+\sum_{|\alpha|+2\ell=3j+2}C_\alpha\big[(\alpha_1+1)!\big]\cdots\big[(\alpha_\ell+1)!\big]\Bigg)(a-x)^{-(3j+2)}\tag{2.3}$$

or

$$\partial_t^j u(x,0)=\big[(3j+1)!+b_j\big](a-x)^{-(3j+2)},\tag{2.4}$$

where $b_j\geq 0$.

Finally, using (2.4), we see that

$$\big|\partial_t^j u(x,0)\big|\geq|a-x|^{-(3j+2)}(3j)!.\tag{2.5}$$

Inequality (2.5) shows that $u(x,t)$ cannot be analytic near $t=0$ for any fixed $x\neq a$.

Observe that if $a\in\mathbb{R}$, then $u(x,0)=(a-x)^{-2}$ is real-valued and analytic in $\mathbb{R}-\{a\}$. So, one may ask if there are nonanalytic solutions to KdV when the initial data are analytic everywhere in $\mathbb{R}$.

Globally analytic data. If $a \in \mathbb{C} - \mathbb{R}$, then $u(x,0) = (a-x)^{-2}$ is analytic in $\mathbb{R}$. In particular, if we choose $a = i$ and $x = 0$, then we have

$$\partial_t^j u(0,0) = i^{-(3j+2)}\left[(3j+1)! + b_j\right]. \tag{2.6}$$

However, in this case, the KdV solution is complex-valued. Thus, one may ask the question if we can have real-valued initial data which are analytic on $\mathbb{R}$ and for which the KdV solution is not analytic in t.

Real-valued globally analytic data. Next we choose

$$u(x,0) = \Re(i-x)^{-2}. \tag{2.7}$$

Then

$$\partial_x^k u(x,0) = (k+1)!\Re(i-x)^{-2-k}, \tag{2.8}$$

$$\partial_x^k u(0,0) = -(k+1)!\Re i^{-k} = \begin{cases} -1, & k = 4j, \\ 1, & k = 4j+2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.9}$$

Using (1.15) and (2.8), we have

$$\begin{aligned} &\partial_t^j u(x,0) \\ &\quad = \Re(3j+1)!(i-x)^{-(3j+2)} \\ &\qquad + \sum_{|\alpha|+2\ell=3j+2} C_\alpha\left((\alpha_1+1)!\Re(i-x)^{-2-\alpha_1}\right)\cdots\left((\alpha_k+1)!\Re(i-x)^{-2-\alpha_k}\right), \end{aligned} \tag{2.10}$$

so

$$\begin{aligned} &\partial_t^j u(0,0) \\ &\quad = (3j+t)!\Re i^{-(3j+2)} \\ &\qquad + \sum_{|\alpha|+2\ell=3j+2} C_\alpha\left[(\alpha_1+1)!\right]\left[\Re i^{-(2+\alpha_1)}\right]\cdots\left[(\alpha_\ell+1)!\right]\left[\Re i^{-(2+\alpha_\ell)}\right]. \end{aligned} \tag{2.11}$$

We have

$$\begin{aligned} \Re i^{-(3j+2)} &= (-1)^{1/2(3j+2)}, \\ \left[\Re i^{-(2+\alpha_1)}\right]\cdots\left[\Re i^{-(2+\alpha_\ell)}\right] &= (-1)^\ell\left(\Re i^{-\alpha_1}\right)\cdots\left(\Re i^{-\alpha_\ell}\right). \end{aligned} \tag{2.12}$$

If α_m is an odd number for some m, then the last product equals zero, while if α_m is even for all m, then the last product equals

$$(-1)^\ell(-1)^{(1/2)(\alpha_1+\cdots+\alpha_\ell)} = (-1)^{(1/2)(3j+2)}. \tag{2.13}$$

Therefore, if j is even, then

$$\partial_t^j u(0,0) = -(-1)^{3j/2}\left[(3j+1)! + b_j\right], \tag{2.14}$$

where $b_j \geq 0$, which shows that the solution $u(x,t)$ which exists (see, e.g., [8, 9, 10]) cannot be analytic in t near $t = 0$ when $x = 0$.

3. Periodic case

Now, for the periodic case, define

$$g(x) = \frac{-e^{ix}}{2-e^{ix}} = -\sum_{k=1}^{\infty} 2^{-k} e^{ikx}. \tag{3.1}$$

Then

$$g^{(n)}(x) = -\sum_{k=1}^{\infty} 2^{-k}(ik)^n e^{ikx}, \tag{3.2}$$

$$g^{(n)}(0) = i^{n+2} A_n, \tag{3.3}$$

where

$$A_n = \sum_{k=1}^{\infty} 2^{-k} k^n > 2^{-n} n^n. \tag{3.4}$$

Let $u(x,t)$ be a solution to the initial value problem (1.2) and (1.3) with initial data $\phi(x) = g(x)$. Then, by (1.15), we have

$$\begin{aligned} \partial_t^j u(0,0) &= g^{(3j)}(0) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha \left(g^{(\alpha_1)}(0)\right) \cdots \left(g^{(\alpha_\ell)}(0)\right) \\ &= \left(A_{3j} + \sum_{|\alpha|+2\ell=3j+2} C_\alpha A_{\alpha_1} \cdots A_{\alpha_\ell}\right)\left(i^{3j+2}\right), \end{aligned} \tag{3.5}$$

and by (3.4), we have that for any j,

$$\left|\partial_t^j u(0,0)\right| \geq A_{3j} > 2^{-3j}(3j)^{3j} > (j!)^3. \tag{3.6}$$

Therefore, $u(x,t)$ is not analytic in the t-variable at the point $(0,0)$.

Real-valued solutions. Let $u(x,t)$ be the solution to the Cauchy problem (1.2) and (1.3) with initial data $\phi(x) = \Re g(x)$. By (1.15), we have

$$\partial_t^j u(0,0) = \Re g^{(3j)}(0) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha \Re\left(g^{(\alpha_1)}(0)\right) \cdots \Re\left(g^{(\alpha_\ell)}(0)\right). \tag{3.7}$$

Now, by (3.3), we note that

$$\Re g^{(n)}(0) = \begin{cases} g^{(n)}(0), & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}, \end{cases} \tag{3.8}$$

and thus, any product of the form

$$\Re\left(g^{(\alpha_1)}(0)\right)\cdots\Re\left(g^{(\alpha_\ell)}(0)\right) \tag{3.9}$$

must be equal to either 0 or

$$\left(g^{(\alpha_1)}(0)\right)\cdots\left(g^{(\alpha_\ell)}(0)\right). \tag{3.10}$$

Therefore, if we assume that j is even, then for some sequence of real, nonnegative coefficients D_α (specifically where $D_\alpha \in \{0, C_\alpha\}$), we have

$$\begin{aligned}\partial_t^j u(0,0) &= g^{(3j)}(0) + \sum_{|\alpha|+2\ell=3j+2} D_\alpha \left(g^{(\alpha_1)}(0)\right)\cdots\left(g^{(\alpha_\ell)}(0)\right)\\ &= \left(A_{3j} + \sum_{|\alpha|+2\ell=3j+2} D_\alpha A_{\alpha_1}\cdots A_{\alpha_\ell}\right)\left(i^{3j+2}\right).\end{aligned} \tag{3.11}$$

It follows that for any even j,

$$\left|\partial_t^j u(0,0)\right| \geq A_{3j} > 2^{-3j}(3j)^{3j} > (j!)^3. \tag{3.12}$$

Therefore, the solution $u(x,t)$ which exists (see, e.g., [1]) is not analytic in the t-variable at the point $(0,0)$.

4. Concluding remarks

One of the motivations for this work has been the results in [5, 4]. There, it was proved that, unlike the KdV, the Cauchy problem for the evolution equation

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{4.1}$$

with analytic initial data is analytic in both the space and the time variables, globally in x and locally in t. This equation was introduced independently by Fuchssteiner and Fokas [3] and by Camassa and Holm [2] as an alternative to KdV modeling shallow water waves. In the past decade, it has been the subject of extensive studies from the analytic as well as the geometric and algebraic points of view.

Finally, we note that one may obtain nonanalytic solutions to the KdV by using other analytic initial data. For example, G. Łysik in a private communication mentioned that in the nonperiodic case, he can show that the Cauchy problem for the KdV with initial data $\varphi(x) = 1/(1+x^2)$ is not analytic (like in the heat equation). For more results about the analyticity and smoothing effects of the KdV, we refer the reader to the paper of Kato and Ogawa [6] and the references therein.

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Peter Byers: Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618, USA

E-mail address: byers.8@nd.edu

A. Alexandrou Himonas: Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618, USA

E-mail address: himonas.1@nd.edu