

ATTRACTORS FOR NONAUTONOMOUS MULTIVALUED EVOLUTION SYSTEMS GENERATED BY TIME-DEPENDENT SUBDIFFERENTIALS

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In a real separable Hilbert space, we consider nonautonomous evolution equations including time-dependent subdifferentials and their nonmonotone multivalued perturbations. In this paper, we treat the multivalued dynamical systems associated with time-dependent subdifferentials, in which the solution is not unique for a given initial state. In particular, we discuss the asymptotic behaviour of our multivalued semiflows from the viewpoint of attractors. In fact, assuming that the time-dependent subdifferential converges asymptotically to a time-independent one (in a sense) as time goes to infinity, we construct global attractors for nonautonomous multivalued dynamical systems and its limiting autonomous multivalued dynamical system. Moreover, we discuss the relationship between them.

1. Introduction

In [8, 12], we considered a nonlinear evolution equation in a real Hilbert space H of the form

$$u'(t) + \partial\varphi^t(u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s (\geq 0), \quad (1.1)$$

where $\partial\varphi^t$ is the subdifferential of a time-dependent proper lower semicontinuous (l.s.c.) and convex function φ^t on H , $g(t, \cdot)$ is a single-valued perturbation which is small relative to φ^t , and f is a given forcing term. Assuming that φ^t , $g(t, \cdot)$, and $f(t)$, respectively, converge to a convex function φ^∞ on H , a single-valued operator $g^\infty(\cdot)$ in H and an element f^∞ in H in appropriate senses as $t \rightarrow +\infty$, we also considered the limiting autonomous system

$$u'(t) + \partial\varphi^\infty(u(t)) + g^\infty(u(t)) \ni f^\infty \quad \text{in } H, \quad t \geq 0. \quad (1.2)$$

In fact, in [12] we showed the existence and global boundedness of the solutions for (1.1) and (1.2). In [8], considering the case when the Cauchy problems (1.1) and (1.2) lose the uniqueness of solutions, we discussed the large-time behaviour of multiple solutions for (1.1) and (1.2). In such a situation, the solution operator $E(t, s)$ ($0 \leq s \leq t < +\infty$) for (1.1) is multivalued. Namely, $E(t, s)$ ($0 \leq s \leq t < +\infty$) is the multivalued operator from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $u_0 \in \overline{D(\varphi^s)}$ the set

$$E(t, s)u_0 := \{z \in H \mid \text{there is a solution } u \text{ of (1.1) on } [s, +\infty) : u(s) = u_0, u(t) = z\}. \tag{1.3}$$

Of course, the solution operator $S(t)$ ($t \geq 0$) for the limiting autonomous system (1.2) is similarly defined as a multivalued operator in $\overline{D(\varphi^\infty)}$, and the family $\{S(t); t \in \mathbb{R}_+\}$ forms a multivalued semigroup, where $\mathbb{R}_+ := [0, +\infty)$. Then, in [8] we showed that there exists a global attractor for multivalued evolution operators $\{E(t, s)\}$ and it is semi-invariant under $S(t)$. Moreover, we gave a sufficient condition in order that the global attractor for (1.1) is invariant under $S(t)$.

In this paper, we consider the asymptotic stability for nonautonomous evolution equations in a separable Hilbert space H of the form

$$u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s (\geq 0), \tag{1.4}$$

where $G(t, \cdot)$ is a multivalued operator which is small relative to φ^t .

Recently, in [6] Kapustian and Valero constructed the global attractor for (1.4) in the case that $\varphi^t \equiv \varphi$, $G(t, \cdot) \equiv G(\cdot)$, and $f(t) \equiv 0$.

In Mel'nik and Valero [9], they constructed the uniform global attractor for (1.4) with $\varphi^t \equiv \varphi$ and $f(t) \equiv 0$, which implies that the domains of solution operators $\{E(t, s)\}$ are independent of time $t, s \in \mathbb{R}_+$.

The main object of this paper is to construct a global attractor for (1.4), which implies that the domains of $\{E(t, s)\}$ move with the time $t, s \in \mathbb{R}_+$. Under suitable convergence assumptions for φ^t , $G(t, \cdot)$, and $f(t)$ as $t \rightarrow +\infty$, we shall construct the global attractors for (1.4) and discuss the relationship to the one for the multiple flows associated with the limiting autonomous system.

Moreover, in Section 6, as a model problem, we consider a parabolic variational inequalities with time-dependent double obstacles. By applying our abstract results, we can discuss the asymptotic stability for the double obstacle problem without the uniqueness of solutions.

Notation. Throughout this paper, let H be a (real) separable Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function φ on H , we use the notation $D(\varphi)$, $\partial\varphi$, and $D(\partial\varphi)$ to indicate the effective domain, subdifferential, and its domain of φ , respectively; for their precise definitions and basic properties see [2].

For two sets A and B in H , we define the so-called Hausdorff semi-distance

$$\text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H. \tag{1.5}$$

2. Preliminaries

In this section, let H be a real Hilbert space and we consider an evolution equation of the form

$$v'(t) + \partial\varphi^t(v(t)) \ni q(t) \quad \text{in } H, \quad t \in J, \tag{2.1}$$

where J is an interval in \mathbb{R}_+ , $\partial\varphi^t$ is the subdifferential of time-dependent proper l.s.c. and convex function φ^t on H , and q is a function given in $L^2_{\text{loc}}(J; H)$.

Definition 2.1. (i) For a compact interval $J := [t_0, t_1] \subset \mathbb{R}_+$ and $q \in L^2(J; H)$, a function $v : J \rightarrow H$ is called a solution of (2.1) on J , if

$$\begin{aligned} v &\in C(J; H) \cap W^{1,2}_{\text{loc}}((t_0, t_1]; H), \quad \varphi^{(\cdot)}(v(\cdot)) \in L^1_{\text{loc}}(J), \\ v(t) &\in D(\partial\varphi^t) \quad \text{for a.e. } t \in J, \\ q(t) - v'(t) &\in \partial\varphi^t(v(t)), \quad \text{a.e. } t \in J. \end{aligned} \tag{2.2}$$

(ii) For any interval J in \mathbb{R}_+ and $q \in L^2_{\text{loc}}(J; H)$, a function $v : J \rightarrow H$ is called a solution of (2.1) on J , if it is a solution of (2.1) on every compact subinterval of J in the sense of (i).

Here let $\{a_r\} := \{a_r; r \in \mathbb{R}_+\}$ and $\{b_r; r \in \mathbb{R}_+\}$ be families of absolutely continuous (real) functions a_r, b_r on \mathbb{R}_+ , with parameter $r \in \mathbb{R}_+$, such that

$$a'_r \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+), \quad b'_r \in L^1(\mathbb{R}_+). \tag{2.3}$$

By (2.3), the limits $a_r(+\infty) := \lim_{t \rightarrow +\infty} a_r(t)$ and $b_r(+\infty) := \lim_{t \rightarrow +\infty} b_r(t)$ exist, so a_r and b_r are considered as continuous functions on $\overline{\mathbb{R}_+} := [0, +\infty]$.

With these families $\{a_r\}$ and $\{b_r\}$, the evolution equation (2.1) is formulated for any family $\{\varphi^t\}$ in the class $\Phi(\{a_r\}, \{b_r\})$ specified as follows.

Definition 2.2. The family $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if φ^t is a proper l.s.c. convex functions on H satisfying the following property:

(*) For each $r \in \mathbb{R}_+, s, t \in [0, +\infty]$, and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$\begin{aligned} |\tilde{z} - z|_H &\leq |a_r(t) - a_r(s)| \left(1 + |\varphi^s(z)|^{1/2}\right), \\ \varphi^t(\tilde{z}) - \varphi^s(z) &\leq |b_r(t) - b_r(s)| \left(1 + |\varphi^s(z)|\right). \end{aligned} \tag{2.4}$$

Given a family $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$, consider the evolution equation (2.1) on J , where J is an interval of the form $[t_0, t_1]$ or $[t_0, t_1)$ with $0 \leq t_0 < t_1 \leq +\infty$.

The Cauchy problem for (2.1) is usually formulated for any given $v_0 \in H$ and interval J with initial time $t_0 (\geq 0)$. A solution v of (2.1) on J satisfying $v(t_0) = v_0$ is called a solution of the Cauchy problem for (2.1) with initial value v_0 .

According to the results in [7], we have the following statement:

(I) The Cauchy problem for (2.1) subject to the initial condition $v(t_0) = v_0 \in \overline{D(\varphi^{t_0})}$ has one and only one solution v on $J = [t_0, t_1]$ or $[t_0, t_1)$, $0 \leq t_0 < t_1 \leq +\infty$, such that $(\cdot - t_0)^{1/2}v' \in L^2(J; H)$, $(\cdot - t_0)\varphi^{(\cdot)}(v(\cdot)) \in L^\infty(J)$, and $\varphi^{(\cdot)}(v(\cdot))$ is absolutely continuous on any compact interval of (t_0, t_1) . In particular, if $v_0 \in D(\varphi^{t_0})$, then the solution v satisfies that $v' \in L^2(J; H)$ and $\varphi^{(\cdot)}(v(\cdot))$ is absolutely continuous on any compact interval in J .

Next, we introduce the notion of convergence of convex functions as follows.

Definition 2.3 (cf. [10]). Let $\psi, \psi_n (n \in \mathbb{N})$ be proper l.s.c. and convex functions on H . Then we say that ψ_n converges to ψ on H as $n \rightarrow +\infty$ in the sense of Mosco, denoted by $\psi_n \Rightarrow \psi$ on H as $n \rightarrow +\infty$, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \rightarrow z$ weakly in H as $k \rightarrow +\infty$, then

$$\liminf_{k \rightarrow +\infty} \psi_{n_k}(z_k) \geq \psi(z), \tag{2.5}$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in H such that

$$z_n \rightarrow z \quad \text{in } H \text{ as } n \rightarrow +\infty, \quad \lim_{n \rightarrow +\infty} \psi_n(z_n) = \psi(z). \tag{2.6}$$

The next statements (II) and (III) are found in [5].

(II) Let $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$. Then

- (i) $\varphi^{t_n} \Rightarrow \varphi^{t_0}$ on H as $n \rightarrow +\infty$ for any $t_0 \in \overline{\mathbb{R}_+}$ and any $\{t_n\} \subset \overline{\mathbb{R}_+}$ with $t_n \rightarrow t_0$ as $n \rightarrow +\infty$; of course, in case $t_0 = +\infty$, the convergence $t_n \rightarrow t_0$ means that $t_n \rightarrow +\infty$;
- (ii) if the level set $\{z \in H; |z|_H^2 + \varphi^t(z) \leq r\}$ is compact in H for any $r > 0$ and $t \geq 0$, then $\bigcup_{t \in \overline{\mathbb{R}_+}} \{z \in H; |z|_H^2 + \varphi^t(z) \leq r\}$ is compact in H , too, for any $r > 0$.

(III) Let $\{\varphi_n^t\}$ be a sequence of families in $\Phi(\{a_r\}, \{b_r\})$, $\{q_n\}$ be a sequence in $L^2_{loc}(J; H)$ and $\{v_{0,n}\}$ be a sequence in H with $v_{0,n} \in \overline{D(\varphi_n^{t_0})}$. Assume that $\varphi_n^t \Rightarrow \varphi^t$ on H for each $t \in \mathbb{R}_+$, $q_n \rightarrow q$ in $L^2_{loc}(J; H)$ and $v_{0,n} \rightarrow v_0$ in H with $v_0 \in \overline{D(\varphi^{t_0})}$ as $n \rightarrow +\infty$. We denote by v_n (resp., v) the solution of (2.1) on J corresponding to the source term q_n (resp., q) and initial value $v_n(t_0) = v_{0,n}$ (resp., $v(t_0) = v_0$). Then for every compact subset J_1 of J ,

$$\begin{aligned} v_n &\rightarrow v \quad \text{in } C(J_1; H), \\ (\cdot - t_0)^{1/2}v'_n &\rightarrow (\cdot - t_0)^{1/2}v' \quad \text{weakly in } L^2(J_1; H), \\ \int_{J_1} \varphi_n^t(v_n(t)) dt &\rightarrow \int_{J_1} \varphi^t(v(t)) dt \quad \text{as } n \rightarrow +\infty. \end{aligned} \tag{2.7}$$

Remark 2.4. In (III), if the level set $\{z \in H; |z|_H^2 + \varphi^t(z) \leq r\}$ is compact in H for every finite $r > 0$ and $t \geq 0$, then the convergence $q_n \rightarrow q$ in $L^2_{loc}(J; H)$ can be replaced by $q_n \rightarrow q$ weakly in $L^2_{loc}(J; H)$.

3. Global boundedness of solutions

Throughout this paper, let H be a real separable Hilbert space. In this section, we consider an evolution equation of the form

$$u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \geq s (\geq 0), \tag{3.1}$$

where $G(t, \cdot)$ is a multivalued operator from a subset $D(G(t, \cdot)) \subset H$ into H for each $t \in \overline{\mathbb{R}_+}$ and $f \in L^2_{loc}([s, +\infty); H)$. Also, the Cauchy problem for (3.1), associated with initial value $u_0 \in H$, is referred to as (3.2), namely

$$\begin{aligned} u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \geq s, \\ u(s) = u_0. \end{aligned} \tag{3.2}$$

Definition 3.1. Let J be any interval in \mathbb{R}_+ with initial time s . Then

- (i) a function $u : J \rightarrow H$ is called a solution of (3.1) on J , if there exists a function $g \in L^2_{loc}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and u is the solution of (2.1) with $q = f - g$ on J ;
- (ii) a function $u : J \rightarrow H$ is called a solution of the Cauchy problem (3.2) on J with given initial value $u_0 \in H$, if it is a solution of (3.1) on J satisfying $u(s) = u_0$.

We show the existence and global boundedness of a solution of (3.2) on $[s, +\infty)$ for $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $G(\cdot, \cdot)$, which satisfy some further conditions as follows:

(A1) there exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1 |z|_H^2, \quad \forall t \in \overline{\mathbb{R}_+}, \quad \forall z \in D(\varphi^t); \tag{3.3}$$

(A2) for each $r > 0$ and $t \in \overline{\mathbb{R}_+}$, the level set $\{z \in H; \varphi^t(z) \leq r\}$ is compact in H ;

(A3) $D(\varphi^t) \subset D(G(t, \cdot)) \subset H$ for any $t \in \overline{\mathbb{R}_+}$. And for any interval $J \subset \mathbb{R}_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on J such that

$$g(t) \in G(t, v(t)) \quad \text{for a.e. } t \in J; \tag{3.4}$$

(A4) $G(t, z)$ is a convex subset of H for any $z \in D(\varphi^t)$ and $t \in \overline{\mathbb{R}_+}$;

(A5) there are positive constants C_2, C_3 such that

$$|g|_H^2 \leq C_2 \varphi^t(z) + C_3, \quad \forall t \in \overline{\mathbb{R}_+}, \quad \forall z \in D(\varphi^t), \quad \forall g \in G(t, z); \tag{3.5}$$

- (A6) (demi-closedness) if $z_n \in D(\varphi^{t_n})$, $g_n \in G(t_n, z_n)$, $\{t_n\} \subset \overline{\mathbb{R}_+}$, $\{\varphi^{t_n}(z_n)\}$ is bounded, $z_n \rightarrow z$ in H , $t_n \rightarrow t$ and $g_n \rightarrow g$ weakly in H as $n \rightarrow +\infty$, then $g \in G(t, z)$;
- (A7) for each bounded subset B of H , there exist positive constants $C_4(B)$ and $C_5(B)$ such that

$$\begin{aligned} \varphi^t(z) + (g, z - b)_H &\geq C_4(B)|z|_H^2 - C_5(B), \\ \forall t \in \overline{\mathbb{R}_+}, \forall g \in G(t, z), \forall z \in D(\varphi^t), \forall b \in B. \end{aligned} \tag{3.6}$$

By the same argument in [11, 12], we can get the global existence and boundedness of the solution for (3.1) on $[s, +\infty)$.

THEOREM 3.2 (cf. [11, 12]). *Assume that $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and (A1), (A2), (A3), (A4), (A5), (A6), and (A7) hold. Let $f \in L^2_{\text{loc}}(\mathbb{R}_+, H)$. Then, for each $s \geq 0$ and $u_0 \in \overline{D(\varphi^s)}$, (3.2) has at least one solution u on $[s, +\infty)$.*

Furthermore, assume that

$$S_f := \sup_{t \geq 0} \|f\|_{L^2(t, t+1; H)} < +\infty, \tag{3.7}$$

then, the solution u of (3.2) on $[s, +\infty)$ has the following global estimate:

$$\sup_{t \geq s} \|u(t)\|_H^2 + \sup_{t \geq s} \int_t^{t+1} \varphi^\tau(u(\tau)) \, d\tau \leq N_1 \left(1 + S_f^2 + \|u_0\|_H^2 \right), \tag{3.8}$$

where N_1 is a positive constant independent of f , $s \geq 0$ and $u_0 \in \overline{D(\varphi^s)}$. Moreover, for each $\delta > 0$ and each bounded subset B of H , there is a constant $N_2(\delta, B) > 0$, depending only on $\delta > 0$ and B , such that

$$\sup_{t \geq s+\delta} \|u'\|_{L^2(t, t+1; H)}^2 + \sup_{t \geq s+\delta} \varphi^t(u(t)) \leq N_2(\delta, B) \tag{3.9}$$

for the solution u of (3.2) on $[s, +\infty)$ with $s \geq 0$ and $u_0 \in \overline{D(\varphi^s)} \cap B$.

4. Global attractor for the autonomous multivalued dynamical system

In this section, we assume that the source term $f(t)$ converges to some element $f^\infty \in H$ in the sense of Stepanov as $t \rightarrow +\infty$, that is,

$$\|f(t + \cdot) - f^\infty\|_{L^2(0, 1; H)} \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \tag{4.1}$$

Here by (i) of (II), note that $\{\varphi^t; t \in \overline{\mathbb{R}_+}\} \in \Phi(\{a_r\}, \{b_r\})$ implies that

$$\varphi^t \rightrightarrows \varphi^\infty \quad \text{on } H \text{ in the sense of Mosco [10]} \tag{4.2}$$

as $t \rightarrow +\infty$.

From assumptions (A6), (4.1) and the fact (4.2), it follows that the limiting system for (3.1) is of the form

$$u'(t) + \partial\varphi^\infty(u(t)) + G^\infty(u(t)) \ni f^\infty \quad \text{in } H, t \geq 0, \tag{4.3}$$

where $G^\infty := G(\infty, \cdot)$.

Now, we consider the limiting evolution equation (4.3) on \mathbb{R}_+ and construct a global attractor for (4.3).

The limiting autonomous system (4.3) can be considered as the special case of (3.1) with $\varphi^t \equiv \varphi^\infty$, $G(t, \cdot) \equiv G^\infty(\cdot)$, and $f(t) \equiv f^\infty$. Therefore applying Theorem 3.2, we have the global bounded solution of the Cauchy problem for (4.3) on \mathbb{R}_+ with initial value $u_0 \in \overline{D(\varphi^\infty)}$. So we can define a family $\{S(t); t \in \mathbb{R}_+\}$ of solution operators. But we do not show the uniqueness of solutions for (4.3) on \mathbb{R}_+ with a given initial value u_0 . Hence the solution operator is multivalued. Namely, for each $t \in \mathbb{R}_+$, the solution operator $S(t)$ assigns to any $u_0 \in \overline{D(\varphi^\infty)}$ the set

$$S(t)u_0 := \{v \in \overline{D(\varphi^\infty)} \mid \text{there is the solution } u \text{ of (4.3) on } \mathbb{R}_+ : u(0) = u_0, u(t) = v\}. \tag{4.4}$$

Clearly, the following conditions are satisfied:

(S1) $S(0) = I$ (the identity) on $\overline{D(\varphi^\infty)}$;

(S2) $S(t+s)u_0 = S(t)(S(s)u_0)$, for all $u_0 \in \overline{D(\varphi^\infty)}$, for all $s, t \in \mathbb{R}_+$.

Therefore $\{S(t); t \in \mathbb{R}_+\}$ forms a multivalued semigroup on $\overline{D(\varphi^\infty)}$. Moreover, we see the closedness of $S(\cdot)(\cdot)$ in the following sense.

LEMMA 4.1. *Assume that $t_n, t \in \mathbb{R}_+$ with $t_n \rightarrow t$, $u_{0n}, u_0 \in \overline{D(\varphi^\infty)}$ with $u_{0n} \rightarrow u_0$ in H and an element $z_n \in S(t_n)u_{0n}$ converges to some element z in H as $n \rightarrow +\infty$. Then, $z \in S(t)u_0$.*

Proof. Since $t_n \rightarrow t$ as $n \rightarrow +\infty$, without loss of generality, we may assume that there is a finite time T such that $t, t_n \in [0, T]$ for any $n \in \mathbb{N}$.

Since $z_n \in S(t_n)u_{0n}$, there exists a global solution u_n of (4.3) on \mathbb{R}_+ such that

$$u_n(t_n) = z_n, \quad u_n(0) = u_{0n}. \tag{4.5}$$

Let δ be any positive number. By Theorem 3.2, there is a positive constant N_δ depending only on δ such that

$$\sup_{t \geq \delta} |u_n(t)|_H^2 + \sup_{t \geq \delta} |u'_n|_{L^2(t, t+1; H)}^2 + \sup_{t \geq \delta} \varphi^\infty(u_n(t)) \leq N_\delta. \tag{4.6}$$

Therefore, we observe that there exist a subsequence $\{n_k\} \subset \{n\}$ and a function $u_\delta \in C([\delta, T]; H)$ such that

$$u_{n_k} \rightharpoonup u_\delta \quad \text{in } C([\delta, T]; H), \quad u'_{n_k} \rightharpoonup u'_\delta \quad \text{weakly in } L^2(\delta, T; H) \tag{4.7}$$

as $k \rightarrow +\infty$.

By the convergence result (III) with (A2), (A6) and [Remark 2.4](#), (by taking a subsequence of $\{n_k\}$ if necessary), we see that u_δ is the solution of

$$u'_\delta(t) + \partial\varphi^\infty(u_\delta(t)) + G^\infty(u_\delta(t)) \ni f^\infty \quad \text{in } H, \text{ a.e. } t \in [\delta, T]. \quad (4.8)$$

Here by the usual diagonal argument with respect to δ , we can construct the solution $u \in C((0, T]; H)$ of

$$u'(t) + \partial\varphi^\infty(u(t)) + G^\infty(u(t)) \ni f^\infty \quad \text{in } H, \text{ a.e. } t \in (0, T]. \quad (4.9)$$

Using the same technique of [[11](#), Lemma 3.10], we show that

$$u(t) \longrightarrow u_0 \quad \text{in } H \text{ as } t \longrightarrow 0, \quad (4.10)$$

which implies that u is the solution ([4.3](#)) on $[0, T]$ such that $u \in C([0, T]; H)$ and

$$u_{n_k} \longrightarrow u \quad \text{in } C([0, T]; H) \text{ as } k \longrightarrow +\infty. \quad (4.11)$$

Now, we show that $z = u(t)$, namely, $z \in S(t)u_0$. Here, let ε be any positive number. Since $u \in C([0, T]; H)$ and $t_{n_k} \rightarrow t$ as $k \rightarrow +\infty$, there is a positive number $K_{1,\varepsilon}$ such that

$$|u(t_{n_k}) - u(t)|_H \leq \frac{\varepsilon}{3}, \quad \forall k \geq K_{1,\varepsilon}. \quad (4.12)$$

On the other hand, it follows from ([4.11](#)) that there is a positive number $K_{2,\varepsilon}$ such that

$$|u_{n_k} - u|_{C([0, T]; H)} \leq \frac{\varepsilon}{3}, \quad \forall k \geq K_{2,\varepsilon}. \quad (4.13)$$

Moreover, since $z_{n_k} \rightarrow z$ in H as $k \rightarrow +\infty$, there is a positive number $K_{3,\varepsilon}$ such that

$$|z - z_{n_k}|_H \leq \frac{\varepsilon}{3}, \quad \forall k \geq K_{3,\varepsilon}. \quad (4.14)$$

Noting ([4.5](#)), ([4.12](#)), ([4.13](#)), and ([4.14](#)), we have

$$\begin{aligned} |z - u(t)|_H &\leq |z - z_{n_k}|_H + |z_{n_k} - u(t_{n_k})|_H + |u(t_{n_k}) - u(t)|_H \\ &= |z - z_{n_k}|_H + |u_{n_k}(t_{n_k}) - u(t_{n_k})|_H + |u(t_{n_k}) - u(t)|_H \\ &\leq |z - z_{n_k}|_H + |u_{n_k} - u|_{C([0, T]; H)} + |u(t_{n_k}) - u(t)|_H \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \forall k \geq K_{1,\varepsilon} + K_{2,\varepsilon} + K_{3,\varepsilon}. \end{aligned} \quad (4.15)$$

Since ε is arbitrary, we conclude $z = u(t)$, namely

$$z \in S(t)u_0. \quad (4.16)$$

Thus, [Lemma 4.1](#) is proved. □

Furthermore, we have the following properties of the multivalued semigroup $\{S(t); t \in \mathbb{R}_+\}$, which can be proved by the same way as in [5, Lemma 4.1]. For a detail proof, see [5, Lemma 4.1].

LEMMA 4.2 (cf. [5, Lemma 4.1]). *Suppose the same assumptions of Theorem 3.2 and (4.1). Then*

- (i) *for each bounded subset B of H , the set $\bigcup_{t \in \mathbb{R}_+} S(t)(\overline{D(\varphi^\infty)} \cap B)$ is bounded in H ;*
- (ii) *for each bounded subset B of H and each positive number δ , the set*

$$C_\delta := \bigcup_{t \geq \delta} S(t)(\overline{D(\varphi^\infty)} \cap B) \tag{4.17}$$

- is relatively compact in H , and φ^∞ is bounded on C_δ ;*
- (iii) *there exists a compact and convex subset B_0 of $D(\varphi^\infty)$ such that*

$$\sup_{z \in B_0} \varphi^\infty(z) < +\infty \tag{4.18}$$

and for each bounded subset B of H there is a finite time $T_B > 0$ satisfying

$$S(t)(\overline{D(\varphi^\infty)} \cap B) \subset B_0 \quad \forall t \geq T_B. \tag{4.19}$$

Remark 4.3. By taking into account Lemmas 4.1 and 4.2, we easily see that for each bounded subset B of H and each positive numbers δ ,

$$S(\cdot)(\cdot) \text{ is upper semicontinuous on } J \times (\overline{D(\varphi^\infty)} \cap B), \tag{4.20}$$

where J is any compact subinterval of $[\delta, +\infty)$. For the definition and property of the upper semicontinuous mapping, see [1, 3], for instance.

Now, we mention the existence of a global attractor for the multivalued semigroup $\{S(t); t \in \mathbb{R}_+\}$ associated with (4.3).

THEOREM 4.4. *Suppose the same assumptions of Theorem 3.2 and (4.1). Then, there is a subset \mathcal{A}_∞ of $D(\varphi^\infty)$ such that*

- (i) *\mathcal{A}_∞ is nonempty and compact in H ;*
- (ii) *for each bounded subset B of H and each number $\varepsilon > 0$ there exists $T_{B,\varepsilon} > 0$ such that*

$$\text{dist}_H(S(t)z, \mathcal{A}_\infty) < \varepsilon \tag{4.21}$$

- uniformly in $z \in \overline{D(\varphi^\infty)} \cap B$ and all $t \geq T_{B,\varepsilon}$;*
- (iii) *$S(t)\mathcal{A}_\infty = \mathcal{A}_\infty$ for any $t \in \mathbb{R}_+$.*

We say that \mathcal{A}_∞ is a global attractor for $\{S(t); t \in \mathbb{R}_+\}$, if it possesses properties (i), (ii), and (iii) of Theorem 4.4. Clearly, the global attractor is unique if it exists.

Proof of Theorem 4.4. On account of Lemmas 4.1 and 4.2, we can construct a global attractor \mathcal{A}_∞ for $\{S(t); t \in \mathbb{R}_+\}$ by the standard technique established in [4, 13]. Actually, using the compact absorbing set B_0 obtained by Lemma 4.2(iii), the global attractor \mathcal{A}_∞ can be defined by

$$\mathcal{A}_\infty := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B_0}. \tag{4.22}$$

Now, we show that the set \mathcal{A}_∞ has the properties (i), (ii), and (iii) of Theorem 4.4.

By the global existence of solutions for (4.3) and the compactness of B_0 , it is easily seen that \mathcal{A}_∞ is nonempty and compact in H . Hence, Theorem 4.4(i) holds.

Next we show (ii). Since B_0 is the absorbing set, it suffices to show that

$$\text{dist}_H(S(t)B_0, \mathcal{A}_\infty) \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \tag{4.23}$$

We prove (4.23) by contradiction. Namely, assume that \mathcal{A}_∞ does not attract B_0 . Then there are $\delta_0 > 0$ and sequences $\{t_n\} \subset \mathbb{R}_+$ with $t_n \geq n$, $\{z_n\} \subset B_0$, and $\{w_n\} \subset H$ with $w_n \in S(t_n)z_n$ such that

$$|w_n - y|_H \geq \delta_0, \quad \forall y \in \mathcal{A}_\infty. \tag{4.24}$$

Since B_0 is the compact absorbing set in H , there exists a finite time $T(B_0) > 0$ such that

$$S(t)B_0 \subset B_0, \quad \forall t \geq T(B_0). \tag{4.25}$$

Hence, the set

$$\{w_n \in H; w_n \in S(t_n)z_n, \forall t_n \geq T(B_0)\} \tag{4.26}$$

is relatively compact in H . So, there are a subsequence $\{n_k\} \subset \{n\}$ and a point $w \in H$ such that

$$w_{n_k} \longrightarrow w \quad \text{in } H \text{ as } k \longrightarrow +\infty. \tag{4.27}$$

Since $w_{n_k} \in S(t_{n_k})z_{n_k}$ and $\{z_{n_k}\} \subset B_0$, it follows that

$$w \in \mathcal{A}_\infty. \tag{4.28}$$

This contradicts (4.24). Hence, the set \mathcal{A}_∞ attracts the compact absorbing set B_0 , which means that Theorem 4.4(ii) holds.

Now we prove Theorem 4.4(iii). At first let us show $\mathcal{A}_\infty \subset S(t)\mathcal{A}_\infty$ for any $t \in \mathbb{R}_+$. Let z be any element of \mathcal{A}_∞ . Then, there exist sequences $\{t_n\} \subset \mathbb{R}_+$, $\{x_n\} \subset B_0$ and $\{z_n\} \subset H$ with $z_n \in S(t_n)x_n$ such that

$$t_n \uparrow +\infty, \quad z_n \longrightarrow z \quad \text{in } H \text{ as } n \longrightarrow +\infty. \tag{4.29}$$

Since B_0 is the compact absorbing set in H , there exists a finite time $T(B_0) > 0$ such that

$$S(\tau)B_0 \subset B_0, \quad \forall \tau \geq T(B_0). \tag{4.30}$$

For each $t \in \mathbb{R}_+$, it follows from (S2) that

$$z_n \in S(t)S(t_n - t)x_n \quad \text{for any } n \text{ with } t_n \geq t + T(B_0). \tag{4.31}$$

Hence there is an element $w_n \in S(t_n - t)x_n$ such that

$$z_n \in S(t)w_n. \tag{4.32}$$

By (4.30), we easily see that the set $\{w_n \in H; n \in \mathbb{N} \text{ with } t_n \geq t + T(B_0)\}$ is relatively compact in H . So, there are a subsequence $\{n_k\} \subset \{n\}$ and a point $y \in H$ such that

$$w_{n_k} \longrightarrow y \quad \text{in } H \text{ as } k \longrightarrow +\infty. \tag{4.33}$$

Clearly, $y \in \mathcal{A}_\infty$.

Moreover, by taking a subsequence of $\{n_k\}$ if necessary, it follows from Lemma 4.1, (4.29), (4.30), (4.32), and (4.33) that

$$z \in S(t)y, \tag{4.34}$$

which implies that $z \in S(t)\mathcal{A}_\infty$. Therefore, we have

$$\mathcal{A}_\infty \subset S(t)\mathcal{A}_\infty, \quad \forall t \in \mathbb{R}_+. \tag{4.35}$$

Next we show $S(t)\mathcal{A}_\infty \subset \mathcal{A}_\infty$ for any $t \in \mathbb{R}_+$. By the result as above, we see that for any $t \in \mathbb{R}_+$

$$S(t)\mathcal{A}_\infty \subset S(t)(S(\tau)\mathcal{A}_\infty) = S(t + \tau)\mathcal{A}_\infty, \quad \forall \tau \in \mathbb{R}_+. \tag{4.36}$$

Let y be any element of $S(t)\mathcal{A}_\infty$. By (4.36) we may assume that there are sequences $\{\tau_n\} \subset \mathbb{R}_+$ with $\tau_n \geq n$ and $\{x_n\} \subset \mathcal{A}_\infty$ such that

$$y \in S(t + \tau_n)x_n. \tag{4.37}$$

Since $\mathcal{A}_\infty \subset B_0$, from the attractive property (ii) of Theorem 4.4 it follows that

$$y \in \mathcal{A}_\infty, \tag{4.38}$$

which implies that $S(t)\mathcal{A}_\infty \subset \mathcal{A}_\infty$ for any $t \in \mathbb{R}_+$. Thus, (iii) of Theorem 4.4 holds. □

Remark 4.5. The authors in [6] had already constructed the global attractor for (4.3) with $f^\infty \equiv 0$. However, our situation is different a little. So, we have given the above our proof of Theorem 4.4, which will be used in the next section.

5. Global attractor of the nonautonomous multivalued dynamical system

In this section, we construct a global attractor for the nonautonomous system (3.1).

By Theorem 3.2, we define the solution operator $E(t, s)$ ($0 \leq s \leq t < +\infty$) for (3.1). But we do not have the uniqueness of solutions for the Cauchy problem (3.2) on $[s, +\infty)$, hence $E(t, s)$ is multivalued. Namely, $E(t, s)$ ($0 \leq s \leq t < +\infty$) is the operator from $\overline{D(\varphi^s)}$ into $D(\varphi^t)$ which assigns to each $u_0 \in \overline{D(\varphi^s)}$ the set $E(t, s)u_0$ given by (1.3) for (3.1).

It is easy to check the following properties of $\{E(t, s); 0 \leq s \leq t < +\infty\}$:

- (E1) $E(s, s) = I$ on $\overline{D(\varphi^s)}$ for any $s \geq 0$;
- (E2) $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$.

Moreover, by the same argument of Lemma 4.1, we have the closedness of $E(\cdot)(\cdot)$ in the following sense.

LEMMA 5.1. *Assume that $s_n, s, t_n, t \in \mathbb{R}_+$ with $s_n \rightarrow s$ and $t_n \rightarrow t$, $u_{0n} \in \overline{D(\varphi^{s_n})}$, $u_0 \in \overline{D(\varphi^s)}$ with $u_{0n} \rightarrow u_0$ in H an element $z_n \in E(t_n + s_n, s_n)u_{0n}$ converges to some element z in H as $n \rightarrow +\infty$. Then, $z \in E(t + s, s)u_0$. In particular, if $s = +\infty$, then $z \in S(t)u_0$.*

In order to construct a global attractor for the multivalued evolution operator $E(t, s)$ associated with (3.1), we give a definition of a ω -limit set under $E(t, s)$.

Definition 5.2. Let $\mathcal{B}(H)$ be a family of bounded subsets of H . Then, for each $B \in \mathcal{B}(H)$ the set

$$\omega_E(B) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau, s \geq 0} E(t + s, s)(\overline{D(\varphi^s)} \cap B)} \tag{5.1}$$

is called the ω -limit set of B under $E(t, s)$.

Remark 5.3 (cf. [9, Lemma 1]). By definition of the ω -limit set $\omega_E(B)$, it is easy to see that $x \in \omega_E(B)$ if and only if there exist sequences $\{t_n\} \subset \mathbb{R}_+$ with $t_n \rightarrow +\infty$, $\{s_n\} \subset \mathbb{R}_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{s_n})}$ and $\{x_n\} \subset H$ with $x_n \in E(t_n + s_n, s_n)z_n$ such that

$$x_n \rightarrow x \quad \text{in } H \text{ as } n \rightarrow +\infty. \tag{5.2}$$

Now, we mention our main result in this section, which is the existence of the attracting set for (3.1).

THEOREM 5.4. *Let $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and assume that (A1), (A2), (A3), (A4), (A5), (A6), (A7), and (4.1) hold. Then, the set \mathcal{A}_* ,*

$$\mathcal{A}_* := \overline{\bigcup_{B \in \mathcal{B}(H)} \omega_E(B)} \tag{5.3}$$

satisfies the following:

- (i) $\mathcal{A}_* \subset D(\varphi^\infty)$ and \mathcal{A}_* is nonempty and compact in H ;
- (ii) for each bounded set $B \in \mathcal{B}(H)$,

$$\text{dist}_H(E(t, s)z, \mathcal{A}_*) \rightarrow 0 \quad \text{uniformly in } s \geq 0, z \in \overline{D(\varphi^s)} \cap B \tag{5.4}$$

as $t \rightarrow +\infty$;

- (iii) $\mathcal{A}_* \subset S(t)\mathcal{A}_* \subset \mathcal{A}_\infty$ for any $t \in \mathbb{R}_+$.

To prove [Theorem 5.4](#), we prepare [Lemma 5.5](#).

For the moment, we fix a bounded subset $B \in \mathcal{B}(H)$, and using the global boundedness result obtained in [Theorem 3.2](#), choose constants $r_B > 0$ and $M_B > 0$ so that

$$\begin{aligned} |w|_H \leq r_B \quad & \text{for any } s, t \geq 0, z \in \overline{D(\varphi^s)} \cap B, w \in E(t+s, s)z, \\ \varphi^{t+s}(w) \leq M_B \quad & \text{for any } s \geq 0, t \geq 1, z \in \overline{D(\varphi^s)} \cap B, w \in E(t+s, s)z. \end{aligned} \tag{5.5}$$

Next, we observe from property (*) of time-dependence that for each $s \geq 0$, $z \in \overline{D(\varphi^s)} \cap B$, $t \geq 0$, and $w \in E(t+s, s)z$ there is $\tilde{z} := \tilde{z}_{s,z,t,w} \in D(\varphi^\infty)$ satisfying

$$\begin{aligned} |\tilde{z} - w|_H &\leq \left(\int_{t+s}^\infty |a'_{r_B}(\sigma)| d\sigma \right) (1 + M_B^{1/2}), \\ \left(\text{hence } |\tilde{z}|_H &\leq r_B + \left(\int_0^\infty |a'_{r_B}(\sigma)| d\sigma \right) (1 + M_B^{1/2}) =: r'_B \right), \\ \varphi^\infty(\tilde{z}) &\leq M_B + \left(\int_{t+s}^\infty |b'_{r_B}(\sigma)| d\sigma \right) (1 + M_B) \\ &\leq M_B + \left(\int_0^\infty |b'_{r_B}(\sigma)| d\sigma \right) (1 + M_B) =: M'_B. \end{aligned} \tag{5.6}$$

Now, we define the set \tilde{B} by

$$\tilde{B} := \{z \in H; |z|_H \leq r'_B\} \cap \overline{D(\varphi^\infty)}. \tag{5.7}$$

Since B_0 is the absorbing set for $\{S(t); t \in \mathbb{R}_+\}$, there exists a finite time $\tilde{T} := \tilde{T}_{\tilde{B}} > 0$ such that

$$S(t)\tilde{B} \subset B_0, \quad \forall t \geq \tilde{T}. \tag{5.8}$$

By using the above facts, we show a key lemma to prove [Theorem 5.4](#).

LEMMA 5.5. *Let B_0 be the absorbing set for $\{S(t); t \in \mathbb{R}_+\}$ obtained in Lemma 4.2(iii). Then*

$$\omega_E(B) \subset B_0, \quad \forall B \in \mathcal{B}(H). \tag{5.9}$$

Proof. For each $B \in \mathcal{B}(H)$, let x be any element of $\omega_E(B)$. Then, by Remark 5.3, we see that there exist sequences $\{t_n\} \subset \mathbb{R}_+$ with $t_n \rightarrow +\infty$, $\{s_n\} \subset \mathbb{R}_+$, $\{z_n\} \subset B$ with $z_n \in D(\varphi^{s_n})$ and $\{x_n\} \subset H$ with $x_n \in E(t_n + s_n, s_n)z_n$ such that

$$x_n \rightarrow x \quad \text{in } H \text{ as } n \rightarrow +\infty. \tag{5.10}$$

Let \tilde{T} be the positive finite time obtained in (5.8). It follows from (E2) that

$$x_n \in E(t_n + s_n, t_n + s_n - \tilde{T})E(t_n + s_n - \tilde{T}, s_n)z_n \quad \text{for any } n \text{ with } t_n \geq \tilde{T} + 1. \tag{5.11}$$

By (5.11), there is an element $y_n \in E(t_n + s_n - \tilde{T}, s_n)z_n$ such that

$$x_n \in E(t_n + s_n, t_n + s_n - \tilde{T})y_n. \tag{5.12}$$

Here note that

$$|y_n|_H \leq r_B, \quad \varphi^{t_n + s_n - \tilde{T}}(y_n) \leq M_B \quad \text{for any } n \text{ with } t_n \geq \tilde{T} + 1, \tag{5.13}$$

where r_B and M_B are positive constants in (5.5).

From property (*) of time-dependence that for $y_n \in E(t_n + s_n - \tilde{T}, s_n)z_n$ there is $\tilde{y}_n \in D(\varphi^\infty)$ satisfying

$$\begin{aligned} |\tilde{y}_n - y_n|_H &\leq \left(\int_{t_n + s_n - \tilde{T}}^\infty |a'_{r_B}(\sigma)| d\sigma \right) (1 + M_B^{1/2}), \\ \left(\text{hence } |\tilde{y}_n|_H \leq r_B + \left(\int_0^\infty |a'_{r_B}(\sigma)| d\sigma \right) (1 + M_B^{1/2}) = r'_B \right), \\ \varphi^\infty(\tilde{y}_n) &\leq M_B + \left(\int_{t_n + s_n - \tilde{T}}^\infty |b'_{r_B}(\sigma)| d\sigma \right) (1 + M_B) \\ &\leq M_B + \left(\int_0^\infty |b'_{r_B}(\sigma)| d\sigma \right) (1 + M_B) = M'_B. \end{aligned} \tag{5.14}$$

Clearly, $\{\tilde{y}_n \in D(\varphi^\infty); n \in \mathbb{N} \text{ with } t_n \geq \tilde{T} + 1\}$ is relatively compact in H , hence we assume that

$$\tilde{y}_n \rightarrow \tilde{y}_\infty \quad \text{in } H \text{ as } n \rightarrow +\infty \tag{5.15}$$

for some $\tilde{y}_\infty \in H$; it is easily seen that $\tilde{y}_\infty \in \tilde{B}$ and

$$y_n \rightarrow \tilde{y}_\infty \quad \text{in } H \text{ as } n \rightarrow +\infty. \tag{5.16}$$

Here, applying Lemma 5.1, it follows from (5.10), (5.12), and (5.16) that

$$x \in S(\tilde{T})\tilde{y}_\infty, \tag{5.17}$$

which implies that

$$x \in S(\tilde{T})\tilde{B} \subset B_0. \tag{5.18}$$

Therefore, we observe that

$$\omega_E(B) \subset B_0. \tag{5.19}$$

□

Proof of Theorem 5.4. By Lemma 5.5, we observe that $\mathcal{A}_* \subset B_0$. Therefore, Theorem 5.4(i) holds. Also, it follows from (5.3) and Remark 5.3 that Theorem 5.4(ii) holds.

Now, we show Theorem 5.4(iii). At first, we show that $\mathcal{A}_* \subset S(t)\mathcal{A}_*$ for any $t \in \mathbb{R}_+$. To do so, let x be any element of \mathcal{A}_* . By the definition of \mathcal{A}_* , we may assume that there exist sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_E(B_n)$ such that

$$x_n \longrightarrow x \quad \text{in } H \text{ as } n \longrightarrow +\infty. \tag{5.20}$$

It follows from Remark 5.3 that for each n , there exist sequences $\{t_{n,j}\} \subset \mathbb{R}_+$ with $t_{n,j} \rightarrow +\infty$, $\{s_{n,j}\} \subset \mathbb{R}_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{s_{n,j}})}$, and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(t_{n,j} + s_{n,j}, s_{n,j})z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \quad \text{in } H \text{ as } j \longrightarrow +\infty. \tag{5.21}$$

Let t be any time in \mathbb{R}_+ . We note that

$$v_{n,j} \in E(t_{n,j} + s_{n,j}, t_{n,j} + s_{n,j} - t)E(t_{n,j} + s_{n,j} - t, s_{n,j})z_{n,j} \tag{5.22}$$

for j with $t_{n,j} \geq t + 1$. Hence, there is a $w_{n,j} \in E(t_{n,j} + s_{n,j} - t, s_{n,j})z_{n,j}$ such that

$$v_{n,j} \in E(t_{n,j} + s_{n,j}, t_{n,j} + s_{n,j} - t)w_{n,j}. \tag{5.23}$$

For each n , by global estimates obtained in Theorem 3.2, $\{w_{n,j} \in H; j \in \mathbb{N} \text{ with } t_{n,j} \geq t + 1\}$ is relatively compact in H . So, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_\infty \in H$ as $j \rightarrow +\infty$. Clearly, $\tilde{w}_\infty \in \omega_E(B_n)$. Moreover, from Lemma 4.1, (5.21), and (5.23), we observe that

$$x_n \in S(t)\tilde{w}_\infty \subset S(t)\omega_E(B_n), \tag{5.24}$$

which implies that

$$x_n \in \bigcup_{n \geq 1} S(t)\omega_E(B_n), \quad \forall n \geq 1. \tag{5.25}$$

Moreover, by [Lemma 5.1](#), we observe that for each $X \subset B_0$,

$$\overline{S(t)X} \subset S(t)\overline{X}, \quad \forall t \in \mathbb{R}_+. \tag{5.26}$$

Since [\(5.20\)](#), [\(5.25\)](#), and [\(5.26\)](#), we see that

$$\begin{aligned} x &\in \overline{\bigcup_{n \geq 1} S(t)\omega_E(B_n)} \\ &= \overline{S(t)\bigcup_{n \geq 1} \omega_E(B_n)} \\ &\subset \overline{S(t)\bigcup_{n \geq 1} \omega_E(B_n)} \\ &\subset S(t)\mathcal{A}_*. \end{aligned} \tag{5.27}$$

Hence, we observe that \mathcal{A}_* is semi-invariant under the multivalued semigroup $S(t)$, namely

$$\mathcal{A}_* \subset S(t)\mathcal{A}_*, \quad \forall t \in \mathbb{R}_+. \tag{5.28}$$

Next, we show that $S(t)\mathcal{A}_* \subset \mathcal{A}_\infty$ for any $t \in \mathbb{R}_+$. By [\(5.28\)](#), for each $t \in \mathbb{R}_+$

$$S(t)\mathcal{A}_* \subset S(t)S(\tau)\mathcal{A}_* = S(t + \tau)\mathcal{A}_*, \quad \forall \tau \in \mathbb{R}_+. \tag{5.29}$$

Since $\mathcal{A}_* \subset B_0$, from [\(5.29\)](#) and the attractive property (ii) of [Theorem 4.4](#), it follows that

$$S(t)\mathcal{A}_* \subset \mathcal{A}_\infty, \quad \forall t \in \mathbb{R}_+, \tag{5.30}$$

hence, we conclude that

$$\mathcal{A}_* \subset S(t)\mathcal{A}_* \subset \mathcal{A}_\infty, \quad \forall t \in \mathbb{R}_+. \tag{5.31}$$

□

[Theorem 5.4](#) says that the attracting set \mathcal{A}_* for [\(3.1\)](#) is semi-invariant under $S(t)$ associated with the limiting autonomous system [\(4.3\)](#), in general.

In order to get the invariance of \mathcal{A}_* under $S(t)$, we use the concept of a regular approximation which was introduced in [\[8\]](#).

Definition 5.6. Let $z \in D(\varphi^\infty)$. Then, we say that $S(t)z$ is regularly approximated by $E(t + s, s)$ as $s \rightarrow +\infty$, if for each finite $T > 0$ there are sequences $\{s_n\} \subset \mathbb{R}_+$ with $s_n \rightarrow +\infty$ and $\{z_n\} \subset H$ with $z_n \in D(\varphi^{s_n})$ and $z_n \rightarrow z$ in H satisfying the following property: for any function $u \in W^{1,2}(0, T; H)$ satisfying $u(t) \in S(t)z$ for all $t \in [0, T]$ there is a sequence $\{u_n\} \subset W^{1,2}(0, T; H)$ such that $u_n(t) \in E(t + s_n, s_n)z_n$ for all $t \in [0, T]$ and $u_n \rightarrow u$ in $C([0, T]; H)$ as $n \rightarrow +\infty$.

Using the above concept, we can show that the invariance of \mathcal{A}_* under $S(t)$ and the relationship between \mathcal{A}_* and \mathcal{A}_∞ .

THEOREM 5.7. Suppose that all the assumptions in [Theorem 5.4](#) hold true.

(i) Assume that for any point z of \mathcal{A}_* , $S(t)z$ is regularly approximated by $E(t, s)$ as $s \rightarrow +\infty$. Then,

$$\mathcal{A}_* = S(t)\mathcal{A}_* \subset \mathcal{A}_\infty \quad \text{for any } t \in \mathbb{R}_+. \tag{5.32}$$

(ii) Assume that for any point z of \mathcal{A}_∞ , $S(t)z$ is regularly approximated by $E(t, s)$ as $s \rightarrow +\infty$. Then,

$$\mathcal{A}_* = \mathcal{A}_\infty. \tag{5.33}$$

Proof. By the same way in [[8](#), [Theorem 3.2](#)], we can prove [Theorem 5.7\(i\)](#). For the detailed proof, see [[8](#), [Theorem 3.2](#)].

Similarly, we can show [Theorem 5.7\(ii\)](#) by the similar argument of (i). In fact, let x be any element of \mathcal{A}_∞ . It follows from [Theorem 4.4\(iii\)](#) that

$$x \in \mathcal{A}_\infty = S(t)\mathcal{A}_\infty, \quad \forall t \in \mathbb{R}_+. \tag{5.34}$$

Let $t \in \mathbb{R}$ be fixed. Then there exists an element $w \in \mathcal{A}_\infty$ such that

$$x \in S(t)w. \tag{5.35}$$

Now, by our assumption, we see that there are sequences $\{s_n\} \subset \mathbb{R}_+$, $\{w_n\} \subset H$ and $\{x_n\} \subset H$ such that

$$\begin{aligned} s_n \longrightarrow +\infty, \quad w_n \in D(\varphi^{s_n}), \quad w_n \longrightarrow w \quad \text{in } H, \\ x_n \in E(t+s_n, s_n)w_n, \quad x_n \longrightarrow x \quad \text{in } H \end{aligned} \tag{5.36}$$

as $n \rightarrow +\infty$.

Since $\{w_n\}$ is bounded, namely $\{w_n\} \subset B$ for some $B \in \mathcal{B}(H)$, [\(5.36\)](#) implies that

$$x \in \omega_E(B). \tag{5.37}$$

Therefore, we have

$$\mathcal{A}_\infty \subset \omega_E(B) \subset \mathcal{A}_*. \tag{5.38}$$

Hence, it follows from [\(5.38\)](#) and [Theorem 5.4\(iii\)](#) that

$$\mathcal{A}_* = \mathcal{A}_\infty. \tag{5.39}$$

□

By [Theorem 5.7](#), we get the invariance of \mathcal{A}_* under $S(t)$. Therefore, we say that the set \mathcal{A}_* is the global attractor for nonautonomous systems [\(3.1\)](#).

Remark 5.8. If the solution operator $S(t)$ is single-valued, namely the solution of the Cauchy problem for (4.3) is unique, the assumptions of Theorem 5.7 always hold. Thus, Theorems 5.4 and 5.7 include the abstract results obtained in [12] which were concerned with the asymptotic stability for the single-valued dynamical system associated with time-dependent subdifferentials.

6. Application to an obstacle problem for PDEs

Let Ω be a bounded domain in \mathbb{R}^N ($1 \leq N < +\infty$) with smooth boundary $\Gamma = \partial\Omega$ and p be a fixed number with $2 \leq p < +\infty$. We use the notation

$$a_p(v, z) := \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla z \, dx, \quad \forall v, z \in W^{1,p}(\Omega) \tag{6.1}$$

and denote by (\cdot, \cdot) the usual inner product in $L^2(\Omega)$.

The obstacle functions σ_0 and σ_1 are prescribed so that

$$\begin{aligned} \sigma_i &\in L^\infty(\mathbb{R}_+; W^{1,p}(\Omega)) \cap L^\infty(\mathbb{R}_+ \times \Omega), \\ \frac{d\sigma_i}{dt} &\in L^1(\mathbb{R}_+; W^{1,p}(\Omega)) \cap L^2(\mathbb{R}_+; W^{1,p}(\Omega)) \cap L^1(\mathbb{R}_+; L^\infty(\Omega)) \cap L^2(\mathbb{R}_+; L^\infty(\Omega)), \end{aligned} \tag{6.2}$$

for $i = 0, 1$, and

$$\sigma_1 - \sigma_0 \geq c_0 \quad \text{a.e. on } \mathbb{R}_+ \times \Omega \tag{6.3}$$

for some constant $c_0 > 0$. For each $t \in [0, +\infty]$, we define

$$K(t) := \{z \in W^{1,p}(\Omega); \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \text{ a.e. on } \Omega\}, \tag{6.4}$$

where $\sigma_i(+\infty, \cdot) = \lim_{t \rightarrow \infty} \sigma_i(t, \cdot)$ in $W^{1,p}(\Omega)$, $i = 0, 1$.

Also, let f be a function in $L^2_{loc}(\mathbb{R}_+; L^2(\Omega))$ and $f^\infty \in L^2(\Omega)$ such that

$$\|f(t + \cdot) - f^\infty\|_{L^2(0,1;L^2(\Omega))} \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \tag{6.5}$$

Furthermore, let h be a nonnegative continuous function on $\mathbb{R}_+ \times \mathbb{R}$ satisfying that there is a positive constant L_h such that

$$|h(t, z_1) - h(t, z_2)| \leq L_h |z_1 - z_2| \tag{6.6}$$

for all $t \in \mathbb{R}_+$, $z_i \in \mathbb{R}$, and $i = 1, 2$. For any $z \in \mathbb{R}$, $h^\infty(z) := \lim_{t \rightarrow +\infty} h(t, z)$ exists.

Now, under the above assumptions, for given $\mathbf{b} \in L^\infty(\Omega)^\mathbb{N}$, we consider an interior double obstacle problem (6.7): find functions $u, q : [s, +\infty) \rightarrow L^2(\Omega)$ such that

$$\begin{aligned} u &\in C([s, +\infty); L^2(\Omega)) \cap L^p_{\text{loc}}((s, +\infty); W^{1,p}(\Omega)) \cap W^{1,2}_{\text{loc}}((s, +\infty); L^2(\Omega)); \\ q &\in L^\infty((s, +\infty); L^2(\Omega)); \\ u(t) &\in K(t) \quad \text{for a.e. } t \geq s; \\ 0 &\leq q(t, x) \leq h(t, u(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega; \\ (u'(t) + q(t) + \mathbf{b} \cdot \nabla u(t) - f(t), u(t) - z) + a_p(u(t), u(t) - z) &\leq 0 \\ &\text{for any } z \in K(t), \text{ a.e. } t \geq s. \end{aligned} \tag{6.7}$$

And we also consider the limiting (autonomous) problem (6.8) for (6.7): find functions $u, q : \mathbb{R}_+ \rightarrow L^2(\Omega)$ such that

$$\begin{aligned} u &\in C(\mathbb{R}_+; L^2(\Omega)) \cap L^p_{\text{loc}}(\mathbb{R}_+; W^{1,p}(\Omega)) \cap W^{1,2}_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)); \\ q &\in L^\infty((0, +\infty); L^2(\Omega)); \\ u(t) &\in K(\infty) \quad \text{for a.e. } t \in \mathbb{R}_+; \\ 0 &\leq q(t, x) \leq h^\infty(u(t, x)) \quad \text{a.e. on } \mathbb{R}_+ \times \Omega; \\ (u'(t) + q(t) + \mathbf{b} \cdot \nabla u(t) - f^\infty, u(t) - z) + a_p(u(t), u(t) - z) &\leq 0 \\ &\text{for any } z \in K(\infty), \text{ a.e. } t \in \mathbb{R}_+. \end{aligned} \tag{6.8}$$

In order to apply the abstract results in Sections 3, 4, and 5, we choose $L^2(\Omega)$ as a real separable Hilbert space H . And we define a family $\{\varphi^t\}$ of proper l.s.c. convex functions φ^t on $L^2(\Omega)$ by

$$\varphi^t(z) = \begin{cases} \frac{1}{p} \int_{\Omega} |\nabla z|^p dx & \text{if } z \in K(t), \\ +\infty & \text{if } z \in L^2(\Omega) \setminus K(t). \end{cases} \tag{6.9}$$

Also, we define a multivalued operator $G(\cdot, \cdot)$ from $\mathbb{R}_+ \times H^1(\Omega)$ into $L^2(\Omega)$ by

$$\begin{aligned} G(t, z) := \{g \in L^2(\Omega) : g = l + \mathbf{b} \cdot \nabla z \text{ in } L^2(\Omega), \\ 0 \leq l(x) \leq h(t, z(x)) \text{ a.e. on } \Omega\} \end{aligned} \tag{6.10}$$

for all $t \in \mathbb{R}_+$ and $z \in H^1(\Omega)$. And G^∞ is also defined by replacing $h(t, z(x))$ by $h^\infty(z(x))$ in (6.10).

By the same calculations in [14, Lemma 5.1], we get the following lemma.

LEMMA 6.1 (cf. [14, Lemma 5.1]). (i) For any $r \geq 0$ and $t \in \mathbb{R}_+$,

$$a_r(t) = b_r(t) := M \int_0^t \{ |\sigma'_0|_{L^\infty(\Omega)} + |\sigma'_0|_{W^{1,p}(\Omega)} + |\sigma'_1|_{L^\infty(\Omega)} + |\sigma'_1|_{W^{1,p}(\Omega)} \} d\tau, \quad (6.11)$$

where M is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$.

(ii) Assumptions (A1), (A2), (A3), (A4), (A5), (A6), and (A7) hold for φ^t and $G(\cdot, \cdot)$.

Clearly, the obstacle problem (6.7) can be reformulated as an evolution equation (3.1) involving the subdifferential of φ^t given by (6.9) and the multivalued operator $G(t, \cdot)$ defined by (6.10). Similarly, the limiting system (6.8) is also rewritten in the form (4.3). Therefore, by Lemma 6.1, we can apply abstract results in Sections 3, 4, and 5. Namely, we can obtain the existence of the global solutions for (6.7) and (6.8). Moreover, there are the global attractors \mathcal{A}_* for (6.7) and \mathcal{A}_∞ for (6.8) such that

$$\mathcal{A}_* \subset S(t)\mathcal{A}_* \subset \mathcal{A}_\infty, \quad \forall t \in \mathbb{R}_+, \quad (6.12)$$

where $S(t)$ is the solution operator associated with (6.8).

Additionally, we assume that $\sigma_0(t, x)$ is nondecreasing and $\sigma_1(t, x)$ (resp., $h(t, x)$) is nonincreasing with respect to $t \in \overline{\mathbb{R}_+}$ for any $x \in \Omega$ (resp., $x \in \mathbb{R}$), and $f(t) \equiv f^\infty$. Then, we easily see that the assumption of Theorem 5.7 holds, so we have

$$\mathcal{A}_* = \mathcal{A}_\infty. \quad (6.13)$$

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