EXISTENCE PROBLEMS FOR HOMOCLINIC SOLUTIONS

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The problem $\dot{x} = f(t,x)$, $x(-\infty) = x(+\infty)$, where $x(\pm\infty) := \lim_{t\to\pm\infty} x(t) \in \mathbb{R}^n$, is considered. Some existence results for this problem are established using the fixed point method and topological degree theory.

1. Introduction

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function; consider the boundary value problem

$$\dot{x} = f(t,x), \quad x(-\infty) = x(+\infty), \tag{1.1}$$

where

$$x(\pm\infty) := \lim_{t\to\pm\infty} x(t) \in \mathbb{R}^n. \tag{1.2}$$

The solutions of problem (1.1) are often called, by Poincaré, *homoclinic solutions*. They appear in certain celeste mechanics and cosmogony problems.

Problem (1.1) can be considered as a generalization of the boundary value problem

$$\dot{x} = f(t,x), \quad x(a) = x(b), \tag{1.3}$$

when $a \to -\infty$ and $b \to +\infty$.

The boundary value problems on compact intervals have been studied in numerous papers but the boundary value problems on noncompact intervals have been less studied. A first substantial approach of these problems, using functional methods are due to Kartsatos [8]. Last time, this type of results has been published in [2, 3, 4, 5, 6].

For problem (1.3), Mawhin obtained many existence results through topological degree theory; in [9, 10, 11] the reader can find the fundamental ideas of the

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method developed by Mawhin, the main results, and a rich bibliography in this field. Some approaches of Mawhin dedicated to problem (1.3) can be adjusted for problem (1.1).

The present paper is dedicated to the existence of solutions for problem (1.1); the used method will be the reduction of problem (1.1) to a fixed point problem for a convenient operator defined in a suitable functional space. Such a space is

$$C_l := \{x : \mathbb{R} \longrightarrow \mathbb{R}^n,\ \exists x(\pm\infty) \in \mathbb{R}^n\}. \tag{1.4}$$

Section 2 deals especially to praise the main properties of the space C_l. The specified isomorphism between C_l and $C([a,b],\mathbb{R}^n)$ permits to obtain a compactness criterion in C_l (see [1]). We define in C_l the notion of an associated operator to problem (1.1) and indicate the construction method of such operator together with its main properties. An associated operator for problem (1.1) is an operator whose fixed points are solutions for (1.1).

In Section 3, assuming the existence and uniqueness on $\mathbb{R}$ of the solutions for the problem

$$\dot{x} = f(t,x), \quad x(0) = y, \tag{1.5}$$

one builds up associated operators mapping in $\mathbb{R}^n$; consequently, their topological degree will be a Brouwer one.

In Section 4, the continuation method is presented (see Proposition 4.1). Through this method we obtain existence results for perturbed equations. The starting equation is chosen such that the topological degree of its associated operator is easy to be evaluated, and the perturbation is done through homogeneous or "small" functions.

For further details about the construction of the associated operators, the reader can consult [12]. For the topological degree theory we recommend the delightful book [13].

2. General hypotheses and preliminary results

2.1. Introduction. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping; consider problem (1.1) where $x(\pm\infty) := \lim_{t\to\pm\infty} x(t) \in \mathbb{R}^n$ (notation used throughout this paper).

It is clear from the introduction that the aim of this paper is to find sufficient conditions to assure the existence of solutions for problem (1.1). The method will be the reduction of the existence solutions for problem (1.1) to the existence of fixed points for an adequate operator which maps in an adequate functional space.

In this section, we present the principal function spaces, their main properties, the notations, and the principal theoretical results needed in what follows.

2.2. Function spaces. Denote by $|\cdot|$ an arbitrary norm in $\mathbb{R}^n$ and

$$C_c := \{x : \mathbb{R} \longrightarrow \mathbb{R}^n,\ x \text{ continuous}\}. \tag{2.1}$$

As is well known, C_c is a Fréchet space endowed with the uniform convergence on compact subsets of $\mathbb{R}$ with the usual topology. Let C_c^1 denote the linear subspace of C^1 functions in C_c.

The principal function spaces are

$$\begin{aligned} C_l &: = \{x \in C_c,\ \exists x(\pm\infty) \in \mathbb{R}^n\}, \\ C_{ll} &: = \{x \in C_l,\ x(-\infty) = x(+\infty)\}, \end{aligned} \tag{2.2}$$

where C_l and C_{ll} are Banach spaces with respect to the norm

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \{|x(t)|\}, \tag{2.3}$$

where $\mathbb{R}^n$ will be identified naturally with the constant functions subspace. Consider $C_l^1 := C_l \cap C_c^1$, $C_{ll}^1 := C_{ll} \cap C_c^1$.

Another function space, interesting only as linear space, is the space of all Riemann integrable functions on $\mathbb{R}$,

$$C_R := \left\{x \in C_c;\ \int_{-\infty}^{+\infty} x(t)\,dt < +\infty\right\}, \tag{2.4}$$

where

$$\int_{-\infty}^{+\infty} x(t)\,dt := \lim_{A\to-\infty} \int_A^0 x(t)\,dt + \lim_{A\to+\infty} \int_0^A x(t)\,dt. \tag{2.5}$$

Remark 2.1. A function x of class C^1 belongs to C_l if and only if $\dot{x}$ belongs to C_R.

Finally, we use the spaces

$$\begin{aligned} C_{(a,b)} &:= \{x : [a,b] \longrightarrow \mathbb{R}^N,\ x \text{ continuous}\}, \\ C_{[a,b]} &:= \{x \in C_{(a,b)},\ x(a) = x(b)\}, \end{aligned} \tag{2.6}$$

endowed with the usual norm

$$\|x\| := \sup_{t\in[a,b]} \{|x(t)|\}. \tag{2.7}$$

In the case of a Banach space X, where $X = C_l$ or $X = C_{ll}$, set

$$B(\rho) := \{x \in X,\ \|x\|_\infty < \rho\}, \qquad \Sigma(\rho) := \{x \in \mathbb{R}^n,\ |x| < \rho\}. \tag{2.8}$$

2.3. Properties of the space C_l. We state certain properties of the space C_l.

Proposition 2.2. *The spaces C_l and $C_{(a,b)}$ are isomorphic.*

Proof. Indeed, consider $\varphi : (a,b) \to \mathbb{R}$ a continuous and bijective mapping; define the mapping $\Phi : C_l \to C_{(a,b)}$ by the equality

$$(\Phi x)(t) := \begin{cases} x(\varphi(t)), & \text{if } t \in (a,b), \\ x(-\infty), & \text{if } t = a, \\ x(+\infty), & \text{if } t = b. \end{cases} \tag{2.9}$$

It is clear that Φ is an isometric isomorphism and the proof ends. □

Remark 2.3. The same mapping Φ is an isomorphism between C_{ll} and $C_{[a,b]}$.

The property in Proposition 2.2 allows us to obtain a compactness criterion in C_l; obviously, it will work in C_{ll} too, since C_{ll} is a closed subspace of C_l.

Definition 2.4. A family $A \subset C_l$ is called equiconvergent if and only if

$$\begin{aligned} &\forall \varepsilon > 0, \quad \exists T = T(\varepsilon) > 0, \quad \forall x \in A,\ \forall t_1, t_2 \in \mathbb{R},\ t_1 t_2 > 0, \\ &\qquad\quad |t_i| > T(\varepsilon), \qquad |x(t_1) - x(t_2)| < \varepsilon. \end{aligned} \tag{2.10}$$

Proposition 2.5. *A family $A \subset C_l$ is relatively compact if and only if the following three conditions are fulfilled:*

(i) *A is uniformly bounded on $\mathbb{R}$;*
(ii) *A is equicontinuous on every compact interval of $\mathbb{R}$;*
(iii) *A is equiconvergent.*

Proposition 2.5 results immediately from the fact that the isomorphism Φ given by (2.9) transforms a set A, satisfying conditions (i), (ii), and (iii), into an equicontinuous and uniformly bounded set in $C_{(a,b)}$.

Definition 2.6. A family $A \subset C_c$ is called C_R-bounded if and only if there exists a function $\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha(t) \geq 0$ for every $t \in \mathbb{R}$, $\alpha \in C_R$, such that

$$\forall x \in A,\ t \in \mathbb{R}, \quad |x(t)| \leq \alpha(t). \tag{2.11}$$

Corollary 2.7. *A family $A \subset C_l \cap C_c^1$, uniformly bounded on $\mathbb{R}$ having the family of derivatives C_R-bounded, is relatively compact in C_l.*

2.4. Operators. The first operator is the Nemitzky operator, $F : C_c \to C_c$ generated by the continuous function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and defined by

$$(Fx)(t) := f\big(t, x(t)\big). \tag{2.12}$$

Taking into account Remark 2.1, it results that for every solution x of (1.1) it holds

$$x \in C_l \Longleftrightarrow Fx \in C_R. \tag{2.13}$$

Similarly, for every solution x of (1.1),

$$x \in C_{ll} \Longleftrightarrow \int_{-\infty}^{+\infty} (Fx)(s)\,ds = 0. \tag{2.14}$$

In what follows, $X \subseteq C_l$ denotes a closed subspace of C_l and $D \subset X$ is a void set. Define on D an important category of operators called *associated*.

Definition 2.8. The operator $U : D \subset X \to X$ is *associated* to problem (1.1) on the set D if and only if every fixed point of U is a solution for problem (1.1).

By using the formula of a solution for (1.1), it is naturally, in the building of the operator U, to admit

$$FD \subset C_R. \tag{2.15}$$

Remark, in addition, that if U maps in C_l, then only the fixed points satisfy condition (1.1) and if $D \subset C_{ll}$ then we have $UD \subset C_{ll}$.

By Remark 2.1, we can easily obtain associated operators to problem (1.1). Such an operator is, for example,

$$(Ux)(t) := x(b) + \alpha(t)\int_{-\infty}^{+\infty} (Fx)(s)\,ds + \int_{b}^{+\infty} (Fx)(s)\,ds, \tag{2.16}$$

where $b \in \bar{\mathbb{R}}$, and $\alpha : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function with $\alpha(b) \neq 0$, and $\alpha \in C_l$; this operator maps in C_l. If, in addition, $\alpha(-\infty) = \alpha(+\infty)$, then $UC_l \subset C_{ll}$.

Another possibility to construct associated operators in C_{ll} is the next: we search a linear and continuous operator $T : C_{ll} \to C_{ll}$ such that the operator $Lx := \dot{x} + Tx$ is invertible; then

$$U = L^{-1}(F + T). \tag{2.17}$$

Examples of such operators T are $Tx = \theta(\cdot)x(0)$ or $T = \theta(\cdot)x$, where $\theta : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly positive mapping with $\int_{-\infty}^{+\infty} \theta(t)\,dt = 1$.

There exist general procedures to build up the associated operators, like the one from below having a pure algebraic character.

Let X and Z be two linear spaces, $L : D(L) \subset X \to Z$ a linear operator, and $N : D(N) \subset X \to Z$ an arbitrary operator.

If $\dim N(L) = \operatorname{codim} R(L) < \infty$ and $P : X \to X$, $Q : Z \to Z$ are two projectors such that $R(P) = N(L)$, $N(Q) = R(L)$, then $x \in X$ is a solution for the equation

$$Lx = Nx \tag{2.18}$$

if and only if x is a fixed point for the operator

$$U = P + aQN + K(I - Q)N, \tag{2.19}$$

where I is the identity operator in Z, $a \in \mathbb{R}$, $a \neq 0$, and K is the right inverse of L (more precisely $K = (L\,|_{D(L)\cap N(P)})^{-1}$).

This result has been successfully used by Mawhin for the building of associated operators to the boundary value problem (closely related to the periodic solutions problem [9, 10, 11])

$$\dot{x} = f(t,x), \quad x(0) = x(T). \tag{2.20}$$

By this model in the next subsection, we briefly describe how can we construct associated operators to problem (1.1) and their properties.

2.5. Construction of associated operators. The form of associated operators depends firstly on the fundamental space X and next on the space Z and on the choice of the operators L, N and the choice of the projectors P, Q; only after this K can be determined and also the final form of the operator U. Having so many arbitrary elements we can find many associated operators.

In what follows, we sketch the building of associated operators in two important cases: $X = C_l$ and $X = C_{ll}$; further details about the construction can be found in [4, 12].

In the case $X = C_l$ we distinguish three subcases related to L and N; this choice must be made such that the equation (L,N) does contain (1.1). The expression of projectors P and Q depends on the considered case.

In all three cases, we have $Z = C_R \times \mathbb{R}^n$, $D(L) = C_l^1$.

2.5.1. The case L_1.

$$Lx = (\dot{x}, x(+\infty)), \qquad Nx = (Fx, x(-\infty)). \tag{2.21}$$

In this case $P = Q = 0$, so the operator L is invertible and therefore $U = L^{-1}N$. This case gives us the easiest associated operators,

$$Ux = x(+\infty) + \int_{-\infty}^{(\cdot)} (Fx)(s)\,ds \tag{2.22}$$

and the symmetric form

$$Ux = x(-\infty) + \int_{+\infty}^{(\cdot)} (Fx)(s)\,ds. \tag{2.23}$$

2.5.2. The case L_2.

$$Lx = (\dot{x}, 0), \qquad Nx = \big(Fx, x(+\infty) - x(-\infty)\big). \tag{2.24}$$

In this case, since $R(L) = C_R \times \{0\}$ the projector Q may be

$$Q(y, c) = (0, c). \tag{2.25}$$

For P, we take

$$Px = x(b), \quad x \in \bar{\mathbb{R}}, \tag{2.26}$$

or

$$Px = \int_{-\infty}^{+\infty} e(t)x(t)\,dt, \tag{2.27}$$

where

$$e : \mathbb{R} \longrightarrow \mathbb{R}, \quad e \text{ continuous}, \quad \int_{-\infty}^{+\infty} e(s)\,ds = 1. \tag{2.28}$$

For U, we can construct

$$Ux = x(b) + a\big[x(+\infty) - x(-\infty)\big] + \int_b^{(\cdot)} (Fx)(s)\,ds \tag{2.29}$$

or

$$\begin{aligned} Ux &= a\big[x(+\infty) - x(-\infty)\big] \\ &\quad + \int_{-\infty}^{+\infty} \left[x(s) - \frac{1}{2}(Fx)(s)\right] e^{-2|s|}\,ds + \int_0^{(\cdot)} (Fx)(s)\,ds. \end{aligned} \tag{2.30}$$

2.5.3. The case L_3.

$$Lx = \big(\dot{x}, x(+\infty) - x(-\infty)\big), \qquad Nx = (Fx, 0). \tag{2.31}$$

In this case,

$$R(L) = \left\{(y, c) \in C_R \times \mathbb{R}^n \mid c = \int_{-\infty}^{+\infty} y(s)\,ds\right\} \tag{2.32}$$

and hence the projector Q must be changed; we can take for example

$$Q(y, c) = \left(0, c - \int_{-\infty}^{+\infty} y(s)\,ds\right) \tag{2.33}$$

and therefore,

$$Ux = x(b) + a\int_{-\infty}^{+\infty} (Fx)(s)\,ds + \int_b^{(\cdot)} (Fx)(s)\,ds \tag{2.34}$$

and other more complicated forms.

In the case $X = C_{ll}$, we have $D(L) = C^1_{ll}$, $Z = C_R$, $Lx = \dot{x}$ and consequently,

$$R(L) = \left\{ y \in C_R, \int_{-\infty}^{+\infty} y(s)\,ds = 0 \right\}. \tag{2.35}$$

We can take, for example,

$$(Qy)(t) = e(t)\int_{-\infty}^{+\infty} y(s)\,ds. \tag{2.36}$$

In general, in this case the expression of U is more complicated since all its values must be in C_{ll}. For example, for (2.26) we get

$$(Ux)(t) = x(b) + \left[ae^{-2|t|} - \int_b^t e^{-2|s|}\,ds\right] \cdot \int_{-\infty}^{+\infty} (Fx)(s)\,ds + \int_b^t (Fx)(s)\,ds \tag{2.37}$$

and with (2.27), where $e(t) = e^{-2|t|}$,

$$\begin{aligned}(Ux)(t) = &\int_0^t e^{-2|s|}\,ds + ae^{-2|t|} - \int_0^t e^{-2|s|}\left(1 - \frac{1}{2}e^{-2|s|}\right)ds \\ &- \frac{1}{2}\int_{-\infty}^{+\infty} e^{-2|s|}(Fx)(s)\,ds \int_{-\infty}^{+\infty}(Fx)(s)\,ds + \int_0^t (Fx)(s)\,ds.\end{aligned} \tag{2.38}$$

2.6. Admissible operators. It is obvious that this construction of the associated operators has an algebraic character; the condition

$$FD \subset C_R \tag{2.39}$$

is sufficient for the existence of these operators, but it is not sufficient to confer their important topological properties.

Definition 2.9. An associated operator on the set $D \subset X$ to problem (1.1), constructed as in Section 2.5, is called *admissible* if and only if $U : D \subset X \to X$ is compact.

Proposition 2.10. *Let X be a subspace of C_l and $D \subset C_l$ be a bounded subset. If FD is C_R-bounded, then every associated operator constructed as in Section 2.5 is compact.*

The proof of this proposition is complicated in calculus, but it is basically an easy application of the elementary known properties of uniform convergence, which allows to establish immediately the continuity of the operator U which contains finite rank projectors and application of type

$$x \longrightarrow \int_{-\infty}^{+\infty} (Fx)(s)\,ds, \qquad x \longrightarrow \int_b^{(\cdot)} (Fx)(s)\,ds, \quad b \in \bar{\mathbb{R}}. \tag{2.40}$$

The compactness of the operator U is an immediate consequence of Corollary 2.7. At least for the operators U given by (2.22), (2.23), (2.29), (2.30), (2.34), (2.37), and (2.38) the verification of compactity is immediate.

Remark that if f satisfies the condition

$$\left|f(t,x)\right| \leq \theta(t)\cdot\beta(|x|), \tag{2.41}$$

where $\theta \in C_R$, $\theta \geq 0$, $\beta \geq 0$, $\beta : \mathbb{R} \to \mathbb{R}$ is continuous, then FD is C_R-bounded for every bounded set $D \subset X$; indeed, we have

$$\left|(Fx)(t)\right| \leq \rho\theta(t), \tag{2.42}$$

where

$$\rho := \sup\left\{\beta(u),\ |u| \leq r\right\}, \qquad r := \sup\left\{\|x\|_\infty,\ x \in D\right\}. \tag{2.43}$$

The situation is more complicated in the case when (1.1) proceeds from a second-order equation

$$\ddot{y} = h(t, y, \dot{y}), \tag{2.44}$$

where $h : \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k \to \mathbb{R}^k$ is a continuous function. Substituting (2.44) in (1.1), where

$$x = (x_1, x_2),\ x_1 = y,\ x_2 = \dot{y}, \quad f(t,x) = \left(x_2, h(t, x_1, x_2)\right), \tag{2.45}$$

then

$$x \in C_l \Longleftrightarrow y, \quad \dot{y} \in C_l. \tag{2.46}$$

Since

$$y \in C_l \Longleftrightarrow \dot{y} \in C_R, \tag{2.47}$$

it results that

$$\dot{y} \in C_l \cap C_R \tag{2.48}$$

and hence

$$\lim_{t\to\pm\infty} \dot{y}(t) = 0. \tag{2.49}$$

Therefore, the boundary value problem defining the homoclinic solutions for (2.34) has the form

$$\ddot{y} = h(t, y, \dot{y}), \quad y(-\infty) = y(+\infty), \quad \dot{y}(-\infty) = \dot{y}(+\infty) = 0. \tag{2.50}$$

We give an example to obtain the C_R-boundedness of $F(D)$ in this case.

Let $\alpha_1, \alpha_2 \in C_R$, α_1, α_2 be positive; in addition, suppose that $\alpha_2(\pm\infty) = 0$. Let $\gamma, \beta : \mathbb{R} \to \mathbb{R}$ be two continuous and positive functions. We take as fundamental space $X = C_l \times C_{Rl}$, where $C_{Rl} := C_R \cap C_l = \{x \in C_R, x(\pm\infty) = 0\}$.

Let D_1 be a bounded set in C_R and let $D = D_1 \times D_2$, where

$$D_2 = \{x \in C_R, \ |x(t)| \le \alpha_2(t), \ t \in \mathbb{R}\}. \tag{2.51}$$

It is easy to check that if

$$|h(t, x_1, x_2)| \le k_1 \alpha_1(t) \gamma |x_1| + k_2 |x_2| \beta(|x_2|), \tag{2.52}$$

then $F(D)$ is C_R-bounded (more precisely, $C_R \times C_R$-bounded).

The case of second-order equation is different from the first-order equation; this is why it will not be treated here, but it will make the object of a future note.

2.7. Remarks on the topological degree of the admissible operators. Let $\Omega \subset X$ be an open and bounded set, where X is C_l or C_{ll}.

Suppose that $F(\Omega)$ is C_R-bounded, for an admissible operator U, if

$$x \ne Ux, \quad x \in \partial\Omega, \tag{2.53}$$

where $\partial\Omega$ is the boundary of Ω, we can consider its topological degree

$$\deg(I - U, \Omega, 0). \tag{2.54}$$

If this degree is nonzero, then U admits fixed points and so problem (1.1) has solutions.

As we said, the results contained in this section are based on the ones by Mawhin related to the boundary value problem

$$\dot{x} = f(t, x), \qquad x(0) = x(T). \tag{2.55}$$

This author proves that the associated operators to problem (2.55) in the space $C_{(0,T)}$ or $C_{[0,T]}$ are compact on the bounded sets without supplementary conditions on the mapping f as it was to be expected. Moreover, these operators have the same topological degree which does not depend on the choice of L, N, P, Q. In addition, if in particular $f(t,x) = g(x)$, then for each associated operator U to problem (2.55) on the bounded and open set Ω from $C_{(0,T)}$ (or $C_{[0,T]}$), we have

$$\deg(I - U, \Omega, 0) = (-1)^n \deg_B (g, \Omega \cap \mathbb{R}^n, 0), \tag{2.56}$$

where $\deg_B$ denotes the Brouwer degree.

The associated operators to problem (1.1) on Ω from C_l or C_{ll} have the degrees invariant with respect to L, N, P, Q; the proof, based on the invariance of topological degree to homeomorphisms, is essentially simple but complicated to achieve. As we do not use this property in the present paper, we renounce to its proof.

Finally we make only a remark on the isomorphism Φ given by (2.9).

Let $\Omega \subset X$ be an open and bounded set in X ($X = C_l$ or C_{ll}) and let U be an admissible operator on Ω for problem (1.1) fulfilling (2.56).

Set

$$\Omega_\Phi := \Omega(\Phi), \qquad U_\Phi := \Phi U \Phi^{-1}, \tag{2.57}$$

where Φ is given by (2.9) with $a = 0$, $b = T$.

Then Ω_Φ is open and bounded, $\Phi(\partial\Omega) = \partial\Omega_\Phi$ and $\Omega_\Phi \subset C_{(0,T)}$ (resp., $\Omega_\Phi \subset C_{[0,T]}$).

Furthermore, U_Φ is compact and since $\partial\Omega_\Phi = \Phi(\partial\Omega)$, we have

$$x \neq Ux,\ x \in \partial\Omega \Longleftrightarrow x \neq U_\Phi,\ x \in \partial\Omega_\Phi. \tag{2.58}$$

Hence

$$\deg(I - U, \Omega, 0) = \deg\left(I - U_\Phi, \Omega_\Phi, 0\right). \tag{2.59}$$

3. Existence results in the hypothesis of uniqueness of solutions

3.1. Introduction. Let $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a continuous function; consider again problem (1.1).

We research the existence of solutions for problem (1.1) in the hypothesis that the Cauchy problem

$$\dot{x} = f(t,x), \quad x(0) = y \tag{3.1}$$

has a unique solution defined on the whole real axis $\mathbb{R}$, for every G a bounded set in $\mathbb{R}^n$ and for every $y \in G$; denote the solution of (3.1) by

$$x(t;y), \quad y \in G. \tag{3.2}$$

The uniqueness condition is fulfilled in particular if $f(t,x)$ is locally Lipschitz with respect to x. Condition (2.41) is sufficient to assure the existence on $\mathbb{R}$ of the solution (3.2), it is in particular fulfilled in conditions of type (2.41) and even more general.

It is known that the uniqueness condition assures the continuous dependence of the function $x(t;\cdot)$; this property would be stated as: for every $[a,b] \subset \mathbb{R}$ and for every $y_n \in G$, $y_n \to y \in G$, the sequence $x(t;y_n)$ converges uniformly on $[a,b]$ to $x(t;y)$.

In this section, we present certain existence results for problem (1.1), exploiting this continuous dependence with respect to initial data.

3.2. Generalized Poincaré operator. Let $\Omega \subset C_l$ be a bounded and open set; let

$$G := \left\{y \in \mathbb{R}^n,\ x(\cdot;y) \in \Omega\right\}. \tag{3.3}$$

Obviously, G is a bounded and open set.

Theorem 3.1. *Suppose that*

(i) $F\bar{\Omega}$ *is* C_R*-bounded;*

(ii) *for every* $y \in \partial G$ *and for every* $t > 0$

$$x(t;y) \neq x(-t;y); \tag{3.4}$$

(iii) *for every* $y \in \partial G$

$$f(0,y) \neq 0; \tag{3.5}$$

(iv) *for every* $y \in \partial G$

$$\deg_B\big(f(0,y),G,0\big) \neq 0. \tag{3.6}$$

Then problem (1.1) has solutions in $\bar{\Omega}$.

Proof. By hypothesis (i) it results that $x(\cdot;y) \in C_l$, for every $y \in \bar{G}$; set

$$Py = \frac{1}{2}\big[x(+\infty;y) - x(-\infty;y)\big] \tag{3.7}$$

(we call P the *generalized Poincaré operator*). It is easy to check that the solution $x(\cdot;y) \in C_{ll}$ if and only if $Py = 0$.

We want to show that $P : \bar{G} \to \bar{G}$ is continuous; for this aim we remark that

$$Py = \frac{1}{2}\int_{-\infty}^{+\infty} f\big(s,x(s;y)\big)\,ds. \tag{3.8}$$

By hypothesis (i) it results that the integral in (3.8) is uniformly convergent with respect to $y \in \bar{G}$; on the other hand, since the mapping $y \to x(\cdot;y)$ is continuous (as mentioned in the previous paragraph) we conclude the continuity of the mapping $(t,y) \to f(t,x(t;y))$ on every set of type $[-A,A] \times \bar{G}$. Hence the mapping $y \to Py$ is continuous on $\bar{G}$.

Define the application $h : \bar{G} \times [0,1] \to \mathbb{R}^n$ by

$$h(y,\lambda) := \begin{cases} \dfrac{1}{2\lambda}\Big[x\Big(\dfrac{\lambda}{1-\lambda};y\Big) - x\Big(\dfrac{\lambda}{\lambda-1};y\Big)\Big], & \lambda \in (0,1),\ y \in \bar{G}, \\ Py, & \lambda = 1, \\ f(0,y), & \lambda = 0. \end{cases} \tag{3.9}$$

By L'Hospital rule,

$$\lim_{\lambda \downarrow 0} h(y,\lambda) = f(0,\lambda). \tag{3.10}$$

Since

$$\lim_{\lambda \uparrow 1} h(y,\lambda) = Py, \tag{3.11}$$

it follows that h is continuous.

If for $y \in \partial G$ we have $Py = 0$, then $x(\cdot; y)$ is a solution for (1.1).

Suppose then $Py \neq 0$, for every $y \in \partial G$; by hypotheses (ii) and (iii) it results that

$$h(y,\lambda) \neq 0, \quad \forall \lambda \in [0,1], \ \forall y \in \partial G. \tag{3.12}$$

By homotopic invariance property of the topological degree it results that $\deg_B(h(\cdot,\lambda), G, 0)$ is constant for $\lambda \in [0,1]$; in particular,

$$\deg_B\left(h(\cdot,0), G, 0\right) = \deg_B\left(h(\cdot,1), G, 0\right), \tag{3.13}$$

that is,

$$\deg_B(P, G, 0) = \deg_B\left(f(\cdot, y), G, 0\right) \tag{3.14}$$

and hence, by (3.6)

$$\deg_B(P, G, 0) \neq 0, \tag{3.15}$$

which assures the existence of $y \in G$ with $Py = 0$. The theorem is proved. □

3.3. The case Ω connected. The advantage of the previous result is that the topological degrees appearing are Brouwer degrees; the drawback is that condition (3.4) is not easy to be checked. We state now another existence result.

As usual, suppose that $\Omega \subset C_l$ is a bounded and open set; define on $\bar{\Omega}$ the operators

$$Hx = \int_0^{(\cdot)} (Fx)(s)\,ds, \qquad S = I - H. \tag{3.16}$$

Lemma 3.2. *If*

(i) $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz with respect to the second variable;*

(ii) $F(\bar{\Omega})$ *is* C_R*-bounded,*

then $S : \bar{\Omega} \to C_l$ *is injective.*

Proof. Let $x, z \in \bar{\Omega}$ such that

$$S(x) = S(z). \tag{3.17}$$

If $x \neq z$, then there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = z(t_0)$; we can assume that $t_0 > 0$. Let $A > 0$ be such that $t_0 \in [0, A]$ and $r = \max\{\|x\|_\infty, \|y\|_\infty\}$.

Since

$$\begin{gathered} \exists L_r > 0, \ \forall u, v \in \Sigma(r), \quad \left| f(t,u) - f(t,v) \right| \leq L_r |u - v|, \\ \left| x(t) - z(t) \right| \leq L_r \int_0^t \left| u(s) - v(s) \right| ds, \quad t \in [0, A], \end{gathered} \tag{3.18}$$

we obtain by using Gronwall's lemma

$$x(t) = z(t), \quad \forall t \in [0, A]. \tag{3.19}$$

□

Remark 3.3. The mapping $S : \bar{\Omega} \to S(\bar{\Omega})$ is a homeomorphism. In addition, since H is a compact operator, S^{-1} is a compact perturbation of identity, too.

Remark 3.4. If $y \in \mathbb{R}^n \cap S(\bar{\Omega})$, then

$$S^{-1} y = x(\cdot; y). \tag{3.20}$$

Set

$$\Pi x := x(0) - Px. \tag{3.21}$$

Observe that the operator

$$U := \Pi + H \tag{3.22}$$

is just the admissible operator (2.22), where $b = 0$ and $a = 1/2$.

Remark 3.5. The following identity holds:

$$I - U = \left(I - \Pi S^{-1}\right) S. \tag{3.23}$$

Theorem 3.6. *Assume that the following hypotheses are fulfilled:*

(i) $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz;*
(ii) $\Omega \subset C_l$ *is a connected, open, and bounded set;*
(iii) $F(\bar{\Omega})$ *is* C_R*-bounded;*
(iv) *the following relations hold:*

$$\begin{aligned} x &\neq Ux, \quad x \in \partial\Omega, \\ y &\neq Py, \quad y \in \partial\left(S(\bar{\Omega}) \cap \mathbb{R}^n\right). \end{aligned} \tag{3.24}$$

Then

$$\deg(I - U, \Omega, 0) = \pm \deg_B \left(P, S(\Omega) \cap \mathbb{R}^n, 0\right). \tag{3.25}$$

In addition, if

$$\deg_B \left(P, S(\Omega) \cap \mathbb{R}^n, 0\right) \neq 0, \tag{3.26}$$

then problem (1.1) admits solutions.

Proof. By identity (3.22) and applying the Leray-Schauder result for topological degree of product operators, we get

$$\deg(I - U, \Omega, 0) = \deg(S - y, \Omega, 0) \cdot \deg\left(I - \Pi S^{-1}, S(\Omega), 0\right), \tag{3.27}$$

where $y \in S(\Omega)$ is arbitrary. Since $S(\Omega)$ is connected and $S : \bar{\Omega} \to S(\bar{\Omega})$ is a homeomorphism, then

$$\left|\deg(S-y,\Omega,0)\right| = 1, \quad \forall y \in S(\Omega). \tag{3.28}$$

Since ΠS^{-1} takes values in $\mathbb{R}^n$, we have

$$\deg\left(I-\Pi S^{-1}, S(\Omega), 0\right) = \deg_B\left(I-\Pi S^{-1}, S(\Omega)\cap\mathbb{R}^n, 0\right) \tag{3.29}$$

and since

$$\Pi S^{-1} = I - P, \tag{3.30}$$

it results (3.25).

Finally, if (3.26) is satisfied, then U admits fixed points and since U is associated to (1.1), every fixed point is a solution for problem (1.1). □

Remark 3.7. If condition (3.4) is fulfilled for every $t \in \bar{\mathbb{R}}$ and $y \in \partial G$, then

$$\deg(I-U,\Omega,0) = \pm\deg_B\left(f(\cdot,y), G, 0\right). \tag{3.31}$$

Remark 3.8. Formula (3.31) is available for every operator $U : \bar{\Omega} \subset C_l \to C_l$ admissible for problem (1.1).

3.4. Existence results using Miranda's theorem. Let $K = \Pi_{i=1}^n[-l,l] \subset \mathbb{R}^n$ and $\Phi : K \to \mathbb{R}^n$ be a continuous function; denote by Φ_i the ith component of Φ and by y_i the ith component of $y \in \mathbb{R}^n$. Define $L_i^+, L_i^- \subset \mathbb{R}^n$ by

$$\begin{aligned} L_i^+ &:= \left(y_1,\dots,y_{i-1}, l, y_{i+1},\dots,y_n\right), \\ L_i^- &:= \left(y_1,\dots,y_{i-1}, -l, y_{i+1},\dots,y_n\right), \quad i \in \overline{1,n}. \end{aligned} \tag{3.32}$$

Remark that if we take in $\mathbb{R}^n$ the norm

$$|y| = \max_{1\le i\le n}\left\{\left|y_i\right|\right\}, \tag{3.33}$$

then, if $|y_j| \le l$, $j \ne i$, it results that $K = \overline{\Sigma(l)}$ and L_i^+, L_i^- are on two contrary faces of a hypercube K (so $L_i^+, L_i^- \in \partial K$).

Miranda's theorem states that, if

$$\begin{gathered} \Phi_i\left(L_i^+\right) \le 0, \quad \Phi_i\left(L_i^-\right) \ge 0, \quad i \in \overline{1,n}, \\ \left|y_j\right| \le l, \quad j \ne i, \end{gathered} \tag{3.34}$$

then the equation

$$\Phi(y) = 0 \tag{3.35}$$

admits solutions in K.

Suppose that f satisfies the following hypotheses:

(H_1) for every $l > 0$, problem (3.1) has a unique solution defined on the whole $\mathbb{R}$, for every $y \in K$;

(H_2) the functions $\alpha_i(t) := \inf_{x\in\mathbb{R}^n}\{f_i(t,x)\}$, $\beta_i(t) := \sup_{x\in\mathbb{R}^n}\{f_i(t,x)\}$ (where $f = (f_i)_{i\in\overline{1,n}}$) are defined on $\mathbb{R}$ and

$$\alpha_i, \beta_i \in L^1(\mathbb{R}), \quad i \in \overline{1,n}; \tag{3.36}$$

(H_3) there exists a constant $c > 0$ such that for every $i \in \overline{1,n}$ and for every $(t, y) \in \mathbb{R}\times\mathbb{R}^n$ with $|y_i| > c$, we have

$$y_i \cdot f_i(t, y) \geq 0. \tag{3.37}$$

Theorem 3.9. *Assume that the hypotheses (H_1), (H_2), and (H_3) are fulfilled. Then problem (1.1) admits solutions.*

Proof. Consider the operator P on K given by (3.8), that is,

$$Py = \frac{1}{2}\int_{-\infty}^{+\infty} f\big(s;x(s,y)\big)\,ds. \tag{3.38}$$

The operator P is well defined since hypotheses (H_1), (H_2), and (H_3) are assumed; in addition, as remarked, it is continuous on $\mathbb{R}^n$.

Set

$$\begin{aligned} a_i &:= \inf_{t\in\mathbb{R}}\int_0^t \alpha_i(s)\,ds, & b_i &:= \sup_{t\in\mathbb{R}}\int_0^t \beta_i(s)\,ds, \\ a &:= \max_{1\leq i\leq n}\{a_i\}, & b &:= \min_{1\leq i\leq n}\{b_i\}. \end{aligned} \tag{3.39}$$

Then we have for every solution $x(t;y) = (x_i(t;y))_{i\in\overline{1,n}}$,

$$y_i + a \leq x_i(t;y) \leq y_i + b, \quad i \in \overline{1,n}. \tag{3.40}$$

Considering $l \geq 0$ such that

$$l \geq \max\{c-a, c+b\}, \tag{3.41}$$

we obtain

$$x_i\big(t;L_i^+\big) \geq c, \quad x_i\big(t;L_i^-\big) \leq -c, \quad \forall i \in \overline{1,n},\ \forall L_i^+, L_i^- \in \partial K. \tag{3.42}$$

If relation (3.41) is fulfilled, it follows from (H_3),

$$P_i\big(L_i^+\big) \leq 0, \quad P_i\big(L_i^-\big) \geq 0, \quad \forall i \in \overline{1,n},\ \forall L_i^+, L_i^- \in \partial K, \tag{3.43}$$

where $P = (P_i)_{i\in\overline{1,n}}$.

Applying Miranda's theorem, it results that P has a zero in K. The proof is now complete. □

4. Continuation method

4.1. Introduction. In this section $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a continuous function, X is the space C_l or C_{ll} and $\Omega\subset X$ is an open and bounded set. If $F\bar{\Omega}$ is C_R-bounded, as remarked in Section 2, one can associate to problem (1.1) operators $U:\bar{\Omega}\to X$ which are compact and whose fixed points coincide with the solutions of (1.1).

In particular, if

$$x\neq Ux,\quad x\in\partial\Omega,\tag{4.1}$$

then we can define the topological degree of U and if

$$\deg(I-U,\Omega,0)\neq 0,\tag{4.2}$$

then U admits fixed points in Ω.

However, when we face to check condition (4.2), then we can use the so-called *continuation method*, which is based on the well-known homotopic invariance property of the topological degree (used in Section 3).

One of the most used forms of this method is the following. Let $h:\mathbb{R}\times\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ be a continuous and C_R-bounded on $\bar{\Omega}$ function in the sense that there exists $\theta\in C_R$, $\theta>0$, such that for every $x\in\bar{\Omega}$ and for every $\lambda\in[0,1]$ we have $|h(t,x(t),\lambda)|\leq\theta(t)$, $t\in\mathbb{R}$.

Consider the problem

$$\dot{x}=h(t,x,\lambda),\quad x(+\infty)=x(-\infty).\tag{4.3}$$

We can associate to problem (4.3) an operator U_λ which in addition is compact for every λ.

If the condition

$$x\neq U_\lambda x,\quad x\in\partial\Omega,\ \lambda\in[0,1]\tag{4.4}$$

is fulfilled, then we can define the degree $\deg(I-U_\lambda,\Omega,0)$; but a homotopic invariance property tells us that this degree is constant with respect to λ. In particular,

$$\deg\left(I-U_0,\Omega,0\right)=\deg\left(I-U_1,\Omega,0\right).\tag{4.5}$$

Equality (4.5) is useful if U_0 is an associated operator to problem (1.1) ($h(t,x,0)=f(t,x)$) and the degree of $I-U_1$ is easier to be computed, for example, when it is a Brouwer degree.

Condition (4.4) can be formulated under the following form: *for every* $\lambda\in[0,1]$ problem (4.3) *has no solutions* $x(\cdot;\lambda)$ *with* $x\in\partial\Omega$. If this condition is fulfilled, *every* associated operator U_λ satisfies (4.4) because the fixed points of an associated operator coincide with the set of solutions for the problem whose it is associated.

We get therefore the following proposition.

Proposition 4.1. *Assume that*

(i) *there exists* $\theta \in C_R$, $\theta(t) \geq 0$, *such that* $|h(t,x(t),\lambda)| \leq \theta(t)$, *for every* $x \in \bar{\Omega}$, *for every* $\lambda \in [0,1]$;

(ii) *for every* $\lambda \in [0,1]$, *problem (4.3) does not admit solutions* $x(\cdot)$ *with* $x \in \partial\Omega$;

(iii) $h(t,x,1) = f(t,x)$;

(iv) $\deg(I - U_0, \Omega, 0) \neq 0$.

Then problem (1.1) admits solutions.

The question that problem (4.3) has no solutions in $\partial\Omega$ can be formulated under the following form.

"A priori estimates": for every possible solution $x(\cdot)$ of problem (4.3) with $x \in \bar{\Omega}$ we have $x \in \Omega$.

Another form of the same condition is the next.

"A priori bound": there exists a number $r > 0$ such that problem (4.3) does not admit solutions $x(\cdot)$ with $\|x\|_\infty = r$.

In this case we set $\Omega := \{x \in X, \|x\| < r\}$.

Another variant of the same condition is the following.

"Bounded set condition": for every $\lambda \in [0,1]$ for which problem (4.3) has solutions $x(\cdot)$ with $x(t) \in \bar{D}$, $t \in \bar{\mathbb{R}}$, we have $x(t) \in D$, for every $t \in \bar{\mathbb{R}}$.

In this case when $D \subset \mathbb{R}^n$ is an open and bounded set we take $\Omega := \{x \in X, x(t) \in D\}$.

In this section, we indicate certain simple functions candidates to be homotopic linked through h with f, functions for which the computation of their topological degree is more advantageously.

The most difficult problem remains to establish the fact that problem (4.3) has no solutions in $\partial\Omega$; in what follows we consider certain cases when this thing is easy to be checked.

4.2. Homotopy with a linear equation. In this paragraph consider $X = C_{ll}$.

Let $A : \mathbb{R} \to M_n(\mathbb{R})$ be a continuous quadratic matrix; denote by $|\cdot|$ an arbitrary norm for the constant matrices.

Consider the system

$$\dot{x} = A(t)x \tag{4.6}$$

and denote by $X = X(t)$ its fundamental matrix with $X(0) = I$. In [5], the following result is proved.

Proposition 4.2. *Assume that*

$$\int_{-\infty}^{+\infty} |A(t)|\, dt < \infty, \tag{4.7}$$

then there exists $X(\pm\infty) = \lim_{t\to\pm\infty} X(t)$. *If in addition*

$$\operatorname{rank}\left[X(+\infty) - X(-\infty)\right] = n, \tag{4.8}$$

then the problem

$$\dot{x} = A(t)x, \quad x(+\infty) = x(-\infty) \tag{4.9}$$

admits only the zero solution.

Theorem 4.3. *Assume that*

(i) $A : \mathbb{R} \to M_n(\mathbb{R})$ *is a continuous matrix such that conditions (4.7), (4.8) are fulfilled;*

(ii) $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function fulfilling the condition*

$$\left|f(t,x)\right| \le \theta(t) \cdot \omega(|x|), \tag{4.10}$$

where $\theta : \mathbb{R} \to \mathbb{R}$, $\omega : \mathbb{R} \to \mathbb{R}$ *are continuous and positive functions,* $\theta \in C_R$;

(iii) *there exists* $r > 0$ *such that for every* $\lambda \in [0,1]$ *the problem*

$$\dot{x} = (1-\lambda)A(t)x + \lambda f(t,x), \quad x(+\infty) = x(-\infty) \tag{4.11}$$

has no solution $x(\cdot)$ *such that* $\|x\|_\infty = r$.

Then problem (1.1) admits solutions.

Proof. Set

$$\begin{gathered} h(t,x,\lambda) = (1-\lambda)A(t)x + \lambda f(t,x), \\ \Omega = B(r) := \left\{x \in C_{ll},\ \|x\|_\infty < r\right\}. \end{gathered} \tag{4.12}$$

For every $x \in \bar{\Omega}$, we have

$$\left|h(t,x(t),\lambda)\right| \le \rho\left|A(t)\right| + \omega(\rho)\theta(t), \tag{4.13}$$

where

$$\rho = \sup_{|u|\le r} \omega(u) \tag{4.14}$$

and so hypothesis (i) of Proposition 4.1 is satisfied; obviously (iii) is satisfied, too.

For $\lambda = 0$, problem (4.3) becomes

$$\dot{x} = A(t)x, \quad x(+\infty) = x(-\infty) \tag{4.15}$$

which, by Proposition 4.2, admits only the zero solution; that means every operator U_0 attached to problem (4.15) is injective. Since U_0 is linear and compact, then after a known property,

$$\deg\left(I - U_0, \Omega, 0\right) = \pm 1. \tag{4.16}$$

This ends the proof. □

Hypothesis (ii) of Proposition 4.1 is difficult to be checked in practice. In the next theorems it will be fulfilled.

Consider the problem

$$\dot{x} = A(t)x + g(t,x) + p(t), \quad x(-\infty) = x(+\infty), \tag{4.17}$$

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.

Theorem 4.4. *Assume that*

(i) *conditions (4.7), (4.8) are fulfilled;*
(ii) $p \in L^1(\mathbb{R}) \cap C_c$;
(iii) *there exists* $\alpha \in (0,1)$ *such that*

$$g(t,x) = k^\alpha g(t,x), \quad \forall k > 0,\ \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n; \tag{4.18}$$

(iv) *the following inequality holds:*

$$|g(t,x)| \le \theta(t), \quad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n,\ |x| \le 1, \tag{4.19}$$

where $\theta \in C_R$, $\theta(t) \ge 0$, $(\forall) t \in \mathbb{R}$.

Then problem (4.17) admits solutions.

Proof. Set in Proposition 4.1

$$\begin{gathered} h(t,x,\lambda) := (1-\lambda)A(t)x + \lambda\big((A(t)x) + p(t) + h(t,x)\big), \\ \Omega = B(\rho) := \{x \in C_{ll},\ \|x\|_\infty < \rho\}. \end{gathered} \tag{4.20}$$

By (4.18) it results that

$$(|x| \le \rho) \Longrightarrow \big(|g(t,x)| \le \rho^\alpha \theta(t)\big), \tag{4.21}$$

which shows that for $x \in \bar{\Omega}$,

$$|h(t,x,\lambda)| \le 2\rho|A(t)| + \rho^\alpha \theta(t) + |p(t)| \in C_R. \tag{4.22}$$

Obviously, to apply Proposition 4.1, it remains to check only hypothesis (ii). For this aim, we will show that there exists $\rho_0 > 0$ such that for every $\lambda \in [0,1]$ and for every $\rho > \rho_0$ problem (4.3) has no solution $x(\cdot)$ with $\|x\|_\infty = \rho$.

Indeed, if not, then we could find a sequence $\lambda_k \in [0,1]$, a sequence $\rho_k \to \infty$, such that the problem

$$\dot{x} = h(t,x,\lambda_k), \quad x(-\infty) = x(+\infty) \tag{4.23}$$

admits solutions $x_k(\cdot)$ with

$$\|x_k\|_\infty = \rho_k. \tag{4.24}$$

Setting

$$u_k := \frac{1}{\|x_k\|_\infty} x_k = \frac{1}{\rho_k} x_k, \tag{4.25}$$

we have

$$\begin{gathered} \|u_k\|_\infty = 1, \\ \dot{u}_k = (1-\lambda_k)A(t)u_k + \lambda_k\big[A(t)u_k + \rho_k^{\alpha-1} g(t,u_k) + \rho_k^{-1} p(t)\big]. \end{gathered} \tag{4.26}$$

By Corollary 2.7, we get the compactness of the sequence $(u_k)_k$ in C_{ll}.

Let $u \in \overline{(u_k)_k}$, $\lambda \in \overline{(\lambda_k)_k}$; by using the classical properties of uniform convergence we obtain, after computations,

$$\dot{u} = A(t)u, \quad u(-\infty) = u(+\infty),\ \|u\|_\infty = 1, \tag{4.27}$$

which contradicts Proposition 4.2. □

4.3. Auxiliary results. In Section 4.2, the homotopy has been achieved through a linear mapping for which it was easy to evaluate its topological degree. We give rise to another case when the topological degree computation is not too difficult in the sense that it becomes a Brouwer degree. This result will be a consequence of a more general result which links the existence of solutions for problem (1.1) to the existence of solutions for the problems of the type

$$\dot{y} = g(t,y), \quad y(0) = y(T),\ 0 < T < \infty. \tag{4.28}$$

Let $\theta : \mathbb{R} \to \mathbb{R}$, $\theta \in C_R$, $\theta(t) > 0$, for every $t \in \mathbb{R}$; set

$$\psi(t) := \int_{-\infty}^{t} \theta(s)\,ds, \qquad \varphi := \psi^{-1}, \qquad T := \int_{-\infty}^{+\infty} \theta(s)\,ds. \tag{4.29}$$

Obviously, through (2.9), $\varphi : (0,T) \to \mathbb{R}$ determines by (2.9) an isomorphism between C_l and $C_{(0,T)}$ (or between C_{ll} and $C_{[0,T]}$).

Proposition 4.5. *Suppose that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that*

$$\lim_{t\to\pm\infty} \frac{1}{\theta(t)} f(t,y) = \gamma_\pm(y), \quad y \in \mathbb{R}^n, \tag{4.30}$$

the convergence being uniform with respect to y on every compact subset of $\mathbb{R}^n$.

Let

$$g(t,y) := \begin{cases} \dot{\varphi}(t) f(\varphi(t),y), & \text{if } t \in (0,T),\ y \in \mathbb{R}^n, \\ \gamma_-(y), & \text{if } t = 0,\ y \in \mathbb{R}^n, \\ \gamma_+(y), & \text{if } t = 1,\ y \in \mathbb{R}^n. \end{cases} \tag{4.31}$$

Then problem (1.1) admits solutions if and only if (4.28) admits solutions.

Proof. Remark that if $D \subset C_l$ is a bounded set, then $FD \subset C_R$; indeed, for $|t| \geq A$, we have

$$\left|f(t,y)\right| \leq (a+m)\theta(t), \tag{4.32}$$

where

$$m := \max\left\{\sup \gamma_+(u),\ \gamma_-(u),\ u \in D \cap \mathbb{R}^n\right\}. \tag{4.33}$$

Set

$$r := \sup\left\{\|x\|_\infty,\ x \in D\right\}, \qquad \alpha(t) := \sup_{|y|\leq r}\left\{\left|f(t,y)\right|\right\}, \quad t \in [-A,A],$$
$$\beta(t) := \begin{cases} \alpha(t), & |t| < A, \\ (a+m)\theta(t), & |t| \geq A. \end{cases} \tag{4.34}$$

We obtain $\beta \in C_R$ and

$$\left|f(t,x)\right| \leq \beta(t), \quad \forall t \in \mathbb{R},\ \forall x \in D. \tag{4.35}$$

Let $x(t)$ be a solution for (1.1); then $y = \Phi(x)$ is a solution for the differential equation appearing in (4.28) on the interval $(0,T)$. Since $y(t)$ has limits in 0 and T, it can be prolonged as solution on $[0,T]$; but by definition of $y(t)$ it follows that

$$y(0) = \varphi\big(x(-\infty)\big) = \varphi\big(x(+\infty)\big) = y(T). \tag{4.36}$$

The converse is proved by using the isomorphism Φ^{-1}.

Let $\Omega \subset C_l$ be an open and bounded set. Hypothesis (4.30) allows us, as remarked, to associate to problem (1.1) the operator

$$U : \bar{\Omega} \subset C_l \longrightarrow C_l, \quad Ux = x(+\infty) + \int_{-\infty}^{(\cdot)} (Fx)(s)\,ds, \tag{4.37}$$

which, from (4.35), is compact.

The operator U_Φ defined in (2.15) is

$$U_\Phi : \overline{\Omega_\Phi} \subset C_{(0,T)} \longrightarrow C_{(0,T)}, \quad U_\Phi y := y(T) + \int_0^{(\cdot)} g\big(\tau, y(\tau)\big)\,d\tau. \tag{4.38}$$

But the operator U_Φ is associated to problem (4.28). By using the remarks from 2.4 we obtain the following result.

Corollary 4.6. *If $x \neq Ux$, for every $x \in \partial\Omega$, then*

$$\deg(I - U, \Omega, 0) = \deg\big(I - U_\Phi, \Omega_\Phi, 0\big). \tag{4.39}$$

(Obviously, the first degree is computed in C_l, the second in $C_{(0,T)}$.)

An important particular case is

$$f(t,x) = \theta(t)\cdot g(x). \tag{4.40}$$

In this case, when (4.30) is fulfilled, (4.24) becomes

$$\dot{y} = g(y), \quad y(0) = y(T). \tag{4.41}$$

As it is proved in [7], for every associated operator to problem (4.41) in $C_{(0,T)}$ or $C_{[0,T]}$ (so for U_Φ, too) we have

$$\deg\left(I - U_\Phi, \Omega_\Phi, 0\right) = \pm \deg_B\left(g, \Omega\cap\mathbb{R}^n, 0\right). \tag{4.42}$$

We obtain therefore the following proposition.

Proposition 4.7. *Suppose that*

(i) $\theta : \mathbb{R} \to \mathbb{R}$, $\theta \in C_R$, $\theta(t) \geq 0$, *for every* $t \in \mathbb{R}$;
(ii) $g : \mathbb{R}^n \to \mathbb{R}^n$, g *continuous.*

Consider the problem

$$\dot{x} = \theta(t)g(x), \quad x(-\infty) = x(+\infty). \tag{4.43}$$

Let $\Omega \subset C_l$ *be an open and bounded set. If for the operator* U *associated to problem (4.43), we have*

$$x \neq Ux, \quad x \in \partial\Omega, \tag{4.44}$$

then

$$\deg(I - U, \Omega, 0) = \pm \deg_B\left(g, \Omega\cap\mathbb{R}^n, 0\right). \tag{4.45}$$

Furthermore, if

$$\deg_B\left(g, \Omega\cap\mathbb{R}^n, 0\right) \neq 0, \tag{4.46}$$

it results that (4.43) admits solutions. □

4.4. Homotopies with nonlinear equations. We consider the problem

$$\dot{x} = f(t,x) + p(t), \quad x(-\infty) = x(+\infty). \tag{4.47}$$

Suppose that the following hypotheses are fulfilled:

(a_1) $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a continuous function;
(a_2) $|f(t,x)| \leq \beta(t)$, $x \in \mathbb{R}^n$, $|x| \leq 1$, $t \in \mathbb{R}$, $\beta \in C_R\cap C_c$;
(a_3) there exists $\alpha \in (0,1)$, $f(t,kx) = k^\alpha \cdot f(t,x)$, $k > 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$;
(a_4) $|p| \in C_R$.

In addition, let $\theta \in C_R$, $\theta(t) > 0$, with $\int_{-\infty}^{+\infty} \theta(t)\,dt = 1$.
Set

$$g(x) := \int_{-\infty}^{+\infty} f(s,x)\,ds. \tag{4.48}$$

Theorem 4.8. *Assume that hypotheses* (a_1)*,* (a_2)*,* (a_3)*, and* (a_4) *are fulfilled. Consider the problem*

$$\dot{x} = (1-\lambda)\theta(t)g(x) + \lambda\big[f(t,x)+p(t)\big], \quad x(-\infty) = x(+\infty). \tag{4.49}$$

Then

(1) *if*

$$g(y) \neq 0, \quad y \in \mathbb{R}^n,\ \|y\| = 1, \tag{4.50}$$

it results that there exists $\rho > 0$ *such that for every* $\lambda \in [0,1]$, *problem* (4.49) *has no solution* $x(\cdot)$ *with* $\|x\|_\infty = \rho_0$;

(2) *if for this* ρ_0

$$\deg_B\big(g, \Sigma(\rho_0), 0\big) \neq 0, \tag{4.51}$$

then (4.47) *admits solutions.*

Proof. If conclusion (1) is not true, then there would exist the sequences $(\rho_k)_k \subset (0,\infty)$, $(x_k)_k \subset X$, with $\|x_k\|_\infty = \rho_k$, $\lambda_k \in [0,1]$, $\rho_k > k$ and

$$\dot{x}_k = (1-\lambda_k)\theta(t)g(x_k) + \lambda_k\big[f(t,x_k)+p(t)\big], \quad x(-\infty) = x(+\infty). \tag{4.52}$$

Setting

$$u_k = \frac{x_k}{\rho_k}, \tag{4.53}$$

we get

$$\begin{gathered}\dot{u}_k = \rho_k^{-1}\big[(1-\lambda_k)\theta(t)g(u_k) + \lambda_k f(t,u_k)\big] + \rho_k^{-1}\lambda_k p(t), \\ \|u_k\| = 1, \quad u_k(-\infty) = u_k(+\infty).\end{gathered} \tag{4.54}$$

Based on Corollary 2.7, it results that $(u_k)_k$ is relatively compact in X. We can assume, up to subsequences, that $u_k \to u$, $\lambda_k \to \lambda$; we have

$$\|u\|_\infty = 1. \tag{4.55}$$

By (4.50) it results that $\dot{u}_k \to 0$, in X. Therefore $u \in \mathbb{R}^n$.
On the other hand, since

$$\int_{-\infty}^{+\infty} \dot{u}_k(s)\,ds = 0 \tag{4.56}$$

and (4.54), it follows that

$$0=(1-\lambda_k)\int_{-\infty}^{+\infty}\theta(t)g\big(u_k(t)\big)\,dt \\ +\lambda_k\int_{-\infty}^{+\infty}f\big(s,u_k(s)\big)\,ds+\rho_k^{-1}\lambda_k p(t); \tag{4.57}$$

hence, for $k\to\infty$, we get

$$g(u)=0,\quad u\in\mathbb{R}^n,\ \|u\|=1, \tag{4.58}$$

which contradicts (4.50). □

The second part of the theorem follows then by Proposition 4.1 for $\Omega=B(\rho_0)$ and Proposition 4.7.

A similar result can be obtained in the case $\alpha>1$, if $\int_{-\infty}^{+\infty}|p(t)|\,dt<1$.

4.5. Small perturbations. This paragraph deals with the problem

$$\dot{x}=\theta(t)g(x)+e(t,x),\quad x(-\infty)=x(+\infty), \tag{4.59}$$

where $g:\mathbb{R}^n\to\mathbb{R}^n$, $\theta:\mathbb{R}\to\mathbb{R}$, $e:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous functions; in addition $\theta(t)>0$, $t\in\mathbb{R}$, $\theta\in C_R$.

Consider problem (4.43). Let $D\subset\mathbb{R}^n$ be an open and bounded set.

Assume the following hypotheses:

(b_1) for every solution $x(\cdot)$ of problem (4.43) for which $x(t)\in\bar{D}$, $t\in\mathbb{R}$, it results that $x(t)\in D$, $t\in\bar{\mathbb{R}}$;

(b_2) $e(t,x)$ is C_R-bounded on $\bar{\Omega}$, where

$$\Omega:=\{x\in X,\ x(t)\in D,\ t\in\mathbb{R}\}. \tag{4.60}$$

Finally, consider the problem

$$\dot{x}=\theta(t)g(x)+\lambda e(t,x),\quad x(-\infty)=x(+\infty),\ \lambda\in[0,1]. \tag{4.61}$$

Theorem 4.9. *If hypotheses (b_1) and (b_2) are fulfilled, there exists $\varepsilon_0>0$ such that if*

$$\|e(\cdot,y)\|_\infty<\varepsilon_0,\quad \forall y\in\partial D, \tag{4.62}$$

then for every solution $x(\cdot)$ of problem (4.61) for which $x(t)\in\bar{D}$, for every $t\in\mathbb{R}$, it results that $x(t)\in D$, for every $t\in\bar{\mathbb{R}}$.

If, in addition,

$$\deg_B(g,D,0)\neq 0, \tag{4.63}$$

then for every $e(\cdot,\cdot)$ satisfying (4.62), problem (4.59) admits solutions.

Proof. We prove the first part. If the conclusion is not true, then for every $k \in \mathbb{N}^*$ there exists a function $e_k(\cdot,\cdot)$ with $\|e_k(\cdot,y)\|_\infty < 1/k$, for every $y \in \bar{D}$ and a function $x_k(\cdot)$ such that

$$\dot{x}_k = \theta(t)g(x_k) + \lambda_k e(t,x_k), \quad x_k(-\infty) = x_k(+\infty),\ \lambda_k \in [0,1], \tag{4.64}$$

with $x_k(t) \in \bar{D}$, for every $t \in \bar{\mathbb{R}}$, and $x_k(t_k) \notin \partial D$, for an $t_k \in \mathbb{R}$.

Corollary 2.7 assures the compactness of the sequence $(x_k)_k$ in C_l. If $x_k \to x$ in C_l, $\lambda_k \to \lambda$, and $t_k \to t \in \bar{\mathbb{R}}$, one contradicts hypothesis (b_1).

The second part follows by Propositions 4.1 and 4.7 for $\Omega := \{x \in C_l,\ x(t) \in D$, for every $t \in \mathbb{R}\}$. □

4.6. Asymptotically homogeneous systems. Consider again the problems

$$\dot{x} = \theta(t)g(x) + e(t,x), \quad x(-\infty) = x(+\infty), \tag{4.65}$$

$$\dot{x} = \theta(t)g(x), \quad x(-\infty) = x(+\infty), \tag{4.66}$$

$$\dot{x} = \theta(t)g(x) + \lambda e(t,x), \quad x(-\infty) = x(+\infty),\ \lambda \in [0,1]. \tag{4.67}$$

Assume the following hypotheses:

(c_1) $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that

$$g(kx) = g(x), \quad x \in \mathbb{R}^n,\ k > 0; \tag{4.68}$$

(c_2) $\lim_{|x|\to\infty} e(t,x)/|x| = 0$;

(c_3) for every $\rho > 0$, there exists $\alpha_\rho \in C_R$, $\alpha_\rho > 0$, for every $x \in \Sigma(\rho)$,

$$|e(t,x)| \le \alpha_\rho(t), \quad t \in \mathbb{R}; \tag{4.69}$$

(c_4) $\theta \in C_R$, $\theta(t) > 0$, for every $t \in \mathbb{R}$;

(c_5) problem (4.43) admits only the zero solution.

Theorem 4.10. *Assuming that hypotheses (c_1), (c_2), (c_3), (c_4), and (c_5) are fulfilled. Then there exists $\rho_0 > 0$, such that for every solution $x(\cdot)$ for problem (4.67) and for every $\lambda \in [0,1]$*

$$\|x\|_\infty < \rho_0. \tag{4.70}$$

If, in addition,

$$\deg_B(g, \Sigma(\rho_0), 0) \neq 0, \tag{4.71}$$

then problem (4.65) admits solutions.

The proof is analogous with the proofs of Theorems 4.8 and 4.9. If the first conclusion is not true, then one finds $\lambda_k \in [0,1]$ and $x_k \in X$ satisfying

$$\dot{x}_k = \theta(t)g(x_k) + \lambda_k e(t,x_k), \quad x_k(-\infty) = x_k(+\infty),\ \|x_k\|_\infty \longrightarrow \infty. \tag{4.72}$$

Setting again $u_k = x_k/\|x_k\|$, we deduce that $(u_k)_k$ is compact in X; if $u \in \overline{\{x_k\}_k}$, then u satisfies (4.64) and $\|u\|_\infty = 1$, which contradicts hypothesis (c_5).

The second part is an immediate consequence of Propositions 4.1 and 4.7 and of condition (4.71).

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