AN INVERSE PROBLEM FOR EVOLUTION INCLUSIONS

BUI AN TON

Received 18 May 2001

An inverse problem, the determination of the shape and a convective coefficient on a part of the boundary from partial measurements of the solution, is studied using 2-person optimal control techniques.

1. Introduction

Let $H, \mathcal{H}_j, \mathcal{U}_j;\ j = 1,\dots,N$ be Hilbert spaces and let φ be a lower semi-continuous (l.s.c.) function from $H \times \prod_{j=1}^{N} \mathcal{U}_j$ into $\mathbb{R}^+$ with $\varphi(\cdot;u)$ convex on H.

Consider the initial-value problem

$$y' + \partial\varphi(y;u) + f(t,y;u) \ni 0 \quad \text{on } (0,T),\ y(0) = y_0. \tag{1.1}$$

With some conditions on φ and on f, the set $\mathcal{R}(u)$ of all "strong" solutions of (1.1) is nonempty. Let f_j be mappings of $L^2(0,T;\mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$ and associate with (1.1) the cost functionals

$$J_j(y;u) = \int_0^T f_j\big(y(s);u\big)\,ds, \qquad j = 1,\dots,N, \tag{1.2}$$

with $D(\varphi(\cdot,u)) \subset \mathcal{H}_j$ for all $u \in \mathcal{U} = \prod_{j=1}^{N} \mathcal{U}_j$.

The existence of an open loop of (1.1), (1.2) with φ independent of the control u, has been established in Ton [7]. With optimal shape design and with inverse problems in mind, we will consider the case when φ depends on the control u as it appears in the top order term of the partial differential operators involved in the problems.

Optimal design of domains has been investigated by Barbu and Friedman [1], Canadas et al. [2], Gunzburger and Kim [3], Pironneau [6], and others. Inverse

Abstract and Applied Analysis 7:1 (2002) 35–51
2000 Mathematics Subject Classification: 35K60, 49J20, 49N45, 91A06
URL: http://dx.doi.org/10.1155/S1085337502000702

problems have been studied by Canadas et al. [2], Lenhart et al. [4], Lenhart and Wilson [5], and others.

In contrast with all the cited works where a single cost functional is involved, we will consider the N-person optimal control approach. It is well known that for N-control, open and closed loops are two different notions. In this paper, the existence of an open loop of (1.1), (1.2) is established in Section 3, that is, there exists $\tilde{u} \in U$ such that

$$J_j(\tilde{y};\pi_j\tilde{u},\tilde{u}_j) \leq J_j(y;\pi_j\tilde{u},v_j), \quad \forall y \in \mathcal{R}(\pi_j\tilde{u},v_j),\ \forall v_j \in U_j;\ j=1,\dots,N, \tag{1.3}$$

where U_j are given compact convex subsets of the control spaces $\mathcal{U}_j$ and π_j is the projection of $\mathcal{U}$ onto $\prod_{k\neq j}^N \mathcal{U}_k$.

With a cost functional f_j defined by

$$f_j(y) = \|y(\cdot,t)-h(\cdot,t)\|^2_{L^2(0,T;L^2(\Omega))}, \tag{1.4}$$

where Ω is a proper subset of the domain and where h is a measurement of the solution y of (1.1) in the subdomain Ω, then (1.1), (1.2) becomes an inverse problem.

Applications to parabolic inequalities are carried out in Section 4 and the notation and the main assumptions of the paper are given in Section 2.

2. Notation and assumptions

Let H, $\mathcal{H}_j$, $\mathcal{U}_j$; $j=1,\dots,N$ be Hilbert spaces. The norm in H is denoted by $\|\cdot\|$ and $(\cdot,\cdot)$ is the inner product in the space. Throughout, U is a given compact convex subset of the control space $\mathcal{U} = \prod_{j=1}^N \mathcal{U}_j$.

Assumption 2.1. Let φ be a mapping of $H\times\mathcal{U}$ into $\mathbb{R}^+$. We assume that

(1) for each $u\in\mathcal{U}$, $D(\varphi(\cdot;u))$ is dense in H;

(2) $\varphi(y;u)$ is an l.s.c. function from $H\times\mathcal{U}$ into $\mathbb{R}^+$ and is convex on H for each given $u\in\mathcal{U}$;

(3) there exists a positive constant c such that

$$c\|y\|^2 \leq \varphi(y;u), \quad \forall y\in D(\varphi(y;u)),\ \forall u\in\mathcal{U}; \tag{2.1}$$

(4) for each positive constant C,

$$\{y:\varphi(y;u)\leq C\} \tag{2.2}$$

is a compact convex subset of H for each given $u\in\mathcal{U}$;

(5) if $u_n\to u$ in $\mathcal{U}$, then

$$\int_0^T \varphi(y(s);u)\,ds = \lim_n \int_0^T \varphi(y(s);u_n)\,ds, \quad \forall y\in \bigcap_{u_n\in\mathcal{U}} D(\varphi(\cdot;u_n))\cap L^2(0,T;H). \tag{2.3}$$

The subdifferential of $\varphi(y;u)$ at y is the set

$$\partial\varphi(y;u) = \{g : g \in H,\ \varphi(x;u) - \varphi(y;u) \geq (g, x-y),\ \forall x \in D(\varphi(\cdot;u))\}. \tag{2.4}$$

It is known that $A(y;u) = \partial\varphi(y;u)$ is maximal monotone in H. The images of $A(y;u)$ are closed, convex subsets of H.

Let $f(y;u)$ be a mapping of $L^2(0,T;H) \times \mathcal{U}$ into $L^2(0,T;H)$ satisfying the following assumption.

Assumption 2.2. We assume that there exists a constant C such that

$$\|f(y;u)\|_H^2 \leq C\{1 + \|u\|_{\mathcal{U}}^2 + \varphi(y;u)\} \tag{2.5}$$

for all $y \in D(\varphi(\cdot;u))$, all $u \in \mathcal{U}$.

Throughout, the set of solutions of (1.1) is denoted by $\mathcal{R}(u)$.

Assumption 2.3. Let f_j be mappings of $L^2(0,T;\mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$. We assume that

(1) $D(\varphi(\cdot;u)) \subset \mathcal{H}_j$ for all $u \in \mathcal{U}$;

(2) suppose that

$$\begin{aligned} &\varphi(y^n;u^n) + \|(y^n)'\|_{L^2(0,T;H)} \leq C, \\ &\qquad u^n \in U,\ \{y^n, u^n\} \longrightarrow \{y, u\} \quad \text{in } L^2(0,T;H) \times \mathcal{U}, \end{aligned} \tag{2.6}$$

then

$$\int_0^T f_j(y;u)\,dt = \lim_{n\to\infty} \int_0^T f_j(y^n;u^n)\,dt. \tag{2.7}$$

3. Open loop control

The main result of this section is the following theorem.

Theorem 3.1. *Let φ, f be as in Assumptions 2.1 and 2.2, and let f_j be continuous mappings of $L^2(0,T;\mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$. Suppose that $y_0 \in D(\varphi(\cdot;u))$ for all $u \in U$. Then there exists $\{\tilde{y}, \tilde{u}\} \in \{L^2(0,T;H) \cap \mathcal{R}(\tilde{u})\} \times U$ such that*

$$J_j(\tilde{y};\pi_j\tilde{u},\tilde{u}_j) \leq J_j(y;\pi_j\tilde{u},v_j), \quad \forall y \in \mathcal{R}(\pi_j\tilde{u},v_j),\ \forall v_j \in U_j,\ j = 1,\dots,N. \tag{3.1}$$

Moreover, there exists a positive constant C, independent of u such that

$$\begin{aligned} \operatorname{ess\,sup}\varphi(\tilde{y}(t);\tilde{u}) + \|\tilde{y}'\|_{L^2(0,T;H)}^2 + \|A(\tilde{y};\tilde{u})\|_{L^2(0,T;H)}^2 \\ \leq C\Big\{1 + \sup_{u\in U}\varphi(y_0;u)\Big\}, \end{aligned} \tag{3.2}$$

where $A(\tilde{y};\tilde{u})$ is an element of the set $\partial\varphi(\tilde{y};\tilde{u})$.

First, we will show that the set $\mathcal{R}(u)$ is nonempty.

Theorem 3.2. *Suppose all the hypotheses of Theorem 3.1 are satisfied. Then for each given $u \in U$, there exists a solution y of (1.1) with*

$$\|y'\|^2_{L^2(0,T;H)} + \|A(y;u)\|^2_{L^2(0,T;H)} + \operatorname{ess\,sup} \varphi(y(t);u) \le C\{1 + \|u\|^2_{\mathcal{U}}\}. \tag{3.3}$$

The constant C is independent of u and $A(y;u)$ is an element of $\partial\varphi(y;u)$.

Proof. For a given $u \in U$, the existence of a solution y of (1.1) with

$$\{y, y', A(y;u)\} \in L^\infty(0,T;H) \times (L^2(0,T;H))^2 \tag{3.4}$$

is known (cf. Yamada [8]).

We will now establish the estimate of Theorem 3.2. We have

$$(y', \partial\varphi(y;u)) + \|\partial\varphi(y;u)\|^2 + (f(y;u), \partial\varphi(y;u)) = 0. \tag{3.5}$$

With our hypotheses on f, we get

$$\frac{d}{dt}\varphi(y;u) + \|\partial\varphi(y;u)\|^2 \le C\{1 + \|u\|^2_{\mathcal{U}} + \varphi(y(t);u)\}. \tag{3.6}$$

It follows from the Gronwall lemma that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \varphi(y(t);u) + \|\partial\varphi(y;u)\|^2_{L^2(0,T;H)} \le C\{1 + \|u\|^2_{\mathcal{U}}\}. \tag{3.7}$$

The different constants C are all independent of u.

With the estimate (2.1), we deduce from (1.1) and from Assumption 2.2 that

$$\|y'\|^2_{L^2(0,T;H)} \le C\{1 + \|u\|^2_{\mathcal{U}}\}. \tag{3.8}$$

The theorem is thus proved. □

Set

$$\mathcal{B}_C = \left\{ y : \|y'\|_{L^2(0,T;H)} + \sup_{u\in U} \operatorname{ess\,sup} \varphi(y;u) \le C\left(1 + \sup_{u\in U} \|u\|_{\mathcal{U}}\right)\right\}. \tag{3.9}$$

Consider the evolution inclusion

$$y' + \partial\varphi(y;u) + f(x;u) \ni 0 \quad \text{on } (0,T), \; y(0) = y_0 \tag{3.10}$$

with $x \in \mathcal{B}_C$.

In view of Theorem 3.2, inclusion (3.10) has a unique solution which we will write as $y = R(x;u)$.

Denote by

$$J_j(x;y;u) = \int_0^T f_j(y(s);u)\,ds, \quad j = 1,\dots,N, \tag{3.11}$$

the cost functionals associated with (3.10) and where $y = R(x;u)$ is the unique solution of (3.10).

Let

$$\Psi(x;u,v) = \sum_{j=1}^{N} J_j\big(x;y_j;\pi_j u, v_j\big), \tag{3.12}$$

where $y_j = R(x;\pi_j u, v_j)$.

Lemma 3.3. *Suppose all the hypotheses of Theorem 3.1 are satisfied. Then for each given* $\{x,u\} \in \mathcal{B}_C \times U$, *there exists* $v^* \in U$ *such that*

$$\Psi\big(x;u,v^*\big) = d(x;u) = \inf\big\{\Psi(x;u,v) : v \in U\big\}. \tag{3.13}$$

Proof. Let $\{v^n\}$ be a minimizing sequence of (3.13) with

$$d(x;u) \le \Psi\big(x;u,v^n\big) \le d(x;u) + n^{-1}. \tag{3.14}$$

Since $v^n \in U$ and U is a compact subset of $\mathcal{U}$, we obtain by taking subsequences that $v^{n_k} \to v^*$ in $\mathcal{U}$. Let $y_j^n = R(x;\pi_j u, v_j^n)$, then from the estimates of Theorem 3.2 we obtain, by taking subsequences, that

$$\begin{aligned} &\big\{y_j^{n_k}, \big(y_j^{n_k}\big)', A\big(y_j^{n_k};\pi_j u, v^{n_k}\big)\big\} \\ &\qquad \longrightarrow \big\{y_j^*, \big(y_j^*\big)', \chi_j\big\} \quad \text{in } L^2(0,T;H) \times \big(L^2(0,T;H)\big)^2_{\text{weak}}. \end{aligned} \tag{3.15}$$

From the definition of subdifferential, we have

$$\begin{aligned} &\int_0^T \varphi\big(z(t);\pi_j u, v_j^{n_k}\big)\,dt - \int_0^T \varphi\big(y_j^{n_k}(t);\pi_j u, v_j^{n_k}\big)\,dt \\ &\qquad \ge \int_0^T \big(A\big(y_j^{n_k}(t);\pi_j u, v_j^{n_k}\big), z - y_j^{n_k}\big)\,dt, \end{aligned} \tag{3.16}$$

for all $z \in L^2(0,T;H)$.

It follows from Assumption 2.1 that

$$\int_0^T \varphi\big(z(t);\pi_j u, v_j^*\big)\,dt - \int_0^T \varphi\big(y_j^*(t);\pi_j u, v_j^*\big)\,dt \ge \int_0^T \big(\chi_j, z - y_j^*(t)\big)\,dt. \tag{3.17}$$

Hence

$$\chi_j = A\big(y_j^*;\pi_j u, v_j^*\big). \tag{3.18}$$

It is clear that $y_j^* = R(x;\pi_j u, v_j^*)$ and thus,

$$d(x;u) = \Psi\big(x;u,v^*\big) = \sum_{j=1}^{N} J_j\big(x;y_j,\pi_j u, v_j^*\big), \tag{3.19}$$

where $y_j = R(x;\pi_j u, v_j^*)$.

The lemma is proved. □

Let

$$X(x;u) = \{v^* : \Psi(x;u,v^*) \le \Psi(x;u,v),\ \forall v \in U\}. \tag{3.20}$$

Lemma 3.4. *Let g_j be a continuous mapping of U_j into $\mathbb{R}^+$ and suppose that g_j is 1-1. Then there exists a unique $\hat{v} \in X(x;u)$ such that*

$$g_j(\hat{v}_j) = \inf\{g_j(v_j^*) : v^* \in X(x,u)\}. \tag{3.21}$$

Proof. The set $X(x;u)$ is nonempty and with our hypothesis on g_j, it is clear that

$$d_j(x;u) = \inf\{g_j(v_j^*) : v^* \in X(x;u)\} \tag{3.22}$$

exists.

Let v_j^n be a minimizing sequence of the optimization problem (3.22) with

$$d_j(x;u) \le g_j(v_j^n) \le d_j(x;u) + n^{-1}, \quad j = 1,\dots,N, \tag{3.23}$$

and $v^n \in X(x,u)$.

Let $y_j^n = R(x;\pi_j u, v_j^n)$ be the unique solution of (3.10) with controls $\{\pi_j u, v_j^n\}$ and $f(x;\pi_j u, v_j^n)$. Then from the estimates of Theorem 3.2, we obtain, by taking subsequences, that

$$\{y_j^n, (y_j^n)', A(y_j^n;\pi_j u, v_j^n)\} \longrightarrow \{\hat{y}_j, \hat{y}_j', \chi_j\} \quad \text{in } L^2(0,T;H) \times (L^2(0,T;H))^2_{\text{weak}}. \tag{3.24}$$

Since $v^n \in U$, we get by taking subsequences that $v^n \to \hat{v}$ in $\mathcal{U}$.

A proof, as in that of Lemma 3.3, shows that

$$\chi_j = A(\hat{y}_j;\pi_j u, \hat{v}_j), \quad \hat{y}_j = R(x;\pi_j u, \hat{v}_j). \tag{3.25}$$

Hence $\hat{v} \in X(x;u)$. We now have

$$g_j(\hat{v}_j) = d_j(x;u) = \inf\{g_j(v_j^*) : v^* \in X(x;u)\}. \tag{3.26}$$

Since g_j is 1-1, $\hat{v}$ is unique. The lemma is proved. □

Let $\mathcal{L}$ be the nonlinear mapping of $\mathcal{B}_C \times U$ into $\mathcal{B}_C \times U$, defined by

$$\mathcal{L}(x,u) = \{\hat{y}, \hat{v}\}, \tag{3.27}$$

where $\hat{v}$ is the element of U given by Lemma 3.4 and $\hat{y} = R(x;\pi_j u, \hat{v}_j)$ is the unique solution of (3.10) with control $\{\pi_j u, \hat{v}_j\}$ and $f(x;\pi_j u, \hat{v}_j)$.

Lemma 3.5. *Suppose all the hypotheses of Theorem 3.1 are satisfied. Then $\mathcal{L}$, defined by (3.27), has a fixed point, that is, there exists $\{\tilde{y}, \tilde{u}\} \in \mathcal{B}_C \times U$ such that $\mathcal{L}(\tilde{y}, \tilde{u}) = \{\tilde{y}, \tilde{u}\}$.*

Proof. (1) We now show that $\mathcal{L}$ has a fixed point by applying Schauder's theorem. Since $\mathcal{B}_C \times U$ is a compact convex subset of $L^2(0,T;H)\times\mathcal{U}$ and since $\mathcal{L}$ takes $\mathcal{B}_C \times U$ into itself, it suffices to show that $\mathcal{L}$ is continuous.

(2) Let $\{x^n,\ u^n\}$ be in $\mathcal{B}_C \times U$ and let

$$y_j^n = R(x^n;\pi_j u^n, \hat{v}_j^n), \quad \hat{v}^n \text{ as in Lemma 3.4.} \tag{3.28}$$

Since $\{x^n u^n\} \in \mathcal{B}_C \times U$ and $\mathcal{B}_C \times U$ is a compact subset of $L^2(0,T;H)\times\mathcal{U}$, there exists a subsequence such that

$$\{x^n, u^n, \hat{v}^n\} \longrightarrow \{x^*, u^*, \hat{v}\} \quad \text{in } L^2(0,T;H)\times\mathcal{U}\times\mathcal{U}. \tag{3.29}$$

From the estimates of Theorem 3.2, we get

$$\{y_j^n, (y_j^n)', A(y_j^n;u^n)\} \longrightarrow \{y_j^*, (y_j^*)', \chi_j\} \quad \text{in } L^2(0,T;H)\times\left(L^2(0,T;H)\right)^2_{\text{weak}}. \tag{3.30}$$

A proof, as in that of Lemma 3.3, shows that

$$\chi_j = A(y_j^*;u^*), \quad y_j^* = R(x^*;\pi_j u^*, \hat{v}_j). \tag{3.31}$$

(3) We now show that $u^* \in X(x^*, \hat{v})$. Since

$$\mathcal{L}\{u^n, x^n\} = \{v^n, y^n\}, \tag{3.32}$$

it follows from the definition of $\mathcal{L}$ that

$$\begin{gathered}\Psi(x^n;u^n,v^n) \le \Psi(x^n;u^n,v), \quad \forall v\in U,\\ \sum_{j=1}^N J_j(x^n;y_j^n;\pi_j u^n, v_j^n) \le \sum_{j=1}^N J_j(x^n;z_j^n;\pi_j u^n, v_j), \quad \forall v\in U,\end{gathered} \tag{3.33}$$

where $z_j^n = R(x^n;\ \pi_j u^n, v_j)$ is the unique solution of (3.10) with controls $\{\pi_j u^n, v_j\}$ and $f(x^n;\ \pi_j u^n,\ v_j)$.

Again from the estimates of Theorem 3.2, we deduce as above that

$$\{z_j^n, (z_j^n)', A(z_j^n;u^n)\} \longrightarrow \{z_j, z_j', A(z_j;u^*)\} \quad \text{in } L^2(0,T;H)\times\left(L^2(0,T;H)\right)^2_{\text{weak}}. \tag{3.34}$$

It then follows from (3.33) that

$$\sum_{j=1}^N J_j(x^*;y_j^*;\pi_j u^*, \hat{v}_j) \le \sum_{j=1}^N J_j(x^*;z_j;\pi_j u^*;v_j), \quad \forall v\in U, \tag{3.35}$$

that is,

$$\Psi(x^*;u^*,\hat{v}) \le \Psi(x^*;u^*,v), \quad \forall v\in U. \tag{3.36}$$

Hence

$$d(x^*,u^*) = \Psi(x^*;u^*,\hat{v}) = \inf\{\Psi(x^*;u^*,v) : v \in U\}. \tag{3.37}$$

Moreover, we have

$$\lim_n g_j(v_j^n) = g_j(\hat{v}_j), \quad j = 1,\dots,N. \tag{3.38}$$

By hypothesis, g_j is 1-1 and so $\hat{v}$, the unique element of $X(x^*;u^*)$, with

$$g_j(\hat{v}_j) = \inf\{g_j(v_j) : v \in X(x^*;u^*)\}, \tag{3.39}$$

is in $X(x^*;u^*)$. It follows that $\mathcal{L}\{x^*,u^*\} = \{y^*,\hat{v}\}$.

The operator $\mathcal{L}$ is continuous and thus, it has a fixed point by Schauder's theorem. The lemma is thus proved. □

Proof of Theorem 3.1. Let $\mathcal{L}$ be as in (3.33). Then it follows from Lemma 3.5 that $\mathcal{L}$ has a fixed point, that is, there exists $\{\tilde{y},\tilde{u}\}$ with

$$\mathcal{L}\{\tilde{y},\tilde{u}\} = \{\tilde{y},\tilde{u}\}. \tag{3.40}$$

Thus,

$$\tilde{y}' + A(\tilde{y};\tilde{u}) + f(\tilde{y};\tilde{u}) = 0 \quad \text{on } (0,T);\ y(0) = y_0. \tag{3.41}$$

Moreover,

$$\sum_{j=1}^{N} J_j(\tilde{y};\pi_j\tilde{u},\tilde{u}_j) \le \sum_{j=1}^{N} J_j(y_j;\pi_j\tilde{u},v_j), \quad \forall y_j \in \mathcal{R}(\pi_j\tilde{u},v_j),\ \forall v \in U. \tag{3.42}$$

Take $v = (\pi_j\tilde{u},v_j)$ and we obtain from (3.42) that

$$J_j(\tilde{y};\pi_j\tilde{u},\tilde{u}_j) \le J_j(y_j;\pi_j\tilde{u},v_j), \quad \forall y_j \in \mathcal{R}(\pi_j\tilde{u},v_j). \tag{3.43}$$

Repeating the process N times we get the theorem. □

4. Applications

In this section, we give some applications of Theorem 3.1 to parabolic initial boundary value problems. For simplicity, we take $N = 2$.

Let G be a bounded open subset of $\mathbb{R}^2$ with a smooth boundary and let

$$\begin{gathered} Q = G\times(0,2), \qquad \Gamma = G\times\{2\}, \\ Q(u_1) = \{(\xi,\eta) : \xi \in G,\ 0 < \eta < u_1(\xi)\}, \end{gathered} \tag{4.1}$$

where u_1 is a continuous function of G into $[1,2]$. The top of the cylinders $Q(u_1)$, Q are

$$\Gamma(u_1) = \{(\xi,u_1(\xi)) : \xi \in G\}, \quad \Gamma. \tag{4.2}$$

Make the change of variable $\zeta = 2\eta/u_1$ and set

$$y(\xi,\eta) = y\left(\xi, \frac{u_1\zeta}{2}\right) = Y(\xi,\zeta). \tag{4.3}$$

As done in great details in [4, pages 946–948], we get

$$\nabla^2 y = \nabla_{\xi,\zeta} F(\xi,\zeta;u_1)\nabla_{\xi,\zeta} Y(\xi,\zeta) + u_1^{-1} F\nabla Y \cdot \nabla u_1, \tag{4.4}$$

where $F(\xi,\zeta;u_1)$ is the matrix

$$\begin{pmatrix} 1 & 0 & -\zeta(\partial_{\xi_1} u_1)u_1^{-1} \\ 0 & 1 & -\zeta(\partial_{\xi_2} u_1)u_1^{-1} \\ -\zeta(\partial_{\xi_2} u_1)u_1^{-1} & -\zeta(\partial_{\xi_1} u_1)u_1^{-1} & \zeta^2|\nabla u_1|^2 u_1^{-2} + 4u_1^{-2} \end{pmatrix}. \tag{4.5}$$

Set

$$\mu(u_1) = 2u_1^{-1}\sqrt{1+|\nabla u_1|^2}. \tag{4.6}$$

4.1. An inverse problem for a nonlinear heat equation. Consider the initial boundary value problem

$$\begin{aligned} y' - \Delta y &= \tilde{f}(y) && \text{on } Q(u_1)\times(0,T), \\ y &= 0 && \text{on } \partial Q(u_1)/\Gamma\times(0,T), \\ -\frac{\partial y}{\partial n} &\in u_2\beta(y) && \text{on } \Gamma(u_1)\times(0,T), \\ y(\cdot,0) &= y_0 && \text{on } Q(u_1), \end{aligned} \tag{4.7}$$

where $\beta \in \partial j(r)$ and $j(r)$ is an l.s.c. convex function from $\mathbb{R}^+$ to $[0,\infty]$.

Let

$$\begin{aligned} J_1(y;u_1,u_2) &= \int_0^T \int_G |y(\xi,u_1(\xi))|^2\, d\xi\, dt, \\ J_2(y;u_1,u_2) &= \int_0^T \int_\Omega |y - h(\xi,\eta)|^2\, d\xi\, d\eta\, dt \end{aligned} \tag{4.8}$$

be the cost functionals associated with (4.7) and let h be the measurement of the solution y of (4.7) in the sub-region Ω.

We denote

$$U_j = \{u_j : \|u_j\|_{H^3(G)} \le C,\ 1 \le u_1(\xi) \le 2,\ 0 \le u_2(\xi) \le C\} \tag{4.9}$$

and let $\mathcal{U}_j = L^2(G)$. It is clear that the U_j are compact convex subsets of the space of controls $\mathcal{U}_j$.

We will take

$$H = L^2(Q), \qquad \mathcal{H}_1 = L^2(G), \qquad \mathcal{H}_2 = L^2(\Omega), \quad \Omega \subset Q. \tag{4.10}$$

The main result of this subsection is the following theorem.

Theorem 4.1. *Let y_0 be in $H_0^1(Q)$ and let $\tilde{f}$ be a continuous function of y, u with*

$$|\tilde{f}(y;u)| \leq C\{1+|y|+|u|\}. \tag{4.11}$$

Let h be a given function in $L^2(0,T;L^2(\Omega))$ where Ω is a proper subset of Q and let $j(r)$ be an l.s.c. convex function on $\mathbb{R}$ with values in $[0,+\infty]$. Then there exists

$$\begin{aligned} \{\hat{y},\hat{y}',\hat{u}\} \in L^2(0,T;H^1(Q(\hat{u}_1))) \cap L^\infty(0,T;L^2(Q(\hat{u}_1))) \\ \times L^2(0,T;L^2(Q(\hat{u}_1))) \times U \end{aligned} \tag{4.12}$$

such that $\hat{y}$ is a solution of the initial boundary value problem (4.7) in $Q(\hat{u}_1) \times (0,T)$; and

$$\begin{aligned} J_1(\hat{y};\hat{u}_1,\hat{u}_2) &\leq J_1(y;\hat{u}_1,v_2), \quad \forall v_2 \in U_2, \\ J_2(\hat{y};\hat{u}_1,\hat{u}_2) &\leq J_2(x;v_1,\hat{u}_2), \quad \forall v_1 \in U_1, \end{aligned} \tag{4.13}$$

where x, y are the solutions of (4.7) with controls $\{v_1,\hat{u}_2\}$, $\{\hat{u}_1,v_2\}$ in $Q(v_1)\times(0,T)$ and in $Q(\hat{u}_1)\times(0,T)$, respectively.

Problems of type (4.7) arise in the study of heat transfer between solids and gases under nonlinear boundary conditions.

As carried out in [4], we make the change of variable $\zeta = 2u_1^{-1}\eta$ and set $y(\xi,\eta) = Y(\xi,\zeta)$. Then (4.7) is transformed into the following problem:

$$\begin{aligned} Y' - \nabla(F(u_1)\cdot\nabla Y) + u_1^{-1}F\nabla Y\cdot\nabla u_1 = \tilde{f}(Y,u) \quad &\text{on } Q\times(0,T), \\ Y = 0 \quad &\text{on } \partial Q/\Gamma\times(0,T), \\ -\frac{\partial Y}{\partial n} \in \mu(u_1)u_2\beta(Y) \quad &\text{on } \Gamma\times(0,T), \\ Y(\cdot,0) = y_0 \quad &\text{on } Q \end{aligned} \tag{4.14}$$

with cost functionals

$$J_1(Y;u_1,u_2) = \int_0^T\int_G |Y(\xi,2)|^2 d\xi\,dt, \tag{4.15}$$

$$J_2(Y;u_1,u_2) = \int_0^T\int_\Omega \left|Y\left(\xi,\frac{2\eta}{u_1}\right) - h(\xi,\eta;t)\right|^2 d\xi\,d\eta\,dt, \tag{4.16}$$

where μ is as in expression (4.6).

Our aim is to find the controls u_1, u_2 so that the solution y of (4.7), if it is unique, is as close to the measurement h in Ω as possible.

Let φ be the mapping of $H\times U_1\times U_2$ into $\mathbb{R}^+$ given by

$$\varphi(Y;u_1,u_2) = \begin{cases} \frac{1}{2}\|F(u)\nabla Y\|_{L^2(Q)}^2 + \int_\Gamma \mu(u_1)u_2 j(Y)\,d\sigma, & j(Y)\in L^1(\Gamma), \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.17}$$

where $j(r)$ is an l.s.c. convex function from $\mathbb{R}$ to $[0,+\infty]$ with $j(0)=0$.

By abuse of notation, we will write y for $Y(\xi,\zeta,t)$ when there is no confusion possible.

Lemma 4.2. *Let φ be as in (4.17). Then φ satisfies Assumption 2.1.*

Proof. (1) It is clear that $\varphi(y;u)$ is an l.s.c. function from $H\times U$ into $\mathbb{R}^+$ and that $C_0^\infty(Q)\subset D(\varphi(\cdot,u))$ for all $u\in U$.

(2) It was shown in [4, pages 949–952] that

$$\int_Q F(u)|\nabla y|^2 d\xi\, d\zeta \ge c\|y\|^2_{H^1(Q)}, \tag{4.18}$$

for all y with $F(u)\nabla y\in H$, $y=0$ on $\partial Q/\Gamma$.

Since $j(r)$ and μ are both positive functions, we get

$$c\|y\|^2_{H^1(Q)}\le\varphi(y;u),\quad \forall y\in D(\varphi). \tag{4.19}$$

(3) By the Sobolev imbedding theorem, the set

$$\{y:\varphi(y;u)\le C\} \tag{4.20}$$

is a compact subset of $H=L^2(Q)$.

(4) Suppose that $u_1^n\to u_1$ in H with $u_1^n\in U_1$. Since u_1^n is in U_1, it follows from the definition of U_1 and from the Sobolev imbedding theorem that there exists a subsequence such that $u_1^n\to u_1\in H^2(G)$ and in $C^1(\bar{G})$.

With $F(u)$, $\mu(u)$ as above, it is trivial to check that we have

$$\lim_{n\to\infty}\int_0^T\varphi\big(y(s);u_1^n\big)\,ds=\int_0^T\lim_{n\to\infty}\varphi\big(y(s);u_1^n\big)\,ds. \tag{4.21}$$

□

Lemma 4.3. *Let φ be as in (4.16). Then $\partial\varphi(y;u)=-\nabla\cdot(F(u)\nabla y)=A(y;u)$ with*

$$D\big(A(y;u)\big)=\Big\{y:\nabla\cdot\big(F(u)\nabla y\big)\in H,\ y=0 \text{ on } \partial Q/\Gamma,\quad -\frac{\partial y}{\partial n}\in\mu(u_1)u_2\beta(y) \text{ on } \Gamma\Big\}. \tag{4.22}$$

Proof. For $y\in H^1(Q)$ with $\nabla\cdot F(u)\nabla y$ in $L^2(Q)$, we know that $F(u)\nabla y\cdot n\in H^{-1/2,2}(\partial Q)$.

Let $A(y;u)=-\nabla\cdot F(u)\nabla y$ with

$$D\big(A(y;u)\big)=\Big\{y:y\in H,\ \nabla\cdot\big(F(u)\nabla y\big)\in H,\ y=0 \text{ on } \partial Q/\Gamma,\quad -\frac{\partial y}{\partial n}y\in\mu(u_1)u_2 y \text{ on } \Gamma\Big\}. \tag{4.23}$$

We now show that A is maximal monotone on H and that $A\subset\partial\varphi(y;u)$.

(1) It is clear that $A(\cdot;u)$ is monotone in H. For $y \in D(A(\cdot;u))$ and $x \in D(\varphi(\cdot;u))$, we have

$$-(\nabla\cdot F(u)\nabla y, x-y) = (F(u)\nabla y, \nabla x - y) - \left\langle \frac{\partial y}{\partial n}, x-y\right\rangle, \tag{4.24}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between $H^{-1/2,2}(\Gamma)$ and its dual.

It follows that

$$-(\nabla\cdot F(u)\nabla y, x-y) \le \varphi(x;u) - \varphi(y;u). \tag{4.25}$$

Hence $A(y;u) \in \partial\varphi(y;u)$.

(2) To show that $A(y;u)$ is maximal monotone, it suffices to show that $I + A(\cdot;u)$ is onto.

Since $\beta(y) \in \partial j(y)$ is maximal monotone, its resolvent operator $(I+\lambda\beta)^{-1}$ is nonexpansive for all $\lambda > 0$.

Consider the elliptic boundary value problem

$$\begin{aligned} -\nabla\cdot(F(u)\nabla y_\lambda) = f \quad \text{on } Q, \qquad y_\lambda = 0 \quad \text{on } \partial Q/\Gamma, \\ \mu(u_1)u_2 y_\lambda + \lambda\frac{\partial}{\partial n}y_\lambda = \mu(u_1)u_2(I+\lambda\beta)^{-1}x \qquad \text{on } \Gamma. \end{aligned} \tag{4.26}$$

For $(f,x) \in L^2(Q)\times L^2(\Gamma)$, there exists a unique solution y_λ of (4.17) with $y_\lambda \in H^1(Q)$. Let L be the mapping of $L^2(\Gamma)$ into itself given by

$$L\left(\sqrt{\mu(u_1)u_2}\,x\right) = \sqrt{\mu(u_1)u_2}\,y_\lambda|_\Gamma. \tag{4.27}$$

(3) We now show that L is a contraction. Let L be as above, then

$$\int_Q F(u)|\nabla(y_\lambda^1 - y_\lambda^2)|^2 - \left\langle \frac{\partial}{\partial n}(y_\lambda^1 - y_\lambda^2), y_\lambda^1 - y_\lambda^2\right\rangle = 0. \tag{4.28}$$

As shown in [4, pages 949 and 952] we have

$$c\|y_\lambda^1 - y_\lambda^2\|_{H^1(Q)}^2 - \left\langle \frac{\partial}{\partial n}(y_\lambda^1 - y_\lambda^2), y_\lambda^1 - y_\lambda^2\right\rangle \le 0. \tag{4.29}$$

Thus,

$$\begin{aligned} c\|y_\lambda^1 - y_\lambda^2\|_{H^1(Q)}^2 &+ \lambda^{-1}\left\|\sqrt{\mu(u_1)u_2}(y_\lambda^1 - y_\lambda^2)\right\|_{L^2(\Gamma)}^2 \\ &\le \lambda^{-1}\left(\mu(u_1)u_2\{(I+\lambda\beta)^{-1}x^1 - (I+\lambda\beta)^{-1}x^2\}, y_\lambda^1 - y_\lambda^2\right) \\ &\le \left\|\sqrt{\mu(u_1)u_2}\lambda^{-1}(y_\lambda^1 - y_\lambda^2)\right\|_{L^2(\Gamma)}\left\|\sqrt{\mu(u_1)u_2}(x^1 - x^2)\right\|_{L^2(\Gamma)}. \end{aligned} \tag{4.30}$$

We have used the nonexpansive property of $(I+\lambda\beta)^{-1}$ in the above estimate. We know that

$$a\|y_\lambda^1 - y_\lambda^2\|_{L^2(\Gamma)}^2 \le \|y_\lambda^1 - y_\lambda^2\|_{H^1(Q)}^2, \tag{4.31}$$

where a is a positive constant.

Thus,

$$\begin{aligned} &\lambda ac\|y_\lambda^1-y_\lambda^2\|_{L^2(\Gamma)}^2+\left\|\sqrt{\mu(u_1)u_2}(y_\lambda^1-y_\lambda^2)\right\|_{L^2(\Gamma)} \\ &\quad\le \left\|\sqrt{\mu(u_1)u_2}(y_\lambda^1-y_\lambda^2)\right\|_{L^2(\Gamma)}\left\|\sqrt{\mu(u_1)u_2}(x^1-x^2)\right\|_{L^2(\Gamma)}. \end{aligned} \tag{4.32}$$

It follows that

$$\left\|\sqrt{\mu(u_1)u_2}(y_\lambda^1-y_\lambda^2)\right\|_{L^2(\Gamma)}\le \gamma\left\|\sqrt{\mu(u_1)u_2}(x^1-x^2)\right\|_{L^2(\Gamma)} \tag{4.33}$$

with

$$\gamma=\frac{\|\mu(u_1)u_2\|_{L^\infty(G)}}{\lambda ac+\|\mu(u_1)u_2\|_{L^\infty(G)}}<1. \tag{4.34}$$

Thus, L is a contraction mapping. There exists a unique y_λ such that

$$\begin{gathered} -\nabla\cdot(F(u_1)\nabla y_\lambda)=f \quad \text{on } Q, \\ y_\lambda=0 \quad \text{on } \partial Q/\Gamma, \\ \mu(u_1)u_2y_\lambda+\lambda\frac{\partial y_\lambda}{\partial n}=\mu(u_1)u_2(I+\lambda\beta)^{-1}y_\lambda \quad \text{on } \Gamma. \end{gathered} \tag{4.35}$$

(4) By a standard argument, we get from (4.35) the following estimate:

$$\|y_\lambda\|_{H^1(Q)}^2\le C\|f\|_{L^2(Q)}. \tag{4.36}$$

Let $\lambda\to 0^+$, and we get by taking subsequences that $y_\lambda\to y$ in $(H^1(Q))_{\text{weak}}\cap L^2(Q)$. It is clear that $y=0$ on $\partial Q/\Gamma$. On the other hand,

$$-\frac{\partial y_\lambda}{\partial n}=\mu(u_1)u_2\lambda^{-1}\{I-(I+\lambda\beta)^{-1}\}y_\lambda=\mu(u_1)u_2\beta_\lambda(y_\lambda), \tag{4.37}$$

where β_λ is the Yosida approximation of β.

Since

$$\beta_\lambda(y_\lambda)\in\beta((I+\lambda\beta)^{-1}y_\lambda), \quad (I+\lambda\beta)^{-1}y_\lambda\longrightarrow y \quad \text{in } L^2(\Gamma), \tag{4.38}$$

it follows from the maximal monotonicity of β that

$$-\frac{\partial}{\partial n}y\in\mu(u_1)u_2\beta(y). \tag{4.39}$$

The lemma is proved. □

Proof of Theorem 4.1. Consider the optimal control problem

$$\begin{aligned} &Y'-\nabla\cdot(F(u)\nabla Y)+g(Y;u)=0 && \text{on } Q\times(0,T), \\ &Y=0 && \text{on } (\partial Q/\Gamma)\times(0,T), \\ &-\frac{\partial}{\partial n}Y\in\mu(u_1)u_2\beta(Y) && \text{on } \Gamma\times(0,T), \\ &Y(\cdot,0)=y_0 && \text{on } Q \end{aligned} \tag{4.40}$$

with

$$g(Y;u) = -u_1^{-1}F(u_1)\nabla Y\cdot\nabla u_1 - \tilde{f}(Y,u) \tag{4.41}$$

and cost functionals

$$\begin{aligned} J_1(Y;u_1,u_2) &= \int_0^T\int_G |Y(\xi,2;t)|^2\,d\xi\,dt, \\ J_2(Y;u_1,u_2) &= \int_0^T\int_\Omega \left|Y\left(\xi,\frac{2\eta}{u_1},t\right)-h(\xi,\eta,t)\right|^2 d\xi\,d\eta\,dt. \end{aligned} \tag{4.42}$$

It is easy to check that g and J_1, J_2 satisfy Assumptions 2.2 and 2.3, respectively. It follows from Lemmas 4.2 and 4.3 and from Theorem 3.1 that there exists an open loop control $\tilde{u}$ of (4.36) and (4.40), that is, we have

$$\begin{gathered} \tilde{Y}\in L^2(0,T;H^1(Q))\cap L^\infty(0,T;L^2(Q)), \\ \{\tilde{Y}',A(\tilde{Y};\tilde{u})\}\in (L^2(0,T;L^2(Q)))^2, \end{gathered} \tag{4.43}$$

solution of (4.36) with controls $\tilde{u}$. Moreover,

$$\begin{aligned} J_1(\tilde{Y};\tilde{u}_1,\tilde{u}_2) &\le J_1(y;\tilde{u}_1,v_2), \\ J_2(\tilde{Y};\tilde{u}_1,\tilde{u}_2) &\le J_2(x;u_1,\tilde{u}_2), \end{aligned} \tag{4.44}$$

for all $y\in\mathcal{R}(\tilde{u}_1,v_2)$, for all $v_2\in U_2$, all $x\in\mathcal{R}(u_1,\tilde{u}_2)$, and all $u_1\in U_1$.

Now set

$$\hat{y}(\xi,\eta) = \tilde{Y}(\xi,\zeta) = \tilde{Y}\left(\xi,\frac{2\eta}{\tilde{u}_1}\right) \tag{4.45}$$

and we get the stated result. □

4.2. Parabolic variational inequalities. Consider the initial boundary value problem

$$\begin{aligned} &y'-\Delta y = \tilde{f}(y) &&\text{on } Q(u_1)\times(0,T), \\ &y=0 &&\text{on } (\partial Q/\Gamma)\times(0,T), \\ &y(\cdot,t)\ge u_2(\xi) &&\text{on } \Gamma\times(0,T), \\ &y(\cdot,0)=y_0 &&\text{on } Q \end{aligned} \tag{4.46}$$

with cost functionals

$$\begin{aligned} J_1(y;u_1,u_2) &= \int_0^T\int_G |y(\xi,u_1(\xi);t)|^2\,d\xi, \\ J_2(y;u_1,u_2) &= \int_0^T\int_\Omega |y(\xi,\eta;t)-h(\xi,\eta)|^2\,d\xi\,d\eta\,dt, \end{aligned} \tag{4.47}$$

where h is the partial measurement of the solution y of (4.46) in the subdomain $\Omega\times(0,T)$, U_1 is as before and

$$U_2=\{v:\|v\|_{H^3(G)}\leq C,\ 0\leq v \text{ on } G\}. \tag{4.48}$$

The main result of this subsection is the following theorem.

Theorem 4.4. *Let y_0 be an element of $H^1(Q)$ with*

$$y_0=0 \quad \text{on } \frac{\partial Q}{\Gamma}, \quad y_0\geq v\geq 0 \quad \text{on } \Gamma,\ \forall v\in U_2. \tag{4.49}$$

Let $h\in L^2(0,T;L^2(\Omega))$ where Ω is a proper subset of $Q(u_1)$ for all $u_1\in U_1$ and let $\tilde f$ be as in Assumption 2.2. Then there exists

$$\begin{aligned}\{\hat y,\hat y',\hat u\}\in L^2(0,T;H^1(Q(\hat u_1)))\cap L^\infty(0,T;L^2(Q(\hat u_1)))\\ \times L^2(0,T;L^2(Q(\hat u_1)))\times U\end{aligned} \tag{4.50}$$

with

$$\begin{aligned}J_1(\hat y;\hat u_1,\hat u_2)&\leq J_1(y;\hat u_1;v_2),\\ J_2(\hat y;\hat u_1,\hat u_2)&\leq J_2(x;u_1,\hat u_2),\end{aligned} \tag{4.51}$$

for all solutions y of (4.46) with controls $\hat u_1$, v_2 all solutions x of (4.42) with controls u_1, $\hat u_2$ and all $\{u_1,v_2\}\in U_1\times U_2$.

As before, we make the change of variables $\zeta=2\eta/u_1$ and as in Section 4.1, we transform (4.42) into a problem in a fixed domain

$$\begin{aligned}Y'-\nabla\cdot F((u_1)\nabla Y)=\tilde f(Y,u)+u^{-1}F(u_1)\nabla Y\cdot\nabla u_1 \quad &\text{on } Q\times(0,T),\\ Y=0 \quad &\text{on } \partial Q/\Gamma\times(0,T),\\ Y\geq u_2 \quad &\text{a.e. on } \Gamma\times(0,T),\\ Y(\cdot,0)=y_0 \quad &\text{on } Q.\end{aligned} \tag{4.52}$$

The cost functionals become

$$\begin{aligned}J_1(Y;u_1,u_2)&=\int_0^T\int_G|Y(\xi,2;t)|^2\,d\xi\,dt,\\ J_2(Y;u_1,u_2)&=\int_0^T\int_\Omega\left|Y\left(\xi,\frac{2\eta}{u_1};t\right)-h(\xi,\eta;t)\right|^2 d\xi\,d\eta\,dt.\end{aligned} \tag{4.53}$$

Set

$$K(u_2)=\{y:y\in L^2(0,T;L^2(Q)),\ y\geq u_2 \text{ a.e. on } \Gamma\times(0,T)\}. \tag{4.54}$$

Then $K(u_2)$ is a closed convex subset of $L^2(0,T;H)$. Let

$$\varphi(y;u) = \frac{1}{2}\int_0^T \int_Q F(u)|\nabla y|^2 \, d\xi \, d\zeta \, dt + I_{K(u_2)}(y), \tag{4.55}$$

where $I_{K(u_2)}$ is the indicator function of the closed convex set $K(u_2)$ of $L^2(0,T;H)$ and

$$\begin{aligned} D(\varphi(y;u)) = \Big\{ y : y \in L^2(0,T;H^1(Q)),\ y = 0 \text{ on } (\partial Q/\Gamma)\times(0,T), \\ y \geq u_2 \text{ on } \Gamma \times (0,T) \Big\}. \end{aligned} \tag{4.56}$$

Lemma 4.5. *Let φ be as in (4.53). Then φ satisfies Assumption 2.1.*

Proof. As in the proof of Lemma 4.2, we have

$$\varphi(y;u) \geq c\|y\|^2_{H^1(Q)}, \quad \forall y \in D(\varphi(\cdot,u)). \tag{4.57}$$

It is clear that

$$\partial\varphi(y;u) = \nabla\big(F(u)\cdot\nabla y\big) + \partial I_{K(u_2)}(y). \tag{4.58}$$

All the other conditions of Assumption 2.1 can be verified without any difficulty. □

Lemma 4.6. *Suppose all the hypotheses of Theorem 4.4 are satisfied. Then there exists a solution $\tilde{Y}$ of*

$$\tilde{Y}' + \partial\varphi(\tilde{Y};\tilde{u}) \ni \tilde{f}(\tilde{Y},\tilde{u}) + \tilde{u}_1^{-1}F(\tilde{u}_1)\nabla\tilde{Y}\cdot\nabla\tilde{u}_1, \qquad \tilde{Y}(\cdot,0) = y_0, \tag{4.59}$$

$$\begin{aligned} \{\tilde{Y},\tilde{Y}',\partial\varphi(\tilde{Y};\tilde{u}),\cdot\tilde{u}\} \in \big(L^2(0,T;H^1(Q)) \cap L^\infty(0,T;L^2(Q))\big) \\ \times \big(L^2(0,T;L^2(Q))\big)^2 \times U. \end{aligned} \tag{4.60}$$

Moreover,

$$\begin{aligned} J_1(\tilde{Y};\tilde{u}_1,\tilde{u}_2) &\leq J_1(y;\tilde{u}_1,v_2), \\ J_2(\tilde{Y};\tilde{u}_1,\tilde{u}_2) &\leq J_2(x;u_1,\tilde{u}_2), \end{aligned} \tag{4.61}$$

for all solutions y, x of (4.55) with controls $\{\tilde{u}_1,v_2\}$, $\{u_1,\tilde{u}_2\}$, respectively, and for all $\{u_1,v_2\}$ in $U_1\times U_2$.

Proof. The proof is an immediate consequence of Theorem 3.1 and Lemma 4.5. □

Proof of Theorem 4.4. Let $\{\tilde{Y},\tilde{u}\}$ be as in Lemma 4.6 and set $\hat{y}(\xi,\eta;t) = \tilde{Y}(\xi,2\eta/\tilde{u}_1)$. Then $\hat{y}$, $\tilde{u}$ is a solution of (4.52) and (4.53). The theorem is proved. □

References

[1] V. Barbu and A. Friedman, *Optimal design of domains with free-boundary problems*, SIAM J. Control Optim. **29** (1991), no. 3, 623–637.

[2] G. Canadas, F. Chapel, M. Cuer, and J. Zolésio, *Shape interfaces in an inverse problem related to the wave equation*, Inverse Problems: An Interdisciplinary Study (Montpellier, 1986), Adv. Electron. Electron Phys., Suppl., vol. 19, Academic Press, London, 1987, pp. 533–551.

[3] M. D. Gunzburger and H. Kim, *Existence of an optimal solution of a shape control problem for the stationary Navier-Stokes equations*, SIAM J. Control Optim. **36** (1998), no. 3, 895–909.

[4] S. Lenhart, V. Protopopescu, and J. Yong, *Identification of boundary shape and reflectivity in a wave equation by optimal control techniques*, Differential Integral Equations **13** (2000), no. 7-9, 941–972.

[5] S. Lenhart and D. G. Wilson, *Optimal control of a heat transfer problem with convective boundary condition*, J. Optim. Theory Appl. **79** (1993), no. 3, 581–597.

[6] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer Series in Computational Physics, Springer-Verlag, New York, 1984.

[7] B. A. Ton, *An open loop equilibrium strategy for quasi-variational inequalities and for constrained non-cooperative games*, Numer. Funct. Anal. Optim. **17** (1996), no. 9-10, 1053–1091.

[8] Y. Yamada, *On evolution equations generated by subdifferential operators*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 3, 491–515.

Bui An Ton: Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2

E-mail address: bui@math.ubc.ca