A BIFURCATION RESULT FOR EQUATIONS WITH ANISOTROPIC p-LAPLACE-LIKE OPERATORS

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Received 5 July 2001

The paper is concerned with a global bifurcation result for equations whose principal parts are anisotropic p-Laplace-like operators. Using an abstract bifurcation result for equations with multivalued operators in Banach spaces together with topological degree calculations, we show the Rabinowitz alternative for the bifurcating branches.

1. Introduction

This paper is about a global bifurcation result for nonlinear equations whose principal operators are anisotropic p-Laplace-like operators (sometimes called ϕ-Laplacians). The equation is, in the weak form,

$$u \in X : \int_\Omega \sum_{i=1}^{N} \phi_i\big(|\nabla u|\big)\partial_i u \partial_i v\, dx = \int_\Omega F(x,u,\lambda) v\, dx, \quad \forall v \in X, \tag{1.1}$$

where X is an appropriate function space and ϕ_i, F are given functions. The coefficients ϕ_i's are different in general. In the particular case where $\phi_i(|\xi|) = |\xi|^{p-2}$, for all $\xi \in \mathbb{R}^N$, for all $i \in \{1,\dots,N\}$, (1.1) reduces to the p-Laplacian equation and there are several bifurcation results available (cf. [3, 4]). It seems that bifurcation problems for anisotropic elliptic operators have not been addressed in detail.

As is well known in bifurcation theory, a first step is to find a "linearization" of (1.1) such that bifurcation in (1.1) can be studied through the eigenvalues and eigenfunctions of the linearization. Different from equations with compact perturbations of linear operators, we show that (1.1) can be related to a nonlinear but homogeneous equation (called the homogenization of (1.1)). Another difficulty is that since the functions ϕ_i's may have different growths at

Abstract and Applied Analysis 6:4 (2001) 229–251
2000 Mathematics Subject Classification: 58E35, 58J55, 35J60
URL: http://aaa.hindawi.com/volume-6/S1085337501000574.html

zero and infinity, the homogenization of (1.1) is defined properly only on a function space different from X. This makes the usual continuation processes (cf. [6, 12, 15]) to calculate the involved topological degrees and to prove global bifurcation in this case more difficult. Moreover, since the ϕ_i's are different in general, (1.1) does not have a variational structure and the variational approach used recently in [8, 9] (by converting the equation to a variational inequality) is not applicable in our problem here.

To study the bifurcation of (1.1), we employ a resolvent operator to convert (1.1) to a fixed point inclusion. Because the principal operator is not strictly monotone, the topological invariances are calculated in terms of the Leray-Schauder degree for multivalued compact vector fields. We show that under some conditions, a nonlinear equation

$$A(u) = B(u, \lambda) \quad (u \in X,\ \lambda \in \mathbb{R}) \tag{1.2}$$

in a Banach space X can be associated with a simpler, homogeneous equation (in a different space) so that the bifurcation properties of (1.2) are reflected in eigenvalues and topological degrees in that "homogenized" equation (in fact, an inclusion). The maps A and B are in general not compact perturbations of linear mappings. Also, by working on a space larger than X, we are able to bypass the assumption that A is of class (S) or $(S)^+$, which is a useful but sometimes hard to verify assumption on nonlinear elliptic operators (cf. [2, 14]). Our approach is based on the classical ideas of using resolvent (Green's) operators and topological degrees. However, due to the nature of our problem (nonvariational structure, set-valued resolvent operator, principal operator not being in class (S) or $(S)^+$, bifurcation and degree computation being in different function spaces, etc.), new arguments and techniques are needed in the proof of the abstract bifurcation theorem in Section 2. The verification of the general assumptions in the specific case of (1.1) also requires nontrivial estimates and calculations.

The paper is organized as follows. In Section 2, we prove an abstract bifurcation theorem for (1.2). We define the homogenized equation and inclusion of (1.2) and prove (in Theorem 2.9) that if there is a change of topological degrees in the homogenized inclusion, then there is a global bifurcation in (1.2) in the sense of the Rabinowitz alternative. This result extends the well-known Rabinowitz global bifurcation theorem to equations whose operators are not necessarily compact perturbations of linear mappings, and the principal operators may not belong to class (S) or $(S)^+$. In Section 3, we use the general result of Section 2 to show global bifurcation in (1.1). In this particular case, the homogenized inclusion of (1.1) is proved to be the usual p-Laplacian equation. Using degree computations at the principal eigenvalue of the p-Laplacian in [3], we prove that there is bifurcation in (1.1) at that eigenvalue, and the corresponding bifurcation branch satisfies the Rabinowitz alternative.

2. A global bifurcation result

We are concerned in this section with the bifurcation of certain functional equations without variational structure, that is, the principal operators are not necessarily gradients of some potential functionals. Consider the following equation:

$$A(u) = B(u, \lambda) \quad (u \in X,\ \lambda \in \mathbb{R}) \tag{2.1}$$

in a real, reflexive Banach space X. The map A is the principal operator and B is the lower order term, depending on a real parameter λ. The set X^* is the dual space of X. We use $\|\cdot\|$ (or $\|\cdot\|_X$) and $\langle\cdot,\cdot\rangle$ (or $\langle\cdot,\cdot\rangle_{X,X^*}$) to denote the norm of X and the dual pairing between X and X^*. Assume that X is compactly embedded in another Banach space Z (with norm $\|\cdot\|_Z$ and dual pairing $\langle\cdot,\cdot\rangle_{Z,Z^*}$), that is, $X \subset Z$ and the embedding $i : X \hookrightarrow Z$ is compact.

Suppose that $A : X \to X^*$ is monotone, hemicontinuous, bounded, and $A(0) = 0$. Also, A is coercive in the following sense:

$$\lim_{\|x\|\to\infty} \frac{\langle A(x), x\rangle}{\|x\|} = \infty. \tag{2.2}$$

The map B is a bounded, continuous mapping from $Z \times \mathbb{R}$ to Z^* such that $B(0,\lambda) = 0$, for all λ (further assumptions on A and B will be specified later). Hence, 0 is always a solution of (2.1). For convenience, we still use B to denote the restriction of B on $X \times \mathbb{R}$. For each f in X^*, the above assumptions on A imply that the equation

$$A(x) = f \tag{2.3}$$

has at least one solution in X (cf. [10]). We denote by $P(f) = P_A(f)$ the set of all solutions of (2.3). Hence, $P_A(f) \neq \emptyset$. Equation (2.1) is equivalent to the fixed point inclusion

$$u \in P_A\big[B(u,\lambda)\big], \quad u \in X. \tag{2.4}$$

Because of the embedding $i : X \hookrightarrow Z$, (2.4) is also equivalent to the following inclusion in Z:

$$u \in i\,P_A\big[B(u,\lambda)\big], \quad u \in Z. \tag{2.5}$$

In fact, if u satisfies (2.4), then it clearly satisfies (2.5). Conversely, if u is a solution of (2.5), then $u \in X$ (since $Z^* \subset X^*$ and $P_A(X^*) \subset X$). As $i(x) = x$ for $x \in X$, we have (2.4).Therefore, instead of studying (2.1) or (2.4), we consider the bifurcation of the equivalent inclusion (2.5), that is,

$$0 \in u - i\,P_A\big[B(u,\lambda)\big], \quad u \in Z. \tag{2.6}$$

We study the bifurcation of this equation in Z. First, we extend in a natural way the definition of bifurcation points and the basic global bifurcation result from equations to inclusions. Assume that Z is a Banach space and F mapping from $Z\times\mathbb{R}$ to $\mathcal{K}(Z)$, where

$$\mathcal{K}(Z)=\{U\subset Z : U\neq\emptyset,\ U \text{ is closed and convex in } Z\}. \tag{2.7}$$

Consider the inclusion

$$0\in u-F(u,\lambda),\quad u\in Z. \tag{2.8}$$

Assume that $0\in F(0,\lambda)$, for all $\lambda\in\mathbb{R}$. Then, for all λ, 0 is a trivial solution of (2.8). We define the bifurcation points as usual.

Definition 2.1. The point $(0,\lambda_0)$ (or simply λ_0) is called a bifurcation point of (2.8) (in Z) if there exists a sequence $\{(u_n,\lambda_n)\}$ of solutions of (2.8) such that $u_n\neq 0$, for all n, and $u_n\to 0$ in Z, $\lambda_n\to\lambda_0$ in $\mathbb{R}$, as $n\to\infty$. Now, we assume that for each λ, $F(\cdot,\lambda)$ is an upper semicontinuous, compact mapping. In particular, for any bounded subset M of Z, the set

$$F(M,\lambda):=\bigcup_{x\in M}F(x,\lambda) \tag{2.9}$$

is relatively compact in Z. Under these assumptions, the topological degree (for multivalued compact fields) $\mathrm{d}(I-F(\cdot,\lambda),U,a)$ is well defined for all open, bounded subsets U of Z, provided

$$a\notin F(\partial U,\lambda)\left(=\bigcup_{x\in\partial U}F(x,\lambda)\right). \tag{2.10}$$

We refer to [7] or [11] for more detailed discussions and basic properties of topological degrees for multivalued compact fields. Assume λ is not a bifurcation point of (2.8). By definition, there exists an open neighborhood U of 0 such that $0\notin u-F(u,\lambda)$ for all $u\in U\setminus\{0\}$. Thus, the degree $\mathrm{d}(I-F(\cdot,\lambda),B_r(0),0)$ is defined for all $r>0$ sufficiently small. We have the following global bifurcation theorem for (2.8).

Theorem 2.2. *Let $a,b\in\mathbb{R}$ $(a<b)$. Assume $(0,a)$ and $(0,b)$ are not bifurcation points of (2.8) and that*

$$\mathrm{d}_Z\left(I-F(\cdot,a),B_r^Z(0),0\right)\neq\mathrm{d}_Z\left(I-F(\cdot,b),B_r^Z(0),0\right) \tag{2.11}$$

for some $r>0$ sufficiently small ($B_r^Z(0)$ is the open ball in Z with center at 0, radius r). Let

$$\mathcal{S}=\overline{\left[\{(u,\lambda):(u,\lambda)\text{ is a solution of (2.8) with } u\neq 0\}\cup\left(\{0\}\times[a,b]\right)\right]}^{Z} \tag{2.12}$$

and $\mathscr{C}$ be the connected component of $\mathcal{S}$ that contains $\{0\}\times[a,b]$. Then, either

(i) *$\mathscr{C}$ is unbounded in $Z\times\mathbb{R}$, or*

(ii) *$\mathscr{C}\cap(\{0\}\times(\mathbb{R}\setminus[a,b]))\neq\emptyset$.*

(We use the superscript Z in (2.12) to emphasize that the operation below it is taken in Z.)

As usual, the proof of this theorem is based on a separation property in metric spaces (Whyburn's lemma) together with excision and homotopy invariance properties of the topological degree. Since the topological degree of compact, multivalued vector fields has these properties (cf. [7] or [11]), the proof of Theorem 2.2 is a straightforward adaptation and generalization of the corresponding result for single-valued compact vector fields, as presented, for example, in [13].

Now, we use Theorem 2.2 to the particular case of inclusion (2.5). First, we have the following lemma.

Lemma 2.3. *For each $f \in X^*$, $P_A(f)$ is a nonempty, convex subset of X.*

Proof. The proof of this lemma is routine. As noted previously, $P_A(f) \neq \emptyset$. Assume that $u, w \in P_A(f)$, that is,

$$A(u) = A(w) = f \quad \text{in } X^*, \tag{2.13}$$

and that $x = tu + (1-t)w$ for some $t \in [0,1]$. Since A is monotone, one can apply Minty's lemma (cf. [5]) to get

$$\big\langle A(v), v-u \big\rangle \geq \langle f, v-u \rangle, \qquad \big\langle A(v), v-w \big\rangle \geq \langle f, v-w \rangle, \tag{2.14}$$

for all $v \in X$. Multiplying the first inequality by t and the second by $1-t$, and adding the inequalities thus obtained, we get

$$\big\langle A(v), v-x \big\rangle \geq \langle f, v-x \rangle, \quad \forall x \in X. \tag{2.15}$$

Apply again Minty's lemma, we have

$$\big\langle A(x), v-x \big\rangle \geq \langle f, v-x \rangle, \quad \forall x \in X, \tag{2.16}$$

which implies that $A(x) = f$ in X^*. This shows that $x \in P_A(f)$. Thus, $P_A(f)$ is convex. □

Lemma 2.4. *The mapping P_A is bounded from X^* to 2^X, that is, if U is a bounded set in X^*, then $P_A(U) = \cup_{f \in U} P_A(f)$ is a bounded set in X.*

Proof. Let U be a bounded subset of X^*. Assume $P_A(U)$ is not bounded and that there exist sequences $\{u_n\}$ and $\{f_n\}$ such that $\{f_n\} \subset U$, $u_n \in P_A(f_n)$, for all n, and $\|u_n\|_X \to \infty$ as $n \to \infty$. By the definition of P_A, $A(u_n) = f_n$, for all n. Thus,

$$\left| \frac{\langle A(u_n), u_n \rangle}{\|u_n\|} \right| \leq \|f_n\|_* \leq \sup_{f \in U} \|f\|_* (< \infty), \quad \forall n. \tag{2.17}$$

This, however, contradicts the coercivity of A (cf. (2.2)). Hence, $P_A(U)$ must be bounded in X. □

LEMMA 2.5. *For each $f \in X^*$, $P_A(f)$ is closed in Z.*

Proof. Let $\{u_n\}$ be a sequence in $P_A(f)$ such that $u_n \to u$ in Z. From Lemma 2.4, the sequence $\{\|u_n\|\}$ is bounded in X. Hence, by passing to a subsequence, we can assume that there exists $u^* \in X$ such that $u_n \rightharpoonup u^*$ in X, ("$\rightharpoonup$" denotes the weak convergence). Therefore, $u_n \to u^*$ in Z(-strong). It follows that $u = u^* \in X$ and $u_n \rightharpoonup u$ in X. From Minty's lemma (cf. [5]), we have from

$$0 = \big\langle A\big(u_n\big) - f, v - u_n\big\rangle, \quad \forall v \in X \tag{2.18}$$

that

$$\big\langle A(v) - f, v - u_n\big\rangle \geq 0, \quad \forall v \in X. \tag{2.19}$$

Letting $n \to \infty$ in this inequality, we have $\langle A(v) - f, v - u\rangle \geq 0$, for all $v \in X$. Using Minty's lemma once more, we have

$$\big\langle A(u) - f, v - u\big\rangle \geq 0, \quad \forall v \in X. \tag{2.20}$$

This, of course, implies that $A(u) = f$, that is, $u \in P_A(f)$. □

Lemmas 2.3, 2.4, and 2.5 prove, in particular, that $iP_A[B(u,\lambda)] \in \mathcal{K}(Z)$ for every $u \in Z$, $\lambda \in \mathbb{R}$. Another lemma is needed for our definition of the degree of $I - iP_A[B(\cdot,\lambda)]$ in Z (I is the identity mapping on Z).

LEMMA 2.6. (a) *The mapping from Z to $\mathcal{K}(Z)$ defined by $u \mapsto iP_A[B(u,\lambda)]$ is compact, that is, for every bounded subset U of Z, the set $\cup_{u\in U} P_A[B(u,\lambda)]$ is relatively compact in Z.*

(b) *The mapping in (a) is upper semicontinuous from Z to $\mathcal{K}(Z)$, that is, for each $u \in Z$, if V is an open set in Z such that $iP_A[B(u,\lambda)] \subset V$, then there exists an open neighborhood U of u in Z such that*

$$iP_A\big[B(U,\lambda)\big] \equiv \bigcup_{x\in U} iP_A\big[B(x,\lambda)\big] \subset V. \tag{2.21}$$

Proof. (a) Since B is bounded, the set $\{B(u,\lambda) : u \in U\}$ is bounded in Z^* and thus in X^*. From Lemma 2.4, the set $\cup_{u\in U} P_A[B(u,\lambda)]$ is bounded in X. Now, the set

$$\bigcup_{u\in U} iP_A\big[B(u,\lambda)\big] = i\left\{\bigcup_{u\in U} P_A\big[B(u,\lambda)\big]\right\} \tag{2.22}$$

is relatively compact in Z by the compactness of the embedding i.

(b) We use Lemma 2.4 and the arguments in Lemma 2.5. Assume by contradiction that for every open neighborhood U of u, one always has

$$iP_A\big[B(U,\lambda)\big] \not\subset V. \tag{2.23}$$

It follows that there exist sequences $\{y_n\} \subset Z$ and $\{u_n\} \subset U$ such that

$$y_n \in P_A\big[B\big(u_n,\lambda\big)\big], \tag{2.24}$$

$$y_n \notin V, \quad \forall n, \tag{2.25}$$

$$u_n \longrightarrow u \quad \text{in } Z \text{ as } n \longrightarrow \infty. \tag{2.26}$$

It follows that $B(u_n,\lambda) \to B(u,\lambda)$ in Z^*. The set $\{B(u_n,\lambda) : n \in \mathbb{N}\}$ is thus bounded in Z^* and in X^*. From Lemma 2.4 and (2.24), the set $\{y_n : n \in \mathbb{N}\}$ is bounded in X. By passing to a subsequence if necessary, we can assume that

$$y_n \rightharpoonup y \quad \text{in } X. \tag{2.27}$$

Also, (2.24) means that

$$A\big(y_n\big) = B\big(u_n,\lambda\big), \quad \forall n. \tag{2.28}$$

Using the arguments in the proof of Lemma 2.5, one has from (2.26), (2.27), and (2.28) that

$$A(y) = B(u,\lambda), \tag{2.29}$$

that is, $y \in iP_A[B(u,\lambda)]$. On the other hand, it follows from (2.25) that $y \notin V$. Hence, $iP_A[B(u,\lambda)] \not\subset V$. This contradiction proves (b). □

We note from the monotonicity of A that $\langle A(u),u\rangle \geq 0$, for all $u \in X$. If A has the following nondegeneracy property:

$$\big[u_n \longrightarrow 0 \text{ in } Z,\ \big\langle A\big(u_n\big),u_n\big\rangle \longrightarrow 0\big] \Longrightarrow u_n \longrightarrow 0 \quad \text{in } X, \tag{2.30}$$

then λ_0 is a bifurcation point of inclusion (2.8) in Z (cf. Definition 2.1) if and only if λ_0 is a bifurcation point of the corresponding inclusion in X.

In fact, assume that $\{(u_n,\lambda_n)\}$ is a sequence of solutions of (2.8) such that $u_n \to 0$ in Z, $u_n \neq 0$, and $\lambda_n \to \lambda_0$. We have $B(u_n,\lambda_n) \to 0$ in Z^*. Thus,

$$\big\langle B\big(u_n,\lambda_n\big),u_n\big\rangle = \big\langle B\big(u_n,\lambda_n\big),u_n\big\rangle_{Z,Z^*} \longrightarrow 0. \tag{2.31}$$

However,

$$\big\langle A\big(u_n\big),u_n\big\rangle = \big\langle B\big(u_n,\lambda_n\big),u_n\big\rangle. \tag{2.32}$$

Therefore, $\{u_n\}$ satisfies the conditions in (2.30). By the implication in (2.30), we must have $u_n \to 0$ in X, that is, λ_0 is a bifurcation point in X. The converse is trivial.

Now, we consider a homogenization procedure for (2.8). We fix a number $p > 1$ and define, for $\sigma > 0$:

$$A_\sigma(u) = \frac{1}{\sigma^{p-1}} A(\sigma u), \tag{2.33}$$
$$B_\sigma(u, \lambda) = \frac{1}{\sigma^{p-1}} B(\sigma u, \lambda). \tag{2.34}$$

For each $\sigma > 0$, A_σ is a monotone, coercive, bounded mapping from X to X^* and B_σ is a continuous, bounded mapping from $Z \times \mathbb{R}$ to Z^*. It is clear that $A = A_1$ and $B = B_1$. We assume furthermore that X_0 is a reflexive Banach space, with norm $\|\cdot\|_{X_0} = \|\cdot\|_0$, that is continuously embedded in X. We have the continuous embeddings $X_0 \hookrightarrow X \hookrightarrow Z$, and thus $Z^* \hookrightarrow X^* \hookrightarrow X_0^*$. Suppose that there exist two mappings $A_0 : X_0 \to X_0^*$ and $B_0 : Z \times \mathbb{R} \to Z^*$ such that A_0 is monotone, bounded, hemicontinuous, coercive in X_0 (in the sense of (2.2)), $A_0(0) = 0$, B_0 is bounded, continuous, and $B_0(0, \lambda) = 0$ for all λ. We assume that A_σ converges to A_0 and B_σ converges to B_0 as $\sigma \to 0$ in the following sense.

(H1) If $\sigma_n \to 0^+$, $v_n \rightharpoonup v$ in X(-weak), and the sequence $\{A_{\sigma_n}(v_n)\}$ is (strongly) convergent in X^*, then $v \in X_0$, $A_0(v) \in X^*$, and

$$A_0(v) = \lim_{n\to\infty} A_{\sigma_n}(v_n) \quad \text{in } X^*. \tag{2.35}$$

(H2a) If $\sigma_n \to 0^+$, $v_n \to v$ in Z, and $\lambda_n \to \lambda$, then

$$B_{\sigma_n}(v_n, \lambda_n) \longrightarrow B_0(v, \lambda) \quad \text{in } Z^*. \tag{2.36}$$

We also need the following assumption on the boundedness of B_σ for σ small:

(H2b) If $\{v_n\}$ is a bounded sequence in Z, $\lambda_n \to \lambda$ and $\sigma_n \to 0^+$, then the sequence $\{B_{\sigma_n}(u_n, \lambda_n)\}$ is bounded in Z^*.

The family $\{A_\sigma : \sigma > 0\}$ has the following nondegeneracy condition near 0:

(H3) If $\sigma_n \to 0^+$, $v_n \rightharpoonup 0$ in X, and

$$A_{\sigma_n}(v_n) = f_n \longrightarrow 0 \quad \text{in } X^*, \tag{2.37}$$

then

$$v_n \longrightarrow 0 \quad \text{in } X. \tag{2.38}$$

For $\sigma > 0$, we denote by P_{A_σ} the solution mapping of the operator A_σ, that is,

$$u = P_{A_\sigma}(f) \Longleftrightarrow A_\sigma(u) = f. \tag{2.39}$$

By the same proof as in Lemma 2.6, we see that the mapping

$$x \longmapsto i P_{A_\sigma}\big[B_\sigma(x, \lambda)\big] \tag{2.40}$$

is compact and upper semicontinuous from Z to $\mathcal{K}(Z)$. From the assumptions on A_0 and B_0, we see that the mapping in (2.40) also has these properties when $\sigma = 0$. Therefore, for every $\sigma \in [0,1]$, the degree $\mathrm{d}(I - iP_{A_\sigma}[B_\sigma(\cdot,\lambda), U, a)$ is defined provided U is a bounded open subset of Z and $a \notin \cup_{u\in\partial U}(u - P_{A_\sigma}[B_\sigma(u,\lambda)])$. Note that condition (H1) can also be stated equivalently as

(H1′) Assume $\sigma_n \to 0$ and $\{v_n\} \subset X$, $\{f_n\} \subset X^*$ are sequences such that

$$A_{\sigma_n}(v_n) = f_n, \quad f_n \longrightarrow f \quad \text{in } X^*, \qquad v_n \rightharpoonup v \quad \text{in } X, \tag{2.41}$$

then $v \in X_0$ and

$$A_0(v) = f \quad \text{in } X_0^*. \tag{2.42}$$

Concerning (H3), we note that the following condition is more restrictive, yet it is easier to verify

(H3′) If $v_n \rightharpoonup 0$ in X, $\sigma_n \to 0$, and $\langle A_{\sigma_n}(v_n), v_n\rangle \to 0$, then $v_n \to 0$ in X.

We also need the following equi-coercivity condition for the family $\{A_\sigma\}$ for small values of σ:

(H4) If $\sigma_n \to 0^+$ and $\|u_n\| \to \infty$, then

$$\frac{\langle A_{\sigma_n}(u_n), u_n\rangle}{\|u_n\|} \longrightarrow \infty. \tag{2.43}$$

Now, we associate to (2.1) the "homogenized" equation

$$A_0(u) = B_0(u,\lambda), \quad u \in X,\ \lambda \in \mathbb{R}, \tag{2.44}$$

which is equivalent to the fixed point inclusion

$$u \in P_{A_0}\big[B_0(u,\lambda)\big], \quad u \in X. \tag{2.45}$$

This inclusion is equivalent to the following inclusion in Z:

$$u \in iP_{A_0}\big[B_0(u,\lambda)\big], \quad u \in Z. \tag{2.46}$$

The operators A_0 and B_0 introduced above play, in some sense, the roles of the Gâteaux derivatives $A'(0)$ and $\partial_u B(0,\lambda)$ in the cases where A and B are not equivalent to linear operators at 0. In classical situations where A is a linear operator and B is a perturbation of a linear operator

$$B(u,\lambda) = \lambda Lu + g(u), \tag{2.47}$$

with $g(u) = o(\|u\|)$ as $u \to 0$, we have $A_0(u) = A(u)$ and $B_0(u,\lambda) = \lambda Lu$. However, in (1.1), A and B cannot be linearized at 0 and A_0, B_0 are nonlinear in general. Nevertheless, one can define generalized eigenvalues and eigenvectors for the homogeneous equation (2.44) and relate the bifurcation of (2.1) with topological degrees at eigenvalues of (2.44).

As usual, it is expected that A_σ converges to A_0 as $\sigma \to 0^+$ with an additional (and natural) condition:

(H5) For all $v \in X_0$,

$$A_\sigma(v) \longrightarrow A_0(v) \quad \text{in } X_0^* \tag{2.48}$$

when $\sigma \to 0^*$. This condition just means that A_σ converges pointwise to A_0 in X_0. Together with (H1) and (H3), (H5) shows that A_0 and B_0 are uniquely determined and are homogeneous of degree $(p-1)$ in u, that is,

$$\begin{aligned} A_0(\sigma u) &= \sigma^{p-1} A_0(u), \quad \forall u \in X_0,\ \forall \sigma \geq 0, \\ B_0(\sigma u, \lambda) &= \sigma^{p-1} B_0(u, \lambda), \quad \forall u \in Z,\ \forall \sigma \geq 0. \end{aligned} \tag{2.49}$$

However, (H5) is not needed in the proof of our main results later. From the homegeneity of A_0 and B_0, it immediately follows that if (u, λ) is a solution of (2.44), then so is $(\sigma u, \lambda)$ with every $\sigma \geq 0$. This suggests the following definition.

Definition 2.7. In (2.44), λ is called an eigenvalue if (2.44) has a solution (u, λ) with $\lambda \neq 0$.

The following lemma gives a property of numbers which are not eigenvalues of (2.44). This is crucial to the proof of our main bifurcation result.

Lemma 2.8. *Under the assumptions (H1), (H2), (H3), and (H4), if a is not an eigenvalue of (2.44), then there exists an open neighborhood U of 0 in Z such that*

$$0 \notin \left\{u - i P_{A_\sigma}\left[B_\sigma(u, a)\right] : u \in \bar{U}^Z \setminus \{0\},\ \sigma \in [0,1]\right\}. \tag{2.50}$$

Proof. First, we show that for $r > 0$ sufficiently small,

$$0 \notin \left\{u - P_{A_\sigma}\left[B_\sigma(u, a)\right] : u \in \overline{B_r(0)}^X \setminus \{0\},\ \sigma \in [0,1]\right\}. \tag{2.51}$$

Assume otherwise that there are sequences $\{u_n\}$, $\{\sigma_n\}$ such that $\sigma_n \to \sigma \in [0,1]$, $0 \neq \|u_n\|_X \to 0$, and

$$0 \in u_n - P_{A_{\sigma_n}}\left[B_{\sigma_n}\left(u_n, a\right)\right], \quad \forall n. \tag{2.52}$$

Hence, $u_n \in P_{A_{\sigma_n}}[B_{\sigma_n}(u_n, a)]$, that is,

$$A_{\sigma_n}\left(u_n\right) = B_{\sigma_n}\left(u_n, a\right). \tag{2.53}$$

Therefore, $A(\sigma_n u_n) = B(\sigma_n u_n, a)$. Putting $v_n = u_n/\|u_n\|_X$ and dividing both sides of this equation by $(\sigma_n \|u_n\|_X)^{p-1}$, we get

$$A_{\sigma_n \|u_n\|_X}\left(v_n\right) = B_{\sigma_n \|u_n\|_X}\left(v_n, a\right), \quad \forall n. \tag{2.54}$$

Since $\|v_n\|_X = 1$, for all n, by passing to a subsequence if necessary, we can assume that

$$v_n \rightharpoonup v \quad \text{in } X. \tag{2.55}$$

It follows that $v_n \to v$ in Z. Because $\sigma_n \|u_n\|_X \to 0$, (H2a) implies that

$$B_{\sigma_n \|u_n\|_X}(v_n, a) \longrightarrow B_0(v, a) \tag{2.56}$$

in Z^* and then in X^*. From (H1$'$) with $f_n = B_{\sigma_n \|u_n\|_X}(v_n, a)$, we have $v \in X_0$ and

$$A_0(v) = B_0(v, a). \tag{2.57}$$

Since a is not an eigenvalue of (2.44), (2.57) holds only when $v = 0$. Hence,

$$v_n \rightharpoonup 0 \quad \text{in } X, \tag{2.58}$$

and thus $v_n \to 0$ in Z. It follows from (H2a) that

$$B_{\sigma_n \|u_n\|_X}\big(v_n, a\big) \longrightarrow B_0(0, a) = 0 \quad \text{in } Z^*\text{(-strong)} \tag{2.59}$$

and thus in X^*(-strong). Equation (2.54) gives

$$A_{\sigma_n \|u_n\|_X}\big(v_n\big) \longrightarrow 0 \quad \text{in } X^*. \tag{2.60}$$

Equations (2.58), (2.60), and assumption (H3) now imply that $v_n \to 0$ in X. But this is impossible since $\|v_n\|_X = 1$, for all n. This contradiction proves (2.51) for all $r > 0$ sufficiently small. Now, we fix $r > 0$ such that (2.51) is satisfied. We now prove that there exists an open neighborhood $U = U_Z$ of 0 in Z such that

$$\bar{U}^Z \cap X \subset B_r(0)\big(= B_r^X(0)\big). \tag{2.61}$$

We use $\bar{U}^Z$ and $\bar{U}^X$ to denote the closure of U in Z and X, respectively. Also, $B_r(0) = B_r^X(0) = \{x \in X : \|x\| < r\}$, $B_r^Z(0) = \{x \in Z : \|x\|_Z < r\}$, and $\bar{B}_r^Z(0) = \overline{B_r^Z(0)}^Z(0) = \{\|x\|_Z \le r\}$. There always exists an open set V_Z in Z such that $0 \in V_Z \cap X = B_r^X(0)$. Choose $r_1 > 0$ small enough such that $B_{r_1}^Z(0) \subset V_Z$. Now, choose $r_2 \in (0, r_1)$ and put $U = U_Z = B_{r_2}^Z(0)$. We have

$$\bar{U}_Z^Z = \overline{B_{r_2}^Z(0)}^Z = \bar{B}_{r_2}^Z(0)\big(= \big\{z \in Z : \|z\|_Z \le r_2\big\}\big) \subset B_{r_1}^Z(0) \subset V_Z. \tag{2.62}$$

Therefore,

$$\bar{U}_Z^Z \cap X \subset V_Z \cap X = B_r^X(0), \tag{2.63}$$

proving (2.61). Now, assume $u \in \bar{U}^Z \setminus \{0\}$ is such that $0 \in u - i P_{A_\sigma}[B_\sigma(u, a)]$ for some $\sigma \in [0, 1]$. As was discussed previously, we have $u \in P_{A_\sigma}[B_\sigma(u, a)]$, which implies that $u \in X$. Hence, u is in $\bar{U}^Z \cap X \setminus \{0\}$ and thus in $B_r(0) \setminus \{0\}$. This, however, contradicts (2.51). Equation (2.50) is proved. $\square$

Now, we arrive at our main result.

Theorem 2.9. *Assume that A, B, A_0, and B_0 satisfy conditions (H1), (H2), (H3), and (H4). If a and b ($a < b$) are not eigenvalues of (2.44) and if*

$$\mathrm{d}_Z\left(I - iP_{A_0}\left[B_0(\cdot,a)\right], B_r^Z(0), 0\right) \neq \mathrm{d}_Z\left(I - iP_{A_0}\left[B_0(\cdot,b)\right], B_r^Z(0), 0\right), \quad (2.64)$$

for some $r > 0$, small, then for $\mathcal{S}$ and $\mathcal{C}$ as in Theorem 2.2, then the Rabinowitz alternative (i) or (ii) holds.

Proof. Note that if a is not an eigenvalue of (2.44), then for all $R > 0$, all $r \in (0, R)$, we have

$$0 \notin \left\{u - iP_{A_0}\left[B_0(u,a)\right] : r \leq \|u\|_Z \leq R\right\}. \quad (2.65)$$

Hence, by the excision property, the degrees in (2.64) do not depend on r. Moreover, we can replace $B_r^Z(0)$ in (2.64) by any bounded open subset U that contains 0.

Let $U = B_r^Z(0)$ be as in condition (2.11) in Theorem 2.2. By choosing $r > 0$ sufficiently small, U also satisfies condition (2.50) in Lemma 2.8. We consider the family of (multivalued) compact perturbations of the identity $\{I - iP_{A_\sigma}[B_\sigma(\cdot,a)] : \sigma \in [0,1]\}$ in $\bar{U}^Z$. We check that the set

$$\left\{iP_{A_\sigma}\left[B_\sigma(\cdot,a)\right] : u \in \bar{U}^Z,\ \sigma \in [0,1]\right\} \quad (2.66)$$

is relatively compact in Z. In fact, let $\{u_n\} \subset \bar{U}^Z$ and $\{\sigma_n\} \subset [0,1]$. Without loss of generality, we can assume that $\sigma_n \to \sigma \in [0,1]$. Since $\{u_n\}$ is bounded in Z, $\{B_{\sigma_n}(u_n,a)\}$ is bounded in Z^* (if $\sigma > 0$, this is a consequence of the boundedness of B and if $\sigma = 0$, this follows from assumption (H2b)). We show that the sequence $\{P_{A_{\sigma_n}}[B_{\sigma_n}(u_n,a)]\}$ is bounded in X. Assume otherwise that this sequence is unbounded. Hence, by passing to a subsequence if necessary, we have a sequence $\{w_n\}$ in X such that $w_n \in P_{A_{\sigma_n}}[B_{\sigma_n}(u_n,a)]$ and $\|w_n\|_X \to \infty$. We have

$$A_{\sigma_n}\left(w_n\right) = B_{\sigma_n}\left(u_n,a\right), \quad \forall n. \quad (2.67)$$

If $\sigma > 0$, then this is impossible by the coercivity of A (cf. (2.2)). Therefore, $\sigma = 0$, that is, $\sigma_n \to 0$. From (2.67), we have

$$\frac{\left\langle A_{\sigma_n}\left(w_n\right), w_n\right\rangle}{\left\|w_n\right\|_X} = \left\langle B_{\sigma_n}\left(u_n,a\right), \frac{w_n}{\left\|w_n\right\|_X}\right\rangle. \quad (2.68)$$

By (H4), the left-hand side of this equation tends to ∞ as $n \to \infty$. On the other hand, its right-hand side is bounded since $\{B_{\sigma_n}(u_n,a)\}$ is bounded in X^* and $\{w_n/\|w_n\|_X\}$ is bounded in X. This contradiction proves that the sequence

$\{P_{\sigma_n}[B_{\sigma_n}(u_n,a)]\}$ is bounded in X. It follows that the set $\{P_\sigma[B_\sigma(u,a)] : u \in \bar{U}^Z,\ \sigma \in [0,1]\}$ is bounded in X. Because the embedding $i : X \hookrightarrow Z$ is compact, the set in (2.66) is relatively compact in Z.

Now, from Lemma 2.8, one has

$$0 \notin \left\{i - iP_{A_\sigma}\left[B_\sigma(u,a)\right] : \sigma \in [0,1],\ u \in \partial_Z U\right\}. \tag{2.69}$$

Using the homotopy invariance property of the degree for multivalued compact fields (cf. [11]), we have

$$\mathrm{d}_Z\left(I - iP_{A_0}\left[B_0(\cdot,a)\right], U, 0\right) = \mathrm{d}_Z\left(I - iP_{A_1}\left[B_1(\cdot,a)\right], U, 0\right). \tag{2.70}$$

Theorem 2.9 is now a direct consequence of Theorem 2.2. □

3. Bifurcation in equations with anisotropic p-Laplace-like operators

In this section, we apply Theorem 2.9 to a quasilinear elliptic equation that contains an anisotropic p-Laplace-like (ϕ-Laplace) operator. The equation is formulated in the weak form as follows:

$$\int_\Omega \sum_{i=1}^N \phi_i\left(|\nabla u|\right)\partial_i u \partial_i v\,dx = \int_\Omega F(x,u,\lambda)v\,dx, \quad \forall v \in X,\ u \in X, \tag{3.1}$$

where X is a function space to be specified later. If $\phi_i = \phi$, for all $i \in \{1,\dots,N\}$, then (3.1) reduces to the equation

$$\int_\Omega \phi\left(|\nabla u|\right)\nabla u \cdot \nabla v\,dx = \int_\Omega F(x,u,\lambda)v\,dx, \quad \forall v \in X, \tag{3.2}$$

which has a variational structure and was treated previously in [8, 9]. If the ϕ_i's are different then the principal operator in (3.1) does not have any potential functional and the approach used in the quoted papers seems not applicable here.

Assume that for all $i \in \{1,\dots,N\}$, $\phi_i : \mathbb{R} \to \mathbb{R}^+$ is an even continuous function such that $\phi_i|_{\mathbb{R}^+}$ is nondecreasing, $\phi_i(0) = 0$, and $\phi_i > 0$ on $(0,\infty)$. Also, there exist $\gamma > 1$ and $a,b,c,d > 0$ such that

$$a|s|^{\gamma-1} - d \le \phi_i(s)s \le b|s|^{\gamma-1} + c, \quad \forall s \in \mathbb{R},\ i \in \{1,\dots,N\}. \tag{3.3}$$

Moreover, for simplicity of calculations, we assume that the ϕ_i's have the same behavior at 0, that is, there exists $p \in [\gamma,\gamma^*)$ (γ^* is the Sobolev conjugate of γ, defined by $\gamma^* = N\gamma(N-\gamma)^{-1}$ if $\gamma < N$ and $\gamma^* = \infty$ if $\gamma \ge N$) and $\beta \in \mathbb{R}\setminus\{0\}$ such that for all i,

$$g_i(s) = \frac{\phi_i(s)}{|s|^{p-2}} \longrightarrow \beta, \quad \text{as } s \longrightarrow 0. \tag{3.4}$$

Suppose that $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the growth condition

$$\left|F(x,u,\lambda)\right| \le C(\lambda)\left[1 + |u|^{p-2}\right] \tag{3.5}$$

for a.e. $x \in \Omega$, for all $\lambda, u \in \mathbb{R}$, where $C \in L^{\infty}_{\mathrm{loc}}(\mathbb{R})$. Moreover, $F(x,0,\lambda)=0$, for all $\lambda \in \mathbb{R}$, a.e. $x \in \Omega$, and

$$\frac{F(x,s,\lambda)}{|s|^{p-2}s} \longrightarrow \lambda, \quad \text{as } s \longrightarrow 0, \tag{3.6}$$

uniformly for a.e. $x \in \Omega$ and for λ in bounded sets.

From (3.3), we see that the integral in the left-hand side of (3.1) is well defined for u, v in the Sobolev space $W_0^{1,\gamma}(\Omega)$. Moreover, the operator A given by

$$\langle A(u), v\rangle = \int_\Omega \sum_{i=1}^N \phi_i\big(|\nabla u|\big)\partial_i u \partial_i v\, dx, \quad u, v \in W_0^{1,\gamma}(\Omega), \tag{3.7}$$

is well defined from $X := W_0^{1,\gamma}(\Omega)$ into $X^* = W^{-1,\gamma'}(\Omega)$ ($\gamma' = \gamma(\gamma-1)^{-1}$ is the Hölder conjugate of γ). Also, A is continuous and bounded in X with $A(0)=0$. From the first inequality of (3.3) and from (3.4), we have a_1 and R_0 positive such that for all $i = 1, \dots, N$,

$$\big|\phi_i(s)\big| \geq a_1 \begin{cases} |s|^{\gamma-2} & \text{for } |s| > R_0, \\ |s|^{p-2} & \text{for } |s| \leq R_0. \end{cases} \tag{3.8}$$

For $u \in W_0^{1,\gamma}(\Omega)$, put

$$\begin{aligned} \Omega_1 = \Omega_1(u) &= \big\{x \in \Omega : \big|\nabla u(x)\big| > R_0\big\}, \\ \Omega_2 = \Omega_2(u) &= \big\{x \in \Omega : \big|\nabla u(x)\big| \leq R_0\big\}. \end{aligned} \tag{3.9}$$

We have

$$\begin{aligned} \langle A(u), u\rangle &= \int_\Omega \sum_{i=1}^N \phi_i\big(|\nabla u|\big)\big|\partial_i u\big|^2 dx \\ &\geq a_1\left(\int_{\Omega_1(u)} \sum_i |\nabla u|^{\gamma-2}\big|\partial_i u\big|^2 dx + \int_{\Omega_2(u)} \sum_i |\nabla u|^{p-2}\big|\partial_i u\big|^2 dx\right) \\ &= a_1\left(\int_{\Omega_1(u)} |\nabla u|^\gamma dx + \int_{\Omega_2(u)} |\nabla u|^p dx\right) \\ &\geq a_1\left(\int_{\Omega_1(u)} |\nabla u|^\gamma dx - \int_{\Omega_2(u)} R_0^\gamma + \int_{\Omega_2(u)} |\nabla u|^\gamma dx\right) \\ &\geq a_1\left(\int_\Omega |\nabla u|^\gamma dx - R_0^\gamma |\Omega|\right) \\ &= a_1 \|u\|^\gamma_{W_0^{1,\gamma}(\Omega)} - a_1 R_0^\gamma |\Omega|. \end{aligned} \tag{3.10}$$

This shows that

$$\frac{\langle A(u), u\rangle}{\|u\|_X} \longrightarrow \infty, \quad \text{as } \|u\|_X \longrightarrow \infty. \tag{3.11}$$

Hence, A is coercive in X in the sense of (2.2).

We assume that the ϕ_i's satisfy the monotonicity condition

$$\sum_{i=1}^{N} \big[\phi_i(|\xi|)\xi_i - \phi_i(|\eta|)\eta_i\big](\xi_i - \eta_i) \geq 0, \tag{3.12}$$

for all $\xi, \eta \in \mathbb{R}^N$. It follows from (3.12) that A is monotone in $W_0^{1,\gamma}(\Omega)$ (but not strictly monotone in general). We put $Z = L^p(\Omega)$ and let $B : Z \times \mathbb{R} \to Z^*$ be defined by

$$\langle B(u,\lambda), v\rangle = \int_\Omega F(x,u,\lambda)v\,dx, \quad \forall u, v \in L^p(\Omega). \tag{3.13}$$

Since $p < \gamma^*$, the embeddings $X \hookrightarrow Z$ and $Z^* \hookrightarrow X^*$ are compact. Equation (3.5) implies that B is well defined and $B(0,\lambda) = 0$ for all λ. Moreover,

$$\|B(u,\lambda)\|_{Z^*} = \|F(\cdot,u,\lambda)\|_{L^{p'}(\Omega)} \leq C_1(\lambda)\big(1 + \|u\|_Z^p\big)^{1/p'}. \tag{3.14}$$

Using Hölder's inequality, one can prove that there exists $C > 0$ such that

$$\begin{aligned} \|B(u,\lambda) - B(\bar{u},\bar{\lambda})\|_{Z^*} &\leq C\|F(\cdot,u,\lambda) - F(\cdot,\bar{u},\bar{\lambda})\|_{L^{p'}(\Omega)}, \\ &\forall (u,\lambda), (\bar{u},\bar{\lambda}) \in Z \times \mathbb{R}. \end{aligned} \tag{3.15}$$

Together with (3.5) and the dominated convergence theorem, this proves the continuity of B on $Z \times \mathbb{R}$.

Now, we check the background assumptions in Theorem 2.9. First, we check the nondegeneracy condition (2.30). Assume that $\{u_n\} \subset X$ is such that $u_n \to 0$ in $L^p(\Omega)$ and

$$\langle A(u_n), u_n\rangle = \int_\Omega \sum_{n=1}^{N} \phi_i(|\nabla u_n|)|\partial_i u_n|^2 \longrightarrow 0. \tag{3.16}$$

Equations (3.8) and (3.9) imply that

$$\int_{\Omega_1(u_n)} |\nabla u_n|^\gamma dx \longrightarrow 0, \qquad \int_{\Omega_2(u_n)} |\nabla u_n|^p dx \longrightarrow 0. \tag{3.17}$$

On the other hand, Hölder's inequality implies that

$$\int_{\Omega_2(u_n)} |\nabla u_n|^\gamma dx \leq |\Omega|^{1-\gamma/p}\bigg(\int_{\Omega_2(u_n)} |\nabla u_n|^p dx\bigg)^{\gamma/p}. \tag{3.18}$$

Therefore,

$$\int_\Omega \left|\nabla u_n\right|^\gamma \longrightarrow 0, \tag{3.19}$$

that is, $u_n \to 0$ in X. We have proved (2.30). This implies that the definition of bifurcation points for (2.8) given in Definition 2.1 is equivalent to the usual definition of bifurcation points.

Now, we put $X_0 = W_0^{1,p}(\Omega)$ with the usual norm

$$\|u\|_0 = \|u\|_{X_0} = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}. \tag{3.20}$$

Since $p \geq \gamma$, $X_0 \subset X$ and the embedding $X_0 \hookrightarrow X$ is continuous. Let $A_0 : X_0 \to X_0^*$ be given by

$$\left\langle A_0(u), v\right\rangle = \beta \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx. \tag{3.21}$$

It is clear that A_0 is well defined, continuous, bounded, and coercive in X_0. Also, $A_0(0) = 0$. For $\sigma > 0$, define A_σ as in (2.33). We check that A_σ converges to A_0 as $\sigma \to 0^+$ in the sense of (H1′) (or equivalently (H1)). Assume that we have sequences $\{\sigma_n\}$, $\{u_n\}$ such that

$$\sigma_n \longrightarrow 0^+, \quad u_n \rightharpoonup u \quad \text{in } X, \tag{3.22}$$

$$A_{\sigma_n}\left(u_n\right) = f_n \longrightarrow f \quad \text{in } X^*. \tag{3.23}$$

We prove that $u \in X_0$ and $A_0(u) = f$ in X_0^*. We have

$$\left\langle A_\sigma(u), v\right\rangle = \int_\Omega \sum_{i=1}^N \frac{\phi_i\left(\sigma|\nabla u|\right)}{\sigma^{p-1}} \partial_i u \partial_i v\, dx. \tag{3.24}$$

Thus,

$$\begin{aligned}
\left\langle A_{\sigma_n}\left(u_n\right), u_n\right\rangle &= \int_\Omega \sum_i \frac{\phi_i\left(\sigma_n\left|\nabla u_n\right|\right)}{\sigma_n^{p-2}} \left|\partial_i u_n\right|^2 dx \\
&= \int_\Omega \sum_i g_i\left(\sigma_n\left|\nabla u_n\right|\right)\left|\nabla u_n\right|^{p-2}\left|\partial_i u_n\right|^2 dx \quad (g_i \text{ is defined in (3.4)}) \\
&= \int_\Omega G\left(\sigma_n\left|\nabla u_n\right|, \nabla u_n\right) dx,
\end{aligned} \tag{3.25}$$

where

$$G(u,\xi) = |\xi|^{p-2} \sum_{i=1}^N g_i\left(|u|\right)|\xi_i|^2, \tag{3.26}$$

for all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. By putting $g_i(0) = \beta$, we see that G is a Carathéodory function (cf. (3.4)). Moreover, for each $u \in \mathbb{R}$ fixed, the mapping $\xi \mapsto G(u,\xi)$ is convex on $\mathbb{R}^N$. Because $u_n \rightharpoonup u$ in X,

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } \left[L^p(\Omega)\right]^N \text{(-weak)} \tag{3.27}$$

and thus in $[L^1(\Omega)]^N$-weak. On the other hand, since $\{\nabla u_n\}$ is a bounded sequence in $[L^p(\Omega)]^N$ and $\sigma_n \to 0$, we have

$$\sigma_n \left|\nabla u_n\right| \longrightarrow 0 \quad \text{in } L^p(\Omega)\text{(-strong)}. \tag{3.28}$$

Therefore, by passing to a subsequence if necessary we can assume that $\sigma_n|\nabla u_n| \to 0$ a.e. in Ω. Furthermore, since $\phi_i(s) \geq 0$ for $s \geq 0$, we have $G(u,\xi) \geq 0$ for all u, ξ. These observations show that all assumptions of [1, Theorem 5.4] are satisfied, which implies that

$$\int_\Omega G(0,\nabla u)dx \leq \liminf_{n\to\infty} \int_\Omega G\left(\sigma_n\left|\nabla u_n\right|, \nabla u_n\right)dx. \tag{3.29}$$

It follows from (3.4) that

$$G(0,\xi) = |\xi|^{p-2}\sum_{i=1}^N \beta\left|\xi_i\right|^2 = \beta|\xi|^p. \tag{3.30}$$

On the other hand, (3.23) implies that

$$\lim_{n\to\infty}\left\langle A_{\sigma_n}\left(u_n\right), u_n\right\rangle = \lim_{n\to\infty}\left\langle f_n, u_n\right\rangle = \langle f, u\rangle < \infty. \tag{3.31}$$

This, together with (3.25), (3.29), and (3.30), imply that

$$\int_\Omega \beta|\nabla u|^p dx = \int_\Omega G(0,\nabla u)dx \leq \langle f, u\rangle < \infty. \tag{3.32}$$

This means that $u \in X_0 = W_0^{1,p}(\Omega)$. We show that

$$\left\langle A_0(u), v\right\rangle = \langle f, v\rangle, \quad \forall v \in X_0. \tag{3.33}$$

First, note that for $u \in X_0$,

$$A_{\sigma_n}(u) \longrightarrow A_0(u) \quad \text{in } X_0^*. \tag{3.34}$$

In fact, by Hölder's inequality, there is a constant $C > 0$ independent of u and n such that

$$\begin{aligned}\left\|A_{\sigma_n}(u) - A_0(u)\right\|_{X_0^*} &\leq C\sum_{i=1}^N \left\|\frac{\phi_i\left(\sigma_n|\nabla u|\right)}{\left(\sigma_n|\nabla u|\right)^{p-2}}|\nabla u|^{p-2}\partial_i u - \beta|\nabla u|^{p-2}\partial_i u\right\|_{L^{p'}(\Omega)} \\ &= C\sum_{i=1}^N \left\|g_i\left(\sigma_n|\nabla u|\right)|\nabla u|^{p-2}\partial_i u - \beta|\nabla u|^{p-2}\partial_i u\right\|_{L^{p'}(\Omega)}.\end{aligned} \tag{3.35}$$

It follows from (3.3) and (3.4) that g_i is bounded on $\mathbb{R}$, that is, $|g_i(s)| \le M$, for all $s \in \mathbb{R}$, for all i, for some $M > 0$. Hence,

$$\left|g_i\big(\sigma_n|\nabla u|\big)|\nabla u|^{p-2}\partial_i u\right| \le M|\nabla u|^{p-1}, \quad \text{a.e. in } \Omega. \tag{3.36}$$

On the other hand, by (3.4),

$$g_i\big(\sigma_n|\nabla u|\big)|\nabla u|^{p-2}\partial_i u \longrightarrow \beta\sum_{i=1}^{N}|\nabla u|^{p-2}\partial_i u, \quad \text{a.e. in } \Omega. \tag{3.37}$$

Then by the dominated convergence theorem,

$$g_i\big(\sigma_n|\nabla u|\big)|\nabla u|^{p-2}\partial_i u - \beta|\nabla u|^{\beta-2}\partial_i u \longrightarrow 0 \tag{3.38}$$

in $L^{p'}(\Omega)$, for each i, implying that $A_{\sigma_n}(u) \to A_0$ in X_0^*. Now, let $v \in X_0$. We have from the monotonicity of A and the definition of A_σ that

$$\left\langle A_{\sigma_n}\big(u_n\big) - A_{\sigma_n}(v), u_n - v\right\rangle \ge 0 \tag{3.39}$$

and thus

$$\left\langle A_{\sigma_n}\big(u_n\big), u_n - v\right\rangle \ge \left\langle A_{\sigma_n}(v), u_n - v\right\rangle. \tag{3.40}$$

From (3.22), (3.23), and (3.34), we get

$$\langle f, u - v\rangle \ge \left\langle A_0(v), u - v\right\rangle, \quad \forall v \in X_0. \tag{3.41}$$

From Minty's lemma (cf. [5]), this implies

$$\left\langle A_0(u), v - u\right\rangle \le \langle f, v - u\rangle, \quad \forall v \in X_0 \tag{3.42}$$

which is, in its turn, equivalent to (3.33). Since (3.33) is the same as (2.42), (H1′) is proved.

Now, we check (H3). Assume $v_n \rightharpoonup 0$ in $W_0^{1,\gamma}(\Omega)$ and that $A_{\sigma_n}(v_n) \to 0$ in $[W_0^{1,\gamma}(\Omega)]^*$. It follows that

$$\left\langle A_{\sigma_n}\big(v_n\big), v_n\right\rangle = \int_\Omega \sum_i \frac{\phi_i\big(\sigma_n\left|\nabla v_n\right|\big)}{\sigma_n^{p-2}}\left|\partial_i v_n\right|^2 dx \longrightarrow 0 \tag{3.43}$$

as $n \to \infty$. Using the notation in (3.9), we have

$$\int_{\Omega_j(\sigma_n|\nabla v_n|)} \sum_i \frac{\phi_i\big(\sigma_n\left|\nabla v_n\right|\big)}{\sigma_n^{p-2}}\left|\partial_i v_n\right|^2 dx \longrightarrow 0 \tag{3.44}$$

$(j=1,2)$. It follows from (3.8) that

$$\begin{aligned}
0 &\geq \limsup \int_{\Omega_1(\sigma_n|\nabla v_n|)} \sum_i \frac{a_i\left(\sigma_n\left|\nabla v_n\right|\right)^{\gamma-2}}{\sigma_n^{p-2}}\left|\partial_i v_n\right|^2 dx \\
&= a_1 \limsup \int_{\Omega_1(\sigma_n|\nabla v_n|)} \sum_i \frac{\left|\nabla v_n\right|^{\gamma-2}}{\sigma_n^{p-\gamma}}\left|\partial_i v_n\right|^2 dx \\
&\geq a_1 \limsup \int_{\Omega_1(\sigma_n|\nabla v_n|)} \sum_i \left|\nabla v_n\right|^{\gamma-2}\left|\partial_i v_n\right|^2 dx \\
&\qquad \text{(since } p \geq \gamma,\ \sigma_n \in (0,1),\ \forall n \text{ large)} \\
&= a_1 \limsup \int_{\Omega_1(\sigma_n|\nabla v_n|)} \left|\nabla v_n\right|^{\gamma} dx \\
&\geq a_1 \liminf \int_{\Omega_1(\sigma_n|\nabla v_n|)} \left|\nabla v_n\right|^{\gamma} dx \\
&\geq 0.
\end{aligned} \tag{3.45}$$

Hence,

$$\lim_{n\to\infty} \int_{\Omega_1(\sigma_n|\nabla v_n|)} \left|\nabla v_n\right|^{\gamma} dx = 0. \tag{3.46}$$

Similarly,

$$\begin{aligned}
0 &= \lim \int_{\Omega_2(\sigma_n|\nabla u_n|)} \sum_i \frac{\phi_i\left(\sigma_n\left|\nabla v_n\right|\right)}{\sigma_n^{p-2}}\left|\partial_i v_n\right|^2 dx \\
&\geq a_1 \lim \int_{\Omega_2(\sigma_n|\nabla u_n|)} \left|\nabla v_n\right|^{p-2}\left|\partial_i v_n\right|^2 dx,
\end{aligned} \tag{3.47}$$

and $\lim \int_{\Omega_2(\sigma_n|\nabla u_n|)} |\nabla v_n|^p dx = 0$. Since $p \geq \gamma$, Hölder's inequality gives

$$\begin{aligned}
\int_{\Omega_2(\sigma_n|\nabla u_n|)} \left|\nabla v_n\right|^{\gamma} dx &\leq \left(\int_{\Omega_2} \left|\nabla v_n\right|^p dx\right)^{\gamma/p} \left|\Omega_2\right|^{1-\gamma/p} \\
&\leq \left(\int_{\Omega_2} \left|\nabla v_n\right|^p dx\right)^{\gamma/p} |\Omega|^{1-\gamma/p}.
\end{aligned} \tag{3.48}$$

Thus,

$$\lim \int_{\Omega_2(\sigma_n|\nabla u_n|)} \left|\nabla v_n\right|^{\gamma} dx = 0. \tag{3.49}$$

Combining (3.46) and (3.49), we get

$$\lim \int_{\Omega} \left|\nabla v_n\right|^{\gamma} dx = 0, \tag{3.50}$$

that is, $v_n \to 0$ in X. (H3) is proved.

Now, we check (H4). Let $\|v_n\|_X \to \infty$ and $\sigma_n \to 0^+$. As in the proof of (3.46) and (3.49), we have, for all n sufficiently large,

$$\int_{\Omega_1(\sigma_n|\nabla u_n|)} \sum_i \frac{\phi_i\big(\sigma_n|\nabla u_n|\big)}{\sigma_n^{p-2}} \big|\partial_i v_n\big|^2 dx \geq a_1 \int_{\Omega_1(\sigma_n|\nabla u_n|)} \big|\nabla u_n\big|^\gamma dx,$$

$$\begin{aligned}
&\int_{\Omega_2(\sigma_n|\nabla u_n|)} \sum_i \frac{\phi_i\big(\sigma_n|\nabla u_n|\big)}{\sigma_n^{p-2}} \big|\partial_i v_n\big|^2 dx \\
&\quad \geq a_1 \int_{\Omega_2(\sigma_n|\nabla u_n|)} \big|\nabla u_n\big|^p dx \\
&\quad \geq a_1 \bigg(\int_{\Omega_2(\sigma_n|\nabla u_n|)} \big|\nabla u_n\big|^\gamma dx \bigg)^{p/\gamma} \big(|\Omega|^{p/(p-\gamma)}\big)^{1-p/\gamma} \\
&\qquad\qquad \text{(trivial modifications when } p=\gamma) \\
&\quad \geq a_2 \begin{cases} \displaystyle\int_{\Omega_2} \big|\nabla v_n\big|^\gamma dx & \text{if } \displaystyle\int_{\Omega_2} \big|\nabla v_n\big|^\gamma dx \geq 1, \\ 0 & \text{if } \displaystyle\int_{\Omega_2} \big|\nabla v_n\big|^\gamma dx < 1, \end{cases} \\
&\qquad\qquad \text{(with } a_2 = a_1 |\Omega|^{p^2/(p-\gamma)\gamma} > 0).
\end{aligned} \tag{3.51}$$

It follows from (3.51) that

$$\begin{aligned}
&\int_\Omega \sum_i \frac{\phi_i\big(\sigma_n|\nabla u_n|\big)}{\sigma_n^{p-2}} \big|\partial_i v_n\big|^2 dx \\
&\quad \geq \min\{a_1, a_2\} \begin{cases} \displaystyle\int_{\Omega} \big|\nabla v_n\big|^\gamma dx & \text{if } \displaystyle\int_{\Omega_2} \big|\nabla v_n\big|^\gamma dx \geq 1 \\ \displaystyle\int_{\Omega_1} \big|\nabla v_n\big|^\gamma dx & \text{if } \displaystyle\int_{\Omega_2} \big|\nabla v_n\big|^\gamma dx < 1 \end{cases} \\
&\quad \geq \min\{a_1, a_2\} \bigg(\int_\Omega \big|\nabla v_n\big|^\gamma dx - 1 \bigg),
\end{aligned} \tag{3.52}$$

since

$$\int_{\Omega_1} \big|\nabla v_n\big|^\gamma dx = \int_\Omega \big|\nabla v_n\big|^\gamma dx - \int_{\Omega_2} \big|\nabla v_n\big|^\gamma dx \geq \int_\Omega \big|\nabla v_n\big|^\gamma dx - 1 \tag{3.53}$$

if $\int_{\Omega_2} |\nabla v_n|^\gamma dx \leq 1$. This proves that

$$\lim_{n\to\infty} \frac{\big\langle A_{\sigma_n}(v_n), v_n\big\rangle}{\|v_n\|_X} \geq \min\{a_1, a_2\} \lim_{n\to\infty} \frac{1}{\|v_n\|_X} \big(\|v_n\|_X^\gamma - 1\big) = \infty. \tag{3.54}$$

Thus (H4) is verified.

Now, we consider the homogenization of B. Let $B_0 : L^p(\Omega)\times\mathbb{R}(= Z\times\mathbb{R}) \to L^{p'}(\Omega)(= Z^*)$ (p' is the Hölder conjugate of p) be defined by

$$\langle B_0(u,\lambda), v\rangle = \int_\Omega \lambda |u|^{p-2} uv\,dx, \tag{3.55}$$

for $u, v \in L^p(\Omega)$, $\lambda \in \mathbb{R}$. It is clear that B_0 is defined and continuous. We check (H2a). Assume $\sigma_n \to 0^+$, $v_n \to v$ in $L^p(\Omega)$, and $\lambda_n \to \lambda$. Since

$$\langle B_{\sigma_n}(v_n,\lambda_n), v\rangle = \int_\Omega \frac{F(x,\sigma_n v_n,\lambda_n)v}{\sigma_n^{p-1}}dx, \quad \forall v \in L^p(\Omega), \tag{3.56}$$

by using (3.5), (3.6), and the dominated convergence theorem, we immediately have (2.36). Note that (3.56) and (3.5) also imply (H2b). On the other hand, the arguments used in the proof of (H1) also prove (H5). We have just checked all background assumptions so that the bifurcation Theorem 2.9 is applicable.

Now, we consider the homogenized equation associated to (3.1). From (3.21) and (3.55) we see that the homogenized equation (2.44) associated to (2.1) is in our example given by

$$\beta\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx - \lambda\int_\Omega |u|^{p-2}uv\,dx = 0, \quad \forall v \in W_0^{1,p}(\Omega),\ u \in W_0^{1,p}(\Omega). \tag{3.57}$$

This equation is the weak form of the p-Laplacian equation

$$-\frac{\beta}{p}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{3.58}$$

Let λ_1 be the principal eigenvalue of the p-Laplacian,

$$\lambda_1 = \min\left\{\int_\Omega |\nabla u|^p dx : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p dx = 1\right\}. \tag{3.59}$$

Then, as proved in [3],

$$\begin{aligned} &\mathrm{d}_{W_0^{1,p}(\Omega)}\left(I - P_0\left[B_0(\cdot,\lambda)\right], B_r^{W_0^{1,p}(\Omega)}(0,r), 0\right) \\ &\qquad = \begin{cases} 1 & \text{if } 0 < \lambda < \dfrac{\beta\lambda_1}{p}, \\ -1 & \text{if } \lambda > \dfrac{\beta\lambda_1}{p}, \\ & \quad \left(\text{and close to } \dfrac{\beta\lambda_1}{p}\right), \end{cases} \end{aligned} \tag{3.60}$$

where $P_0 : [W_0^{1,p}(\Omega)]^* \to W_0^{1,p}(\Omega)$ is the solution operator associated with (3.57) (or (3.58)), that is, for $f \in [W_0^{1,p}(\Omega)]^*$, $u = P_0(f)$ if and only if

$u \in W_0^{1,p}(\Omega)$ and

$$\beta \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx = \langle f, v\rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.61}$$

Note that i is the embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ and $B_0(\cdot,\lambda)\circ i$ is just the restriction of $B_0(\cdot,\lambda)$ in (3.55) onto $W_0^{1,p}(\Omega)$.

For λ close enough to $\beta\lambda_1/p$, λ is not an eigenvalue of (3.58) and 0 is the unique zero of the mapping $I - P_0 \circ B_0(\cdot,\lambda)\circ i$ $(= I - P_0[B_0(\cdot,\lambda)\circ i])$ and of $I - i\circ P_0 \circ B_0(\cdot,\lambda)$ in $L^p(\Omega)$. By using the excision and commutativity properties of the Leray-Schauder degree for any bounded open sets U_1 and U_2 of $W_0^{1,p}(\Omega)$ and of $L^p(\Omega)$, respectively that contain 0, we always have

$$\mathrm{d}_{W_0^{1,p}(\Omega)}\big(I - P_0\big[B_0(\cdot,\lambda)\circ i\big], U_1, 0\big) = \mathrm{d}_{L^p(\Omega)}\big(I - i\circ P_0\big[B_0(\cdot,\lambda)\big], U_2, 0\big), \tag{3.62}$$

for all λ close to $\beta\lambda_1/p$ and different from $\beta\lambda_1/p$. Hence, by using (3.57) and (3.62) together with the bifurcation Theorem 2.9, we have the following bifurcation for (3.1).

THEOREM 3.1. *There is a branch of nontrivial solutions of (3.1) that bifurcates from* $(0, \beta\lambda_1/p)$ *and either is unbounded in* $L^p(\Omega)\times\mathbb{R}$ *or contains another bifurcation point* $(0,\mu)$ *where* $\mu \neq \beta\lambda_1/p$ *and* μ *is an eigenvalue of (3.58).*

Acknowledgement

The author would like to thank the referee for his valuable and constructive suggestions.

References

[1] J. M. Ball, J. C. Currie, and P. J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal. **41** (1981), no. 2, 135–174. MR 83f:49031. Zbl 459.35020.

[2] F. E. Browder, *Existence theorems for nonlinear partial differential equations*, Global Analysis (Proc. Sympos. Pure Math., Berkeley, Calif., 1968), vol. 16, Amer. Math. Soc., Rhode Island, 1970, pp. 1–60. MR 42#4855. Zbl 211.17204.

[3] M. A. del Pino and R. F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations **92** (1991), no. 2, 226–251. MR 92g:35028. Zbl 781.35017.

[4] P. Drábek, *Solvability and Bifurcations of Nonlinear Equations*, Pitman Research Notes in Mathematics Series, vol. 264, Longman Scientific & Technical, Harlow, 1992. MR 94e:47084. Zbl 753.34002.

[5] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Pure and Applied Mathematics, vol. 88, Academic Press, New York, 1980. MR 81g:49013. Zbl 457.35001.

[6] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Monographs on Pure and Applied Mathematics, vol. 45, The Macmillan, New York, 1964, translated by A. H. Armstrong; translation edited by J. Burlak. A Pergamon Press Book. MR 28#2414. Zbl 111.30303.

[7] W. Krawcewicz and J. Wu, *Theory of Degrees with Applications to Bifurcations and Differential Equations*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, 1997. MR 97j:47083. Zbl 882.58001.

[8] V. K. Le, *Global bifurcation in some degenerate quasilinear elliptic equations by a variational inequality approach*, to appear in Nonlinear Anal.

[9] ______, *A global bifurcation result for quasilinear elliptic equations in Orlicz-Sobolev spaces*, Topol. Methods Nonlinear Anal. **15** (2000), no. 2, 301–327. CMP 1 784 144.

[10] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969 (French). MR 41#4326. Zbl 189.40603.

[11] N. G. Lloyd, *Degree Theory*, Cambridge Tracts in Mathematics, vol. 73, Cambridge University Press, Cambridge, 1978. MR 58#12558. Zbl 367.47001.

[12] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513. MR 46#745.

[13] K. Schmitt, *Analysis methods for the study of nonlinear equations*, Master's thesis, University of Utah, Utah, Lecture Notes, 1993.

[14] I. V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Translations of Mathematical Monographs, vol. 139, Amer. Math. Soc., Rhode Island, 1994. MR 95i:35109. Zbl 822.35001.

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications. I*, Fixed-Point Theorems, Springer-Verlag, New York, 1986, Translated from the German by Peter R. Wadsack. MR 87f:47083. Zbl 583.47050.

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