(r, p)-ABSOLUTELY SUMMING OPERATORS ON THE SPACE $C(T, X)$ AND APPLICATIONS

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We give necessary and sufficient conditions for an operator on the space $C(T, X)$ to be (r, p)-absolutely summing. Also we prove that the injective tensor product of an integral operator and an (r, p)-absolutely summing operator is an (r, p)-absolutely summing operator.

For X and Y Banach spaces we denote by $L(X, Y)$ the Banach space of all linear and continuous operators from X to Y equipped with the operator norm, and by $X \otimes_\varepsilon Y$ the injective tensor product of X and Y, that is, the completion of the algebraic tensor product $X \otimes Y$ with respect to the injective cross-norm $\varepsilon(u) = \sup\{\langle x^* \otimes y^*, u\rangle \mid \|x^*\| \leq 1,\ \|y^*\| \leq 1\}$, $u \in X \otimes Y$. If T is a compact Hausdorff space and X is a Banach space, we denote by $C(T, X)$ the Banach space of all continuous X-valued functions defined on T, equipped with the supremum norm and by $C(T) = C(T, X)$ for $X = \mathbb{R}$ or $\mathbb{C}$. It is well known that $C(T, X) = C(T) \otimes_\varepsilon X$. Also if T is a compact space and X is a Banach space, we denote by Σ the σ-field of Borel subsets of T, $S(\Sigma, X)$ the space of X-valued Σ-simple functions on T, and by $B(\Sigma, X)$ we denote the uniform closure of the space $S(\Sigma, X)$; $B(\Sigma)$ for $X = \mathbb{R}$ or $\mathbb{C}$. We also use that $B(\Sigma, X) \hookrightarrow C(T, X)^{**}$. For the representing theorems of the linear and continuous operators on the space $C(T, X)$, see [1, 3]. Recall only that to each $U \in L(C(T, X), Y)$ correspond a representing measure $G : \Sigma \to L(X, Y^{**})$ and $G(E)x = U^{**}(\chi_E x)$. Also if $U \in L(X, Y)$, $V \in L(X_1, Y_1)$, by $U \otimes_\varepsilon V : X \otimes_\varepsilon Y \to X_1 \otimes_\varepsilon Y_1$ we denote the injective tensor product of the operators U and V. If $U \in L(X \otimes_\varepsilon Y, Z)$, for each $x \in X$ we consider the operator $U^{\#}x : Y \to Z$, $(U^{\#}x)(y) = U(x \otimes y)$, $y \in Y$, and evidently $U^{\#} : X \to L(Y, Z)$ is linear and continuous. For $1 \leq r < \infty$ and $x_1, \ldots, x_n \in X$ we write, $l_r(x_i \mid i = 1, n) = (\sum_{i=1}^n \|x_i\|^r)^{1/r}$ and $w_r(x_i \mid i = 1, n) = \sup\{(\sum_{i=1}^n |x^*(x_i)|^r)^{1/r} \mid x^* \in X^*,\ \|x^*\| \leq\ 1\}$. Let $1 \leq p \leq r < \infty$,

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$U \in L(X,Y)$ is called (r,p)-absolutely summing if there is some $C>0$ such that if $x_1,\dots,x_n \in X$ then $l_r(Ux_i \mid i=\overline{1,n}) \le C w_p(x_i \mid i=\overline{1,n})$. The (r,p)-absolutely summing norm of U is $\|U\|_{r,p} = \inf C$. We observe that, $\|U\|_{r,p} = \sup\{l_r(Ux_i \mid i=\overline{1,n}) \mid x_1,\dots,x_n \in X,\ w_p(x_i \mid i=\overline{1,n}) \le 1\}$. We denote by $\mathrm{As}_{r,p}(X,Y)$ the Banach space of all (r,p)-absolutely summing operators from X into Y equipped with the (r,p)-absolutely summing norm. The $(1,1)$-absolutely summing operators we call absolutely summing and $\mathrm{As}(X,Y) = \mathrm{As}_{1,1}(X,Y)$, $\|\ \|_{\mathrm{as}} = \|\ \|_{1,1}$. For other notions used and not defined we refer the reader to [3, 6].

The following theorem is an extension of [1, Proposition 2.2(ii)], [8, Theorem 2.1], and [5, Theorem 3.1].

Theorem 1. *If $U \in \mathrm{As}_{r,p}(X \otimes_\varepsilon Y, Z)$, then $U^\# x \in \mathrm{As}_{r,p}(Y,Z)$ for each $x \in X$ and $U^\# : X \to \mathrm{As}_{r,p}(Y,Z)$ is an (r,p)-absolutely summing operator with respect to the (r,p)-absolutely summing norm on $\mathrm{As}_{r,p}(Y,Z)$. In addition, $\|U^\#\|_{r,p} \le \|U\|_{r,p}$.*

Proof. For $x \in X$, let $V_x : Y \to X \otimes_\varepsilon Y$, $V_x(y) = x \otimes y$. Then by the hypothesis and the ideal property of the (r,p)-absolutely summing operators it follows that $U^\# x = UV_x$ is an (r,p)-absolutely summing operator. Now let $x_1,\dots,x_n \in X$ with $\|U^\# x_i\|_{r,p} > 0$ and $0 < \varepsilon < \|U^\# x_i\|_{r,p}$, for each $i=\overline{1,n}$. By the definition of the (r,p)-absolutely summing norm it follows that there is $(y_{ij})_{j\in\sigma_i}$, σ_i finite, $\sigma_i \subset N$ such that $\|U^\# x_i\|_{r,p} - \varepsilon < l_r(U^\# x_i(y_{ij}) \mid j \in \sigma_i)$ and $w_p(y_{ij} \mid j \in \sigma_i) \le 1$ for each $i=\overline{1,n}$. Hence $l_r(\|U^\# x_i\|_{r,p} - \varepsilon \mid i=\overline{1,n}) < l_r(U(x_i \otimes y_{ij}) \mid j \in \sigma_i,\ i=\overline{1,n})$. As U is an (r,p)-absolutely summing operator we obtain

$$l_r\big(U\big(x_i \otimes y_{ij}\big) \mid j \in \sigma_i,\ i=\overline{1,n}\big) \le \|U\|_{r,p} w_p\big(x_i \otimes y_{ij} \mid j \in \sigma_i,\ i=\overline{1,n}\big). \quad (1)$$

But we claim that $w_p(x_i \otimes y_{ij} \mid j \in \sigma_i,\ i=\overline{1,n}) \le w_p(x_i \mid i=\overline{1,n})$ and thus we obtain

$$l_r\Big(\big\|U^\# x_i\big\|_{r,p} - \varepsilon \mid i=\overline{1,n}\Big) < \|U\|_{r,p} w_p\big(x_i \mid i=\overline{1,n}\big), \quad (2)$$

that is, $l_r(\|U^\# x_i\|_{r,p} \mid i=\overline{1,n}) \le \|U\|_{r,p} w_p(x_i \mid i=\overline{1,n})$. Now for $x_1,\dots,x_n \in X$, if we denote by $I = \{i=\overline{1,n} \mid \|U^\# x_i\|_{r,p} > 0\}$, then from (2) we have

$$\begin{aligned} l_r\Big(\big\|U^\# x_i\big\|_{r,p} \mid i=\overline{1,n}\Big) &= l_r\Big(\big\|U^\# x_i\big\|_{r,p} \mid i \in I\Big) \\ &\le \|U\|_{r,p} w_p\big(x_i \mid i \in I\big) \\ &\le \|U\|_{r,p} w_p\big(x_i \mid i=\overline{1,n}\big) \end{aligned} \quad (3)$$

and the proof of the theorem will be finished. Now let $\psi \in (X \otimes_\varepsilon Y)^*$, $\|\psi\| \le 1$. Then, as it is well known, there is a regular Borel measure μ on $U_{X^*} \times U_{Y^*} = T$

such that for $x \in X$ and $y \in Y$, $\psi(x,y) = \int_T x^*(x) y^*(y) d\mu(x^*, y^*)$, $\|\psi\| = |\mu|(T) \leq 1$ (see [2] or [3]). Then using the Hölder inequality and the fact that $\|\psi\| = |\mu|(T) \leq 1$ we have

$$\left|\langle x\otimes y,\psi\rangle\right| \leq \left(\int_T \left|x^*(x)\right|^p \left|y^*(y)\right|^p d|\mu|\left(x^*,y^*\right)\right)^{1/p}, \quad \text{for } x\in X,\ y\in Y. \tag{4}$$

Thus

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j\in\sigma_i}\left|\langle x_i\otimes y_{ij},\psi\rangle\right|^p &\leq \int_T \sum_{i=1}^{n}\left|x^*\left(x_i\right)\right|^p \sum_{j\in\sigma_i}\left|y^*\left(y_{ij}\right)\right|^p d|\mu|\left(x^*,y^*\right)\\
&\leq \int_T \sum_{i=1}^{n}\left|x^*\left(x_i\right)\right|^p d|\mu|\left(x^*,y^*\right)\\
&\leq \left[w_p\left(x_i \mid i=1,n\right)\right]^p |\mu|(T),
\end{aligned} \tag{5}$$

since $w_p(y_{ij} \mid j \in \sigma_i) \leq 1$, for each $i = 1,n$. Hence $w_p(x_i \otimes y_{ij} \mid j \in \sigma_i,\ i = 1,n) \leq w_p(x_i \mid i = 1,n)$ and the claim is proved. □

In [5, 7], examples of operators are given which show that the converse of Theorem 1 is not true.

The next theorem is an extension of [1, Theorem 2.5] and the result of Swartz [8, Theorem 2].

THEOREM 2. *Let $U : C(T,X) \to Y$ be a linear and continuous operator, G its representing measure. If U is an (r,p)-absolutely summing operator, then $G(E) \in \mathrm{As}_{r,p}(X,Y)$, for each $E \in \Sigma$ and $G : \Sigma \to \mathrm{As}_{r,p}(X,Y)$ has the property that $\|G\|_{r,p}(T) = \sup\{(\sum_{i=1}^n \|G(E_i)\|_{r,p}^r)^{1/r} \mid \{E_1,\ldots,E_n\} \subset \Sigma$ a finite partition of $T\} \leq \|U\|_{r,p}$.*

Proof. As it is well known, if V is an (r,p)-absolutely summing operator then its bidual V^{**} is also (r,p)-absolutely summing (see [6]). As U is an (r,p)-absolutely summing operator we obtain, using Theorem 1, that $V = U^{\#} : C(T) \to \mathrm{As}_{r,p}(X,Y)$ is (r,p)-absolutely summing and hence V^{**} is also (r,p)-absolutely summing. But on $C(T)$, (r,p)-absolutely summing operators are weakly compact. This follows easily using [3, Theorem 15, page 159]. Hence the representing measure G of U which coincides with that of $V = U^{\#}$ takes its values in $\mathrm{As}_{r,p}(X,Y)$. Because $V^{**} : B(\sum) \to \mathrm{As}_{r,p}(X,Y)$ is an (r,p)-absolutely summing we have

$$l_r\left(V\left(\chi_{E_i}\right) \mid i=1,n\right) \leq \left\|V^{**}\right\|_{r,p} w_p\left(\chi_{E_i} \mid i=1,n\right) = \left\|V^{**}\right\|_{r,p} = \left\|U^{\#}\right\|_{r,p} \tag{6}$$

for each partition $\{E_1,\ldots,E_n\} \subset \sum$ of T. Using Theorem 1, we have

$$\left\|U^{\#}\right\|_{r,p} \leq \|U\|_{r,p}. \tag{7}$$

As $G(E) = V^{**}(\chi_E)$, from (6) and (7) we obtain the theorem. □

The following lemmas show that in the inequality from Theorem 2, we can have both equality and strict inequality.

Lemma 3. *For $1 \le p \le r < \infty$, X and Y Banach spaces, there is $U : C([0,1], X) \to Y$ an (r,p)-absolutely summing operator whose representing measure has the properties $\|G\|_{r,p}([0,1]) = (2^r+1)^{1/r}$, $\|U\|_{r,p} = 3$ and hence if $r \neq 1$, $\|G\|_{r,p}([0,1]) < \|U\|_{r,p}$.*

Proof. Let $x^* \in X^*$ with $\|x^*\| = 1$, $y \in Y$, $\|y\| = 1$. For $t \in [0,1]$, t fixed, we denote $\nu = 2\delta_t - \mu$, where δ_t is the Dirac measure and μ is the Lebesgue measure. Let $U : C([0,1], X) \to Y$, $U(f) = (\int_0^1 x^* f \, d\nu) y$. Evidently $G(E) = (x^* \otimes y)\nu(E)$ is the representing measure of U and $\|G(E)\|_{r,p} = |\nu(E)|$, from where

$$\begin{aligned}
&\|G\|_{r,p}\big([0,1]\big) \\
&\quad = \sup\left\{\left(\sum_{i=1}^{n} \big\|G\big(E_i\big)\big\|_{r,p}^{r}\right)^{1/r} \,\middle|\, \big\{E_1,\ldots,E_n\big\} \subset \Sigma \text{ a finite partition of } T\right\} \\
&\quad = \sup\left\{\left(\sum_{i=1}^{n} \big\|\nu\big(E_i\big)\big\|^{r}\right)^{1/r} \,\middle|\, \big\{E_1,\ldots,E_n\big\} \subset \Sigma \text{ a finite partition of } T\right\} \\
&\quad = \big(2^r+1\big)^{1/r}.
\end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned}
l_r\big(Uf_i \mid i = 1,n\big) &= \left(\sum_{i=1}^{n} \left| \int_0^1 x^* f_i \, d\nu \right|^r\right)^{1/r} \\
&\le w_p\big(f_i \mid i = 1,n\big) |\nu|\big([0,1]\big) \\
&= 3 w_p\big(f_i \mid i = 1,n\big)
\end{aligned} \tag{9}$$

hence, $\|U\|_{r,p} \le 3$. Also, $3 = |\nu|([0,1]) \le \|U\|_{r,p}$ and the lemma is proved. □

Lemma 4. *For $1 \le r < \infty$, X and Y Banach spaces, T a compact Hausdorff space, μ a regular positive finite Borel measure on T, there is $U : C(T, X) \to L_r(\mu, Y)$, an r-absolutely summing operator, whose representing measure G has the property $\|G\|_{r,r}(T) = \|U\|_{r,r}$.*

Proof. Let $J : C(T) \to L_r(\mu)$ be the canonical inclusion. As it is well known and easy to prove (cf. [2, 6]), J is an r-absolutely summing operator with $\|J\|_r = [\mu(T)]^{1/r}$. Also, $F(E) = \chi_E$ is the representing measure of J and

$\|F(E)\|_{r,r} = [\mu(E)]^{1/r}$, thus $\|F\|_{r,r}(T) = [\mu(T)]^{1/r}$. Now let $x^* \in X^*$ with $\|x^*\| = 1$, $y \in Y$, $\|y\| = 1$ and $U : C(T, X) \to L_r(\mu, Y)$, $U(f) = J(x^* f)y$. Then $G(E) = (x^* \otimes y)F(E)$ is the representing measure of U and it is clear that $l_r(Uf_i \mid i = 1, n) \le \|J\|_r w_p(x^* f_i \mid i = 1, n) \le [\mu(T)]^{1/r} w_p(f_i \mid i = 1, n)$, that is, U is an r-absolutely summing operator with $\|G\|_{r,r}(T) = \|U\|_{r,r} = [\mu(T)]^{1/r}$. □

The following theorem is an extension of a result from [1, Proposition 3].

THEOREM 5. *Let $U : C(T, X) \to Y$ be a linear and continuous operator, G its representing measure. If $G(E) \in \mathrm{As}_{r,p}(X, Y)$ for each $E \in \sum$ and $G : \sum \to \mathrm{As}_{r,p}(X, Y)$ has finite variation with respect to the (r, p)-absolutely summing norm on $\mathrm{As}_{r,p}(X, Y)$, then U is an (r, p)-absolutely summing operator.*

Proof. We consider $\hat{U} : B(\sum, X) \to Y$, $\hat{U}(f) = \int_T f\, dG$, $f \in B(\sum, X)$. Since $\hat{U}$ is an extension of U to $B(\sum, X)$ and $S(\sum, X)$ is dense in $B(\sum, X)$ it suffices to prove that $\hat{U}$ is (r, p)-absolutely summing on $S(\sum, X)$. Let $f_1, \dots, f_n \in S(\sum, X)$. Then there is $\{E_1, \dots, E_k\} \subset \sum$, a finite partition of T and $x_{ij} \in X$ such that $f_i = \sum_{j=1}^k \chi_{E_j} x_{ij}$, for each $i = 1, \dots, n$. Then

$$\begin{aligned} l_r\big(\hat{U} f_i \mid i = 1, n\big) &= l_r\left(\sum_{j=1}^k G\big(E_j\big) x_{ij} \mid i = 1, n\right) \\ &\le \sum_{j=1}^k l_r\big(G\big(E_j\big) x_{ij} \mid i = 1, n\big) \\ &\le \sum_{j=1}^k \big\|G\big(E_j\big)\big\|_{r,p} w_p\big(x_{ij} \mid i = 1, n\big), \end{aligned} \tag{10}$$

since G takes its values in $\mathrm{As}_{r,p}(X, Y)$. But $w_p(f_i \mid i = 1, n) \ge \max_{j=1,k} w_p \times (x_{ij} \mid i = 1, n)$ (because if $\|x^*\| \le 1$, $t \in E_j$, $j = 1, k$ then $w_p(f_i \mid i = 1, n) \ge (\sum_{i=1}^n |\langle f_i, x^* \otimes \delta_t\rangle|^p)^{1/p} = (\sum_{i=1}^n |x^* f_i(t)|^p)^{1/p} = (\sum_{i=1}^n |x^*(x_{ij})|^p)^{1/p}$) thus,

$$\begin{aligned} l_r\big(\hat{U} f_i \mid i = 1, n\big) &\le \left(\sum_{j=1}^k \big\|G\big(E_j\big)\big\|_{r,p}\right) w_p\big(f_i \mid i = 1, n\big) \\ &\le |G|_{r,p}(T) w_p\big(f_i \mid i = 1, n\big), \end{aligned} \tag{11}$$

since G has finite variation with respect to the (r, p)-absolutely summing norm on $\mathrm{As}_{r,p}(X, Y)$ (here, $|G|_{r,p}(T)$ is the variation of G with respect to the (r, p)-absolutely summing norm on $\mathrm{As}_{r,p}(X, Y)$). This shows that U is (r, p)-absolutely summing and $\|U\|_{r,p} \le |G|_{r,p}(T)$ and the proof is finished. □

In the next theorems we give two applications of the results of Theorem 5.

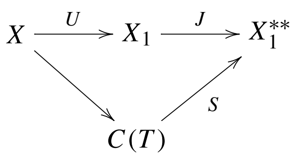
THEOREM 6. *Let* $U : C(T) \to Y$ *be an absolutely summing operator,* $V \in \mathrm{As}_{r,p}(X, Z)$*. Then the injective tensor product* $U \bigotimes_\varepsilon V$ *is an element of* $\mathrm{As}_{r,p}(C(T, X), Y \bigotimes_\epsilon Z)$.

Proof. Let $F \in rcabv(\sum, Y)$ be the representing measure of U, (see [3]). Then $G(E)x = F(E) \bigotimes V(x), x \in X, E \in \sum$ is the representing measure of $U \bigotimes_\epsilon V$. In addition, $G(E) \in \mathrm{As}_{r,p}(X, Y \bigotimes_\epsilon Z)$ and $\|G(E)\|_{r,p} \le \|F(E)\| \, \|V\|_{r,p}$ for $E \in \sum$. Indeed, for $E \in \sum$, let $S_E : Z \to Y \bigotimes_\epsilon Z$, $S_E(z) = F(E) \bigotimes z$. Then $G(E) = S_E V$, hence, because $(\mathrm{As}_{r,p}, \| \, \|_{r,p})$ is a normed ideal of operators and $V \in \mathrm{As}_{r,p}(X, Z)$, we obtain that $G(E) \in \mathrm{As}_{r,p}(X, Y \bigotimes_\epsilon Z)$ and $\|G(E)\|_{r,p} \le \|S_E\| \, \|V\|_{r,p}$. But $\|S_E\| \le \|F(E)\|$ and hence $\|G(E)\|_{r,p} \le \|F(E)\| \, \|V\|_{r,p}$. Now F has bounded variation and hence G satisfies the properties from Theorem 6. Thus, $U \bigotimes_\epsilon V \in \mathrm{As}_{r,p}(C(T, X), Y \bigotimes_\epsilon Z)$. □

In [2, Chapter 34], various results concerning tensor stability and tensor instability of some operator ideals are given. In the next theorem, we prove a result of the same type.

THEOREM 7. *Let* $U : X \to X_1$ *be an integral operator,* $V \in \mathrm{As}_{r,p}(Y, Y_1)$*. Then* $U \bigotimes_\epsilon V \in \mathrm{As}_{r,p}(X \bigotimes_\epsilon Y, X_1 \bigotimes_\epsilon Y_1)$ *and* $\|U \bigotimes_\epsilon V\|_{r,p} \le \|U\|_{\mathrm{int}} \|V\|_{r,p}$.

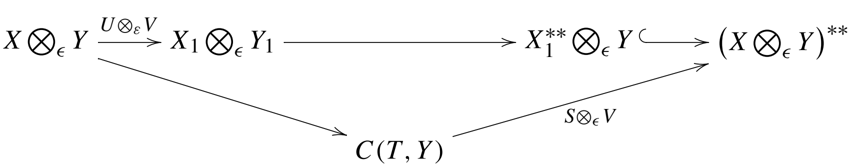
Proof. As U is an integral operator, we have the factorization



(12)

where S is an absolutely summing operator (T being a compact Hausdorff space), (see [2, 3]).

Hence we have the following factorization of $U \bigotimes_\epsilon V$



(13)

(For the inclusion $X_1^{**} \bigotimes_\epsilon Y_1 \hookrightarrow (X_1 \bigotimes_\epsilon Y_1)^{**}$, see [4, Lemma 1].) Using Theorem 6 it follows that $S \bigotimes_\epsilon V \in \mathrm{As}_{r,p}(C(T, Y), X_1^{**} \bigotimes_\epsilon Y_1)$, hence by the ideal property of $\mathrm{As}_{r,p}$ we obtain that $J(U \bigotimes_\epsilon V) \in \mathrm{As}_{r,p}(X \bigotimes_\epsilon Y,$

$X_1 \bigotimes_\epsilon Y_1)^{**}$, where J is the canonical embedding into the bidual, and hence $U \bigotimes_\epsilon V \in \mathrm{As}_{r,p}(X \bigotimes_\epsilon Y, X_1 \bigotimes_\epsilon Y_1)$.

The inequality $\|U \bigotimes_\epsilon V\|_{r,p} \leq \|U\|_{\mathrm{int}} \|V\|_{r,p}$ is also clear. □

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