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Research Article

Certain Subclasses of Bi-Close-to-Convex Functions Associated with Quasi-Subordination

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In the present investigation, we introduce certain new subclasses of the class of biunivalent functions in the open unit disc $U = \{z : |z| < 1\}$ defined by quasi-subordination. We obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in these subclasses. The results present in this paper would generalize and improve those in related works of several earlier authors.

1. Introduction and Preliminaries

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, let S be the class of functions $f(z) \in A$ and univalent in U.

By B, we denote the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the unit disc U and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in U. \tag{2}$$

Firstly, it is necessary to recall some fundamental definitions to acquaint with the main content:

$$S^* = \left\{ f : f \in A, \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0;\ z \in U \right\},$$

the class of starlike functions.

$$K = \left\{ f : f \in A, \operatorname{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0;\ z \in U \right\},$$

the class of convex functions.

$$C = \left\{ f : f \in A, \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0,\ g(z) \in S^*;\ z \in U \right\},$$

the class of close-to-convex functions.

$$C_1 = \left\{ f : f \in A, \operatorname{Re}\left(\frac{(zf'(z))'}{h'(z)}\right) > 0,\ h(z) \in K;\ z \in U \right\},$$

the class of quasi-convex functions.

(3)

The functions in the class S are invertible but their inverse function may not be defined on the entire unit disc U. The Koebe-one-quarter theorem [1] ensures that the image of U under every function $f \in S$ contains a disc of radius 1/4. Thus every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U),$$
$$f(f^{-1}(w)) = w\left(|w| < r_0(f) :\ r_0(f) \geq \frac{1}{4}\right) \tag{4}$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \quad (5)$$

A function $f \in A$ is said to be biunivalent in U if both f and f^{-1} are univalent in U.

Accordingly, a function $f \in A$ is said to be bistarlike, biconvex, bi-close-to-convex, or bi-quasi-convex if both f and f^{-1} are starlike, convex, close-to-convex, or quasi-convex respectively.

Let Σ denote the class of biunivalent functions in U given by (1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \quad (6)$$

and so on. However, the familiar Koebe function $f(z) = z/(1-z)^2$ is not a member of Σ.

Let f and g be two analytic functions in U. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in B$ such that $f(z) = g(u(z))$. This result is known as principle of subordination.

Robertson [2] introduced the concept of quasi-subordination in 1970. For two analytic functions f and ϕ, the function f is said to be quasi-subordinate to ϕ (symbolically $f \prec_q \phi$) if there exist analytic functions k and ω with $|k(z)| \le 1$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{f(z)}{k(z)} \prec \phi(z), \quad (7)$$

or equivalently

$$f(z) = k(z)\phi(\omega(z)). \quad (8)$$

Particularly if $k(z) = 1$, then $f(z) = \phi(\omega(z))$, so that $f(z) \prec \phi(z)$ in U. So it is obvious that the quasi-subordination is a generalization of the usual subordination. The work on quasi-subordination is quite extensive which includes some recent investigations [3–6].

Lewin [7] investigated the class Σ of biunivalent functions and obtained the bound for the second coefficient. Brannan and Taha [8] considered certain subclasses of biunivalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike, and convex functions. They introduced bistarlike functions and biconvex functions and obtained estimates on the initial coefficients. Also the subclasses of bi-close-to-convex functions were studied by various authors [9–11].

Motivated by earlier work on bi-close-to-convex and quasi-subordination, we define the following subclasses.

Also it is assumed that $\phi(z)$ is analytic in U with $\phi(0) = 1$ and let

$$\phi(z) = 1 + B_1z + B_2z^2 + \dots \quad (B_1 \in R^+), \quad (9)$$

$$k(z) = A_0 + A_1z + A_2z^2 + \dots \quad (|k(z)| \le 1, \ z \in U). \quad (10)$$

Definition 1. For $0 \le \alpha \le 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $C_\Sigma(\alpha, \gamma, \phi)$ if there exists a bistarlike function $g(z) = z + \sum_{k=2}^{\infty} b_kz^k$ such that

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} - 1\right] \prec_q (\phi(z) - 1), \quad (11)$$

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{wh'(w)}{j(w)} + \alpha\frac{(wh'(w))'}{j'(w)} - 1\right] \prec_q (\phi(w) - 1), \quad (12)$$

where $h = f^{-1}$, $j = g^{-1}$, and $z, w \in U$.

For $\alpha = 0$, the class $C_\Sigma(\alpha, \gamma, \phi)$ reduces to $C_\Sigma(\gamma, \phi)$, the class of bi-close-to-convex functions of complex order γ defined by quasi-subordination.

Definition 2. For $0 \le \alpha \le 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $C_\Sigma^1(\alpha, \gamma, \psi)$ if there exists a biconvex function $s(z) = z + \sum_{k=2}^{\infty} d_kz^k$ and satisfy the following conditions:

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{zf'(z)}{s(z)} + \alpha\frac{(zf'(z))'}{s'(z)} - 1\right] \prec_q (\psi(z) - 1), \quad (13)$$

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{wv'(w)}{t(w)} + \alpha\frac{(wv'(w))'}{t'(w)} - 1\right] \prec_q (\psi(w) - 1), \quad (14)$$

where $v = f^{-1}$, $t = s^{-1}$, and $z, w \in U$.

It is interesting to note that, for $\alpha = 0$, $C_\Sigma^1(0, \gamma, \psi)$ is the subclass of bi-close-to-convex functions of complex order γ defined by quasi-subordination. Also for $\alpha = 1$, $C_\Sigma^1(1, \gamma, \psi)$ is the class of bi-quasi-convex functions of complex order γ defined by quasi-subordination.

For deriving our main results, we need the following lemmas.

Lemma 3 (see [12]). *If $p \in P$ is family of all functions p analytic in U for which $\mathrm{Re}[p(z)] > 0$ and have the form $p(z) = 1 + p_1z + p_2z^2 + \dots$ for $z \in U$, then $|p_n| \le 2$ for each n.*

Lemma 4 (see [13]). *If $g(z) = z + \sum_{k=2}^{\infty} b_kz^k$ is a starlike function, then*

$$\left|b_3 - b_2^2\right| \le 1. \quad (15)$$

Lemma 5 (see [13]). *If $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is a convex function, then*

$$\left|b_3 - b_2^2\right| \le \frac{1}{3}. \tag{16}$$

Along with the above lemmas, the following well known results are very useful to derive our main results.

Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be an analytic function in A of the form (1), then $|b_n| \le n$, if $g(z)$ is starlike and $|b_n| \le 1$, if $g(z)$ is convex.

2. Coefficient Bounds for the Function Class $C_\Sigma(\alpha,\gamma,\phi)$

Theorem 6. *If $f \in C_\Sigma(\alpha,\gamma,\phi)$, then*

$$\begin{aligned} |a_2| \le \min.&\left[\frac{1}{2}\left[\frac{|A_0\gamma| B_1}{1+\alpha} + 2\right],\right. \\ &\left.\sqrt{\frac{4(1+4\alpha)}{3(1+2\alpha)} + \frac{2(1+3\alpha)|A_0\gamma| B_1}{3(1+\alpha)(1+2\alpha)} + \frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{3(1+2\alpha)}}\right], \end{aligned} \tag{17}$$

$$\begin{aligned} |a_3| \le{}& \frac{(4\alpha^2 + 5\alpha + 2)}{(1+\alpha)^2(1+2\alpha)^2} |A_0\gamma|^2 B_1^2 + \frac{1}{3} \\ &+ \frac{4|A_0\gamma| B_1 + (3|A_0\gamma||B_1 - B_2| + |A_1\gamma| B_1)}{3(1+2\alpha)} + \frac{2|A_0\gamma| B_1}{(1+2\alpha)} \\ &\cdot \sqrt{\frac{(3\alpha^2 + 3\alpha + 1) B_1^2 |\gamma|^2}{(1+2\alpha)^2(1+\alpha)^2} + \frac{|A_0\gamma|(B_1 + |B_1 - B_2|)}{(1+2\alpha)}}. \end{aligned} \tag{18}$$

Proof. As $f \in C_\Sigma(\alpha,\gamma,\phi)$, so by Definition 1 and using the concept of quasi-subordination, there exist Schwarz functions $r(z)$ and $s(z)$ and analytic function $k(z)$ such that

$$\begin{aligned} &\frac{1}{\gamma}\left[(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} - 1\right] \\ &\quad = k(z)(\phi(r(z) - 1), \end{aligned} \tag{19}$$

$$\begin{aligned} &\frac{1}{\gamma}\left[(1-\alpha)\frac{wh'(w)}{j(w)} + \alpha\frac{(wh'(w))'}{j'(w)} - 1\right] \\ &\quad = k(w)(\phi(s(w)) - 1) \end{aligned} \tag{20}$$

where $r(z) = 1 + r_1 z + r_2 z^2 + \ldots$ and $s(w) = 1 + s_1 w + s_2 w^2 + \ldots$.

Define the functions $p(z)$ and $q(z)$ by

$$r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \ldots\right], \tag{21}$$

$$s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2}\left[d_1 z + \left(d_2 - \frac{d_1^2}{2}\right) z^2 + \ldots\right]. \tag{22}$$

Using (21) and (22) in (19) and (20), respectively, it yields

$$\begin{aligned} &\frac{1}{\gamma}\left[(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} - 1\right] \\ &\quad = k(z)\left[\phi\left(\frac{p(z) - 1}{p(z) + 1}\right) - 1\right], \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{1}{\gamma}\left[(1-\alpha)\frac{wh'(w)}{j(w)} + \alpha\frac{(wh'(w))'}{j'(w)} - 1\right] \\ &\quad = k(w)\left[\phi\left(\frac{q(w) - 1}{q(w) + 1}\right) - 1\right]. \end{aligned} \tag{24}$$

But

$$\begin{aligned} &\frac{1}{\gamma}\left[(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} - 1\right] = \frac{1}{\gamma}\left[(1+\alpha)\right. \\ &\quad \cdot (2a_2 - b_2) z + \left((1+2\alpha)(3a_3 - b_3)\right. \\ &\quad \left.\left. + (1+3\alpha)\left(b_2^2 - 2a_2 b_2\right)\right) z^2 + \ldots\right], \end{aligned} \tag{25}$$

$$\begin{aligned} &\frac{1}{\gamma}\left[(1-\alpha)\frac{wh'(w)}{j(w)} + \alpha\frac{(wh'(w))'}{j'(w)} - 1\right] = \frac{1}{\gamma}\left[(1\right. \\ &\quad + \alpha)(b_2 - 2a_2) w \\ &\quad + \left((1+2\alpha)\left[2\left(3a_2^2 - b_2^2\right) - (3a_3 - b_3)\right]\right. \\ &\quad \left.\left. + (1+3\alpha)\left(b_2^2 - 2a_2 b_2\right)\right) w^2 + \ldots\right]. \end{aligned} \tag{26}$$

Again using (9) and (10) in (21) and (22), respectively, we get

$$\begin{aligned} k(z)&\left[\phi\left(\frac{p(z) - 1}{p(z) + 1}\right) - 1\right] = \frac{1}{2} A_0 B_1 c_1 z \\ &+ \left[\frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0 B_2 c_1^2}{4}\right] z^2 \\ &+ \ldots, \end{aligned} \tag{27}$$

$$\begin{aligned} k(w)&\left[\phi\left(\frac{q(w) - 1}{q(w) + 1}\right) - 1\right] = \frac{1}{2} A_0 B_1 d_1 w \\ &+ \left[\frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0 B_2 d_1^2}{4}\right] \\ &\cdot w^2 + \ldots \end{aligned} \tag{28}$$

Using (25) and (27) in (23) and equating the coefficients of z and z^2, we get

$$\frac{(1+\alpha)}{\gamma}(2a_2 - b_2) = \frac{1}{2} A_0 B_1 c_1, \tag{29}$$

$$\frac{(1+2\alpha)\left(3a_3-b_3\right)+(1+3\alpha)\left(b_2^2-2a_2b_2\right)}{\gamma} = \frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2. \quad (30)$$

Again using (26) and (28) in (24) and equating the coefficients of w and w^2, we get

$$\frac{(1+\alpha)}{\gamma}\left(b_2-2a_2\right)=\frac{1}{2}A_0B_1d_1, \quad (31)$$

$$\frac{(1+2\alpha)\left[2\left(3a_2^2-b_2^2\right)-\left(3a_3-b_3\right)\right]+(1+3\alpha)\left(b_2^2-2a_2b_2\right)}{\gamma} = \frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2. \quad (32)$$

From (29) and (31), it is clear that

$$c_1=-d_1, \quad (33)$$

$$a_2=\frac{A_0B_1c_1\gamma}{4(1+\alpha)}+\frac{b_2}{2}=-\frac{A_0B_1d_1\gamma}{4(1+\alpha)}+\frac{b_2}{2}. \quad (34)$$

Therefore on applying triangle inequality and using Lemma 3, (34) yields

$$\left|a_2\right|\le\frac{1}{2}\left[\frac{\left|A_0\gamma\right|B_1}{1+\alpha}+\left|b_2\right|\right]. \quad (35)$$

As $g(z)$ is starlike, so it is well known that $|b_2|\le 2$, (35) gives

$$\left|a_2\right|\le\frac{1}{2}\left[\frac{\left|A_0\gamma\right|B_1}{1+\alpha}+2\right]. \quad (36)$$

Adding (30) and (32), it yields

$$a_2^2=-\frac{2\alpha}{6(1+2\alpha)}b_2^2+\frac{4(1+3\alpha)}{6(1+2\alpha)}a_2b_2 +\frac{A_0B_1\left(c_2+d_2\right)\gamma}{12(1+2\alpha)}+\frac{A_0\left(B_2-B_1\right)\left(c_1^2+d_1^2\right)\gamma}{24(1+2\alpha)}. \quad (37)$$

Using (36) and on applying triangle inequality in (37), we obtain

$$\left|a_2\right|^2\le\frac{(1+4\alpha)}{3(1+2\alpha)}\left|b_2\right|^2+\frac{(1+3\alpha)\left|A_0\gamma\right|B_1\left|b_2\right|}{3(1+\alpha)(1+2\alpha)} +\frac{\left|A_0\gamma\right|\left(B_1+\left|B_2-B_1\right|\right)}{3(1+2\alpha)}. \quad (38)$$

As $g(z)$ is starlike, so using $|b_2|\le 2$ in (38), it yields

$$\left|a_2\right| \le\sqrt{\frac{4(1+4\alpha)}{3(1+2\alpha)}+\frac{2(1+3\alpha)\left|A_0\gamma\right|B_1}{3(1+\alpha)(1+2\alpha)}+\frac{\left|A_0\gamma\right|\left(B_1+\left|B_2-B_1\right|\right)}{3(1+2\alpha)}}. \quad (39)$$

So, result (17) can be easily obtained from (36) and (39).

Now subtracting (32) from (30), we obtain

$$a_3=\frac{b_3-b_2^2}{3}+a_2^2 +\frac{A_1B_1\left(c_1-d_1\right)+A_0B_1\left(c_2-d_2\right)}{12(1+2\alpha)}\gamma. \quad (40)$$

Applying triangle inequality and using Lemma 3 in (40), it yields

$$\left|a_3\right|\le\frac{\left|b_3-b_2^2\right|}{3}+\left|a_2\right|^2+\frac{\left(\left|A_1\gamma\right|+\left|A_0\gamma\right|\right)B_1}{3(1+2\alpha)}. \quad (41)$$

Again adding (30) and (32) and applying triangle inequality, we get

$$\left|a_2\right|^2\le\frac{2\left|A_0\gamma\right|B_1}{(1+2\alpha)}\left[\frac{\left|A_0\gamma\right|B_1}{(1+2\alpha)} +\sqrt{\frac{\left(3\alpha^2+3\alpha+1\right)B_1^2\left|\gamma\right|^2}{(1+2\alpha)^2(1+\alpha)^2}+\frac{\left|A_0\gamma\right|\left(B_1+\left|B_1-B_2\right|\right)}{(1+2\alpha)}}\right] +\frac{\alpha}{(1+\alpha)^2(1+2\alpha)}\left|A_0\gamma\right|^2B_1^2 +\frac{\left|A_0\gamma\right|\left(B_1+\left|B_1-B_2\right|\right)}{(1+2\alpha)}. \quad (42)$$

Using (42) in (41), it gives

$$\left|a_3\right|\le\frac{\left(4\alpha^2+5\alpha+2\right)}{(1+\alpha)^2(1+2\alpha)^2}\left|A_0\gamma\right|^2B_1^2+\frac{\left|b_3-b_2^2\right|}{3} +\frac{4\left|A_0\gamma\right|B_1+\left(3\left|A_0\gamma\right|\left|B_1-B_2\right|+\left|A_1\gamma\right|B_1\right)}{3(1+2\alpha)} +\frac{2\left|A_0\gamma\right|B_1}{(1+2\alpha)} \cdot\sqrt{\frac{\left(3\alpha^2+3\alpha+1\right)B_1^2\left|\gamma\right|^2}{(1+2\alpha)^2(1+\alpha)^2}+\frac{\left|A_0\gamma\right|\left(B_1+\left|B_1-B_2\right|\right)}{(1+2\alpha)}}. \quad (43)$$

On applying Lemma 4 in (43), the result (18) is obvious. □

For $\alpha=0$, Theorem 6 gives the following result.

Corollary 7. *If $f(z)\in C_\Sigma(0,\gamma,\phi)$, then*

$$\left|a_2\right|\le\min.\left[\frac{1}{2}\left[\left|A_0\gamma\right|B_1+2\right], \sqrt{\frac{4}{3}+\frac{2\left|A_0\gamma\right|B_1}{3}+\frac{\left|A_0\gamma\right|\left(B_1+\left|B_2-B_1\right|\right)}{3}}\right], \quad (44)$$

$$\begin{aligned}|a_3| \le {} & 2|A_0\gamma|^2 B_1^2 + \frac{1}{3} \\ & + \frac{4|A_0\gamma| B_1 + (3|A_0\gamma||B_1 - B_2| + |A_1\gamma| B_1)}{3} \\ & + 2|A_0\gamma| B_1 \sqrt{B_1^2|\gamma|^2 + |A_0\gamma|(B_1 + |B_1 - B_2|)}.\end{aligned} \tag{45}$$

3. Coefficient Bounds for the Function Class $C_\Sigma^1(\alpha,\gamma,\phi)$

Theorem 8. *If $f \in C_\Sigma^1(\alpha,\gamma,\phi)$, then*

$$\begin{aligned}|a_2| \le \min.\Bigg[& \frac{1}{2}\left[\frac{|A_0\gamma| B_1}{1+\alpha} + 1\right], \\ & \sqrt{\frac{(1+4\alpha)}{3(1+2\alpha)} + \frac{(1+3\alpha)|A_0\gamma| B_1}{3(1+\alpha)(1+2\alpha)} + \frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{3(1+2\alpha)}}\Bigg],\end{aligned} \tag{46}$$

$$\begin{aligned}|a_3| \le {} & \frac{(4\alpha^2 + 5\alpha + 2)}{(1+\alpha)^2(1+2\alpha)^2}|A_0\gamma|^2 B_1^2 + \frac{1}{9} \\ & + \frac{4|A_0\gamma| B_1 + (3|A_0\gamma||B_1 - B_2| + |A_1\gamma| B_1)}{3(1+2\alpha)} + \frac{2|A_0\gamma| B_1}{(1+2\alpha)} \\ & \cdot \sqrt{\frac{(3\alpha^2 + 3\alpha + 1) B_1^2 |\gamma|^2}{(1+2\alpha)^2(1+\alpha)^2} + \frac{|A_0\gamma|(B_1 + |B_1 - B_2|)}{(1+2\alpha)}}.\end{aligned} \tag{47}$$

Proof. On applying Lemmas 3 and 5 and following the arguments as in Theorem 6, the proof of this theorem is obvious. □

On putting $\alpha = 0$, Theorem 8 gives the following result.

Corollary 9. *If $f(z) \in C_\Sigma^1(0,\gamma,\phi)$, then*

$$\begin{aligned}|a_2| \le \min.\Bigg[& \frac{1}{2}\left[|A_0\gamma| B_1 + 1\right], \\ & \sqrt{\frac{1}{3} + \frac{|A_0\gamma| B_1}{3} + \frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{3}}\Bigg],\end{aligned} \tag{48}$$

$$\begin{aligned}|a_3| \le {} & 2|A_0\gamma|^2 B_1^2 + \frac{1}{9} \\ & + \frac{4|A_0\gamma| B_1 + (3|A_0\gamma||B_1 - B_2| + |A_1\gamma| B_1)}{3} \\ & + 2|A_0\gamma| B_1 \sqrt{B_1^2|\gamma|^2 + |A_0\gamma|(B_1 + |B_1 - B_2|)}.\end{aligned} \tag{49}$$

On putting $\alpha = 1$, Theorem 8 gives the following result.

Corollary 10. *If $f(z) \in C_\Sigma^1(1,\gamma,\phi)$, then*

$$\begin{aligned}|a_2| \le \min.\Bigg[& \frac{1}{2}\left[\frac{|A_0\gamma| B_1}{2} + 1\right], \\ & \sqrt{\frac{5}{9} + \frac{2|A_0\gamma| B_1}{9} + \frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{9}}\Bigg],\end{aligned} \tag{50}$$

$$\begin{aligned}|a_3| \le {} & \frac{11}{36}|A_0\gamma|^2 B_1^2 + \frac{1}{9} \\ & + \frac{4|A_0\gamma| B_1 + (3|A_0\gamma||B_1 - B_2| + |A_1\gamma| B_1)}{9} \\ & + \frac{2|A_0\gamma| B_1}{3} \\ & \cdot \sqrt{\frac{7}{36} B_1^2 |\gamma|^2 + \frac{|A_0\gamma|(B_1 + |B_1 - B_2|)}{3}}.\end{aligned} \tag{51}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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