Research Article

Multiple of Solutions for Nonlocal Elliptic Equations with Critical Exponent Driven by the Fractional p-Laplacian of Order s

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In this paper, we study the existence of infinitely many weak solutions for nonlocal elliptic equations with critical exponent driven by the fractional p-Laplacian of order s. We show the above result when $\lambda > 0$ is small enough. We achieve our goal by making use of variational methods, more specifically, the Nehari Manifold and Lusternik-Schnirelmann theory.

1. Introduction

This work is concerned with the existence of weak solutions of the following critical fractional p-Laplacian problem:

$$(-\Delta)_p^s u = |u|^{p_s^*-2} u + \lambda |u|^{r-2} u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a smoothly bounded domain of $\mathbb{R}^N$, $N \geq sp$, $0 < s < 1$, $1 < r < p < p_s^* := Np/(N-sp)$ is the fractional critical Sobolev exponent, and λ is positive parameter.

$(-\Delta)_p^s$ denotes the fractional p-Laplacian operator defined on smooth functions by

$$(-\Delta)_p^s u(x)$$

$$= 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy, \tag{2}$$

$$x \in \mathbb{R}^N.$$

This definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ when $p = 2$. Let us recall the weak formulation of problem (1). Let us set

$$[u]_{p,s} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx\, dy \right)^{1/p} \tag{3}$$

by the Gagliardo seminorm of the measurable function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{p,s} < \infty \right\} \tag{4}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = \left(|u|_p^p + [u]_{p,s}^p \right)^{1/p}, \tag{5}$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We work in the closed linear subspace

$$X_p^s(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}, \tag{6}$$

equivalently renormed by setting $\|\cdot\|_{s,p} = [\cdot]_{s,p}$, which is uniformly convex Banach space. We note that the embedding

$$X_p^s(\Omega) \hookrightarrow L^r(\Omega) \tag{7}$$

is continuous for $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$, where $p_s^* = pN/(N - ps)$ is the fractional critical Sobolev

exponent (note that when $s = 1$ the above exponent reduces to the classical Sobolev exponent 2^*). As in the classical case the technical problem posed by such an exponent is that the Sobolev embedding is not compact. Denote by $S_{s,p}$ the best Sobolev constant of the immersion $X_p^s(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$, that is,

$$S_{s,p} := \inf \left\{ \|u\|_{s,p}^p : |u|_{p_s^*} = 1, u \in X_p^s(\Omega) \right\}. \tag{8}$$

The dual space of $X_p^s(\Omega)$ is $X_p^s(\Omega)^*$.

Recently, a great attention has been focused on the study of the fractional Laplacian and nonlocal operators of elliptic; this type arises in both pure mathematical research and concrete applications, such as the thin obstacle problem [1, 2], minimal surfaces [3], phase transitions [4], crystal dislocation [5], Markov processes [6], and fractional quantum mechanics [7]. This is one of the reasons why nonlocal fractional problems are widely studied in the literature in many different contexts (see [8]).

When $s = 1$, our problem becomes a scalar quasilinear elliptic equation as follows:

$$-\Delta_p u = |u|^{p^*-2} u + \lambda |u|^{r-2} u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{9}$$

This has been widely studied by many authors. For example, when Ω is bounded, the existence of nontrivial solution was studied (see, e.g., [9, 10]). The typical difficulty in dealing with (9) is that the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact, so the usual "Convex-Compact" method does not apply directly. As for the existence of infinitely many solutions to (9), by using Lusternik-Schnirelmans theory, J. G. Azorero and I. P. Aloson proved in [11] that if Ω is bounded, $1 < r < p$, and $\lambda > 0$ is small, then (9) has infinitely many solutions. Also the main result of [11] was extended to the equation driven by the operator $-\Delta_p u - \Delta_q u$ by G. Li and X. Liang in [12]. G. M. Figueiredo in [13] generalized the same result of [11] to the elliptic equation generated by the operator $-\text{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)$. Several works have been devoted to study some existence and multiplicity results for fractional problems involving the p-Laplacian operator of the type (1), generalizing therefore some classical results obtained in the scalar case. The reader can find a lot of papers in the literature involving this subject; we cite [14–17]. Our goal is to generalize the results of Garcia Azorero and Peral in [11] to the case of the fractional p-Laplacian $(-\Delta)_p^s$ on a bounded domain.

Two major difficulties arise which have to be dealt with in order to reach the desirable conclusions.

First off, it is hard to prove the existence of infinitely many negative energy solutions for our equation by using the variational method because $\mathcal{F}_\lambda$ does not satisfy the (PS) conditions, more precisely, because the problem in question incorporates critical exponents.

Secondly, the functional $\mathcal{F}_\lambda$ is not bounded from below, so in order to comfortably follow through with our plan, we have to introduce an appropriate truncation to the problem, the choice of which is of utmost importance to the results we get in this paper.

Theorem 1. *Assume that $1 < r < p < p_s^*$. Then there exists $\lambda_0 \in \mathbb{R}_+^*$ such that, for each $\lambda \in (0, \lambda_0)$, problem (1) has infinitely many solutions with negative energy.*

Theorem 1 is new as far as we know and it generalizes a similar result in [11] for the fractional p-Laplacian $(-\Delta)_p^s$ type problem. We mainly follow the way in [11] to prove our main result. The paper is organized as follows. In Section 2, we show that the $(PS)_c$ conditions hold for the related energy functional in certain critical levels. That is, we give in a precise range of compactness for the energy functional related. In Section 3, under the assumptions of Theorem 1 and by application of Ljusternik-Schnirelmann methods, we establish the existence of infinitely many solutions with $\lambda > 0$ small enough.

2. The $(PS)_c$ Condition for the Associated Functional

We recall that a weak solution for problem (1) is a function $u : \Omega \longrightarrow \mathbb{R}$, $u \in X_p^s(\Omega)$ such that

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx\,dy$$
$$- \lambda \int_\Omega |u|^{r-2} u\varphi dx - \int_\Omega |u|^{p_s^*-2} u\varphi dx = 0, \quad \forall \varphi \in X_p^s(\Omega). \tag{10}$$

Now let us consider the functional $\mathcal{F}_\lambda : X_p^s(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{F}_\lambda(u) = \frac{1}{p}\|u\|_{s,p}^p - \frac{\lambda}{r}|u|_r^r - \frac{1}{p_s^*}|u|_{p_s^*}^{p_s^*}. \tag{11}$$

Note that the functional $\mathcal{F}_\lambda \in C^1(X_p^s(\Omega), \mathbb{R})$ and its derivative at $u \in X_p^s(\Omega)$ are given by

$$\langle \mathcal{F}_\lambda'(u), \varphi \rangle$$
$$= \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx\,dy$$
$$- \lambda \int_\Omega |u|^{r-2} u\varphi dx - \int_\Omega |u|^{p_s^*-2} u\varphi dx, \tag{12}$$

for every $\varphi \in X_p^s(\Omega)$. Thus, the weak solutions of problem (1) are precisely the critical points of the energy functional $\mathcal{F}_\lambda$. Since problem (1) has a variational structure, the proof of the main result (Theorem 1 and its consequences) reduces to finding critical points of the functional by using suitable abstract approaches. As usual in the critical case, the difficulty related to the variational formulation of (1) is the lack of compactness of the injection of the fractional Sobolev space $X_p^s(\Omega)$ in $L^{p_s^*}(\Omega)$. To overcome this difficulty in treating problem (1), we show that even if the functional $\mathcal{F}_\lambda$ does not verify globally the Palais-Smale condition, it satisfies such a condition in a suitable range related to the best fractional critical Sobolev constant $S_{s,p}$ noted in (8).

The Nehari manifold associated with $\mathcal{F}_\lambda$ is given by

$$\mathcal{N}_\lambda := \left\{ u \in X_p^s(\Omega) \setminus \{0\} : \left\langle \mathcal{F}_\lambda'(u), u \right\rangle = 0 \right\}, \quad (13)$$

where $\mathcal{F}_\lambda'$ denotes the Gâteaux derivative of $\mathcal{F}_\lambda$.

Definition 2.

(i) For $c \in \mathbb{R}$, a sequence $\{u_n\} \subset X_p^s(\Omega)$ is a $(PS)_c$ for $\mathcal{F}_\lambda$ if $\mathcal{F}_\lambda(u_n) = c + o(1)$ and $\mathcal{F}_\lambda'(u_n) = o(1)$ strongly in $X_p^s(\Omega)^*$ as $n \longrightarrow +\infty$, where $X_p^s(\Omega)^*$ is the dual of $X_p^s(\Omega)$.

(ii) $\mathcal{F}_\lambda$ satisfies the $(PS)_c$ condition in $X_p^s(\Omega)$ if any $(PS)_c$ sequence for $\mathcal{F}_\lambda$ contains a convergent subsequence.

The first step for the $(PS)_c$ sequence to hold is bounded.

Lemma 3. *Let $c \in \mathbb{R}$. If $\{u_n\}$ is $(PS)_c$- sequence for $\mathcal{F}_\lambda$, then $\{u_n\}$ is bounded in $X_p^s(\Omega)$.*

Proposition 4. *There exists a $K > 0$ such that, for any $\lambda > 0$ and*

$$c \leq \frac{s}{N} S_{s,p}^{N/sp} - K\lambda^{p_s^*/p_s^* - r}, \quad (14)$$

the functional $\mathcal{F}_\lambda$ satisfies $(PS)_c$ condition.

Proof. Let $\{u_n\}$ be a sequence in $X_p^s(\Omega)$ such that

$$\mathcal{F}_\lambda(u_n) = \frac{1}{p} \|u_n\|_{s,p}^p - \frac{\lambda}{r} |u_n|_r^r - \frac{1}{p_s^*} |u_n|_{p_s^*}^{p_s^*} = c + o(1), \quad (15)$$

$$\left\langle \mathcal{F}_\lambda'(u_n), v \right\rangle$$
$$= \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+sp}} dx\, dy \quad (16)$$
$$- \lambda \int_\Omega |u_n|^{r-2} u_n v dx - \int_\Omega |u_n|^{p_s^*-2} u_n v dx = o\left(\|u_n\|_{s,p}\right),$$

as $n \longrightarrow \infty$, for all $v \in X_p^s(\Omega)$. Then

$$\frac{s}{N} |u_n|_{p_s^*}^{p_s^*} - \frac{1}{\lambda}\left(\frac{1}{r} - \frac{1}{p}\right)|u_n|_r^r$$
$$= \mathcal{F}_\lambda(u_n) - \frac{1}{p}\left\langle \mathcal{F}_\lambda'(u_n), u_n \right\rangle \quad (17)$$
$$= o\left(\|u_n\|_{s,p}\right) + O(1).$$

From (17) and the Hölder inequality, it is implied that (u_n) is bounded in $X_p^s(\Omega)$. Up to a subsequence, this implies the following:

$$u_n \rightharpoonup u \text{ in } X_p^s(\Omega),$$
$$u_n(x) \longrightarrow u(x) \text{ a.e in } \Omega,$$
$$|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))/|x - y|^{N+sp/p'} \text{ is bounded in } L^{p'}(\mathbb{R}^{2N}),$$
$$u_n \longrightarrow u \text{ in } L^r(\Omega), \ 1 \leq r < p_s^*,$$

$|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))/|x - y|^{N+sp/p'}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$ and converges to $|u(x) - u(y)|^{p-2}(u(x) - u(y))/|x - y|^{N+sp/p'}$, a.e in $\mathbb{R}^{2N}$.

So

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+sp}} dx\, dy$$
$$\longrightarrow \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dx\, dy \quad (18)$$

as $n \longrightarrow \infty$.

Moreover

$$\int_\Omega |u_n|^{r-2} u_n v dx \longrightarrow \int_\Omega |u|^{r-2} uv dx$$
$$\text{and } \int_\Omega |u_n|^{p_s^*-2} u_n v dx \longrightarrow \int_\Omega |u|^{p_s^*-2} uv dx \quad (19)$$

So passing to the limit in (16) shows that $u \in X_p^s(\Omega)$ is a weak solution of (1). Setting $v_n = u_n - u$, we have

$$\|v_n\|_{s,p}^p = \|u_n\|_{s,p}^p - \|u\|_{s,p}^p + o(1) \quad (20)$$

By Brezis-Lieb's Lemma [18], we get

$$|v_n|_{p_s^*}^{p_s^*} = |u_n|_{p_s^*}^{p_s^*} - |u|_{p_s^*}^{p_s^*} + o(1). \quad (21)$$

$$c + o(1) = \frac{1}{p} \|v_n\|_{s,p}^p + \frac{1}{p} \|u\|_{s,p}^p - \frac{1}{p_s^*} |v_n|_{p_s^*}^{p_s^*}$$
$$- \frac{1}{p_s^*} |u|_{p_s^*}^{p_s^*} - \frac{\lambda}{r} |u|_r^r, \quad (22)$$

as $n \longrightarrow +\infty$. Taking $v = u_n$ in (16) and Brezis-Liebs Lemma again, we have

$$\|v_n\|_{s,p}^p = \lambda |u|_r^r + |v_n|_{p_s^*}^{p_s^*} + |u|_{p_s^*}^{p_s^*} + o(1) \quad (23)$$

Since u_n is bounded in $X_p^s(\Omega)$ and converges to u in $L^p(\Omega)$, testing (16) with $v = u$ gives

$$\|u\|_{s,p}^p = \lambda |u|_r^r + |u|_{p_s^*}^{p_s^*}. \quad (24)$$

It follows from (23) and (24) that

$$\|v_n\|_{s,p}^p = |v_n|_{p_s^*}^{p_s^*} + o(1) \quad (25)$$

We suppose that

$$\lim_{n \longrightarrow \infty} \|v_n\|_{s,p}^p = l = \lim_{n \longrightarrow \infty} |v_n|_{p_s^*}^{p_s^*} \quad (26)$$

By the definition of the best constant $S_{s,p}$ given in (8), we have

$$\|v_n\|_{s,p}^p \geq S_{s,p} |v_n|_{p_s^*}^p, \quad (27)$$

so

$$l \geq S_{s,p} l^{p/p_s^*}. \tag{28}$$

If $l = 0$, then the lemma is proved. If $l > 0$, then (28) implies that

$$l \geq S_{s,p}^{N/sp}. \tag{29}$$

From (29) and (22), we have

$$
\begin{aligned}
c &= \frac{1}{p}l + \frac{1}{p}\|u\|_{s,p}^p - \frac{1}{p_s^*}l - \frac{1}{p_s^*}|u|_{p_s^*}^{p_s^*} - \frac{\lambda}{r}|u|_r^r \\
&= \frac{s}{N}l + \left(\frac{1}{p} - \frac{1}{p_s^*}\right)|u|_{p_s^*}^{p_s^*} + \lambda\left(\frac{1}{p} - \frac{1}{r}\right)|u|_r^r \\
&\geq \frac{s}{N}S_{s,p}^{N/sp} + \frac{s}{N}|u|_{p_s^*}^{p_s^*} + \lambda\left(\frac{1}{p} - \frac{1}{r}\right)|u|_r^r \\
&\geq \frac{s}{N}S_{s,p}^{N/sp} + \frac{s}{N}|u|_{p_s^*}^{p_s^*} \\
&\quad + \lambda\left(\frac{1}{p} - \frac{1}{r}\right)|\Omega|^{p_s^* - r/p_s^*}|u|_{p_s^*}^r.
\end{aligned}
\tag{30}
$$

We consider the following function $f(x) = (s/N)x^{p_s^*} - \lambda(1/r - 1/p)|\Omega|^{p_s^* - r/p_s^*}x^r$. This function obtains this absolute minimum for $x > 0$ at point $x_0 = \left(\lambda((p-r)/(p_s^*-p))|\Omega|^{p_s^*-r/p_s^*}\right)^{1/p_s^*-r}$, that is,

$$
\begin{aligned}
f(x) \geq f(x_0) &= -\lambda^{p_s^*/p_s^*-r}|\Omega|\left(\frac{p-r}{p_s^*-p}\right)^{r/(p_s^*-r)} \\
&\quad \cdot \left(\frac{p-r}{p}\right)\left(\frac{p_s^*-r}{rp_s^*}\right),
\end{aligned}
\tag{31}
$$

and let the constant $K = |\Omega|((p-r)/(p_s^*-p))^{r/(p_s^*-r)}((p-r)/p)((p_s^*-r)/rp_s^*)$ be strictly positive because $1 < r < p < p_s^*$.

$$
\begin{aligned}
c &\geq \frac{s}{N}S_{s,p}^{N/sp} + \frac{s}{N}|u|_{p_s^*}^{p_s^*} + \lambda\left(\frac{1}{p} - \frac{1}{r}\right)|\Omega|^{p_s^* - r/p_s^*}|u|_{p_s^*}^r, \\
&\geq \frac{s}{N}S_{s,p}^{N/sp} - K\lambda^{p_s^*/p_s^*-r}.
\end{aligned}
\tag{32}
$$

This leads to a contradiction with (9). Therefore $l = 0$ and the proof is complete. □

3. Proof of the Main Result

Under the hypothesis $1 < r < p < N$, using Sobolev's inequality we obtain

$$\mathscr{F}_\lambda(u) \geq h\left(\|u\|_{s,p}\right) \tag{33}$$

where

$$h(x) = \frac{1}{p}x^p - \frac{1}{p_s^* S_{s,p}^{p_s^*/p}}x^{p_s^*} - \frac{\lambda}{r}C_{p,r}x^r \tag{34}$$

and where $C_{r,p}$ is a positive constant independent of $u \in X_p^s(\Omega)$. An easy computation shows that, for all $0 < \lambda < \lambda_0 = C_{p,r}^{-1}((p_s^* - p)/(p_s^* - r))\left[((p-r)/(p_s^* - r))S_{s,p}^{p_s^*/p}\right]^{(p-r)/(p_s^*-p)}$, the real valued function $x \longmapsto h(x)$ has exactly two positive zeros denoted by R_0 and R_1, and the point R is where h attains its nonnegative maximum and verifies $R_0 < R < R_1$.

We now introduce the following truncation of the functional $\mathscr{F}_\lambda$. Take the nonincreasing function $\tau : \mathbb{R}^+ \longrightarrow [0,1]$ and $C^\infty(\mathbb{R}^+)$ such that

$$
\begin{aligned}
\tau(x) &= 1 \quad \text{if } x \leq R_0, \\
\tau(x) &= 0 \quad \text{if } x \geq R_1.
\end{aligned}
\tag{35}
$$

Let $\varphi(u) = \tau(\|u\|_{s,p})$. We consider the truncated functional

$$
\begin{aligned}
\widetilde{\mathscr{F}}_\lambda(u) &= \frac{1}{p}\|u\|_{s,p}^p - \frac{\lambda}{r}\int_\Omega |u|^r \, dx \\
&\quad - \frac{1}{p_s^*}\int_\Omega |u|^{p_s^*}\varphi(u)\,dx.
\end{aligned}
\tag{36}
$$

Similar to (33), we have

$$\widetilde{\mathscr{F}}_\lambda(u) \geq \overline{h}\left(\|u\|_{s,p}\right) \tag{37}$$

where

$$\overline{h}(x) = \frac{1}{p}x^p - \frac{1}{p_s^* S_{s,p}^{p_s^*/p}}x^{p_s^*}\tau(x) - \frac{\lambda}{r}C_{p,r}x^r \tag{38}$$

Clearly,

$$\overline{h}(x) \geq h(x) \tag{39}$$

for $x \geq 0$ and $\overline{h}(x) = h(x)$ if $0 \leq x \leq R_0$, $\overline{h}(x) \geq 0$, if $R_0 < x \leq R_1$ and if $x > R_1$, $\overline{h}(x) = x^r((1/p)x^{p-r} - (\lambda/r)C_{p,r})$ is strictly increasing and so $\overline{h}(x) > 0$, if $x > R_1$. Consequently

$$\overline{h}(x) \geq 0 \quad \text{for } x \geq R_0. \tag{40}$$

We have the following result.

Lemma 5. *This lemma can be expressed as three assertions:*

(1) *$\widetilde{\mathscr{F}}_\lambda \in \mathscr{C}^1(X_p^s(\Omega), \mathbb{R})$ is even.*

(2) *If $\widetilde{\mathscr{F}}_\lambda(u_0) \leq 0$ then $\|u_0\|_{s,p} < R_0$. Moreover, $\widetilde{\mathscr{F}}_\lambda(u) = \mathscr{F}_\lambda(u)$ for all u in a small enough neighborhood of u_0.*

(3) *There exists $\lambda_0 > 0$, such that if $0 < \lambda < \lambda_0$, then $\widetilde{\mathscr{F}}_\lambda$ verifies a local Palais-Smale condition for $c \leq 0$.*

Proof. Since $\varphi \in \mathscr{C}^\infty$ and $\varphi(u) = 1$ for u near 0, $\widetilde{\mathscr{F}}_\lambda \in \mathscr{C}^1(X_p^s(\Omega), \mathbb{R})$ and assertion (1) holds.

By taking $\widetilde{\mathscr{F}}_\lambda(u_0) \leq 0$, we can deduce from (37) that

$$\overline{h}\left(\|u_0\|_{s,p}\right) \leq 0, \tag{41}$$

and by (40) and (41) we have

$$\|u_0\|_{s,p} < R_0, \tag{42}$$

and (2) holds.

For the proof of (3), let $\{u_n\} \subset X_p^s(\Omega)$ be a $(PS)_c$ sequence $\widetilde{\mathscr{F}}_\lambda$, with $c < 0$. Then we may assume that $\widetilde{\mathscr{F}}_\lambda(u_n) < 0$, $\widetilde{\mathscr{F}}_\lambda'(u_n) \longrightarrow 0$. By (8) in Lemma 5 there exists $\lambda_0 > 0$ such that $0 < \lambda < \lambda_0$, $\|u_n\|_{s,p} < R_0$, so $\widetilde{\mathscr{F}}_\lambda(u_n) = \mathscr{F}_\lambda(u_n)$ and $\widetilde{\mathscr{F}}_\lambda'(u_n) = \mathscr{F}_\lambda'(u_n)$. By Proposition 4, $\mathscr{F}_\lambda$ satisfies $(PS)_c$ condition for $c < 0$, so there is a subsequence $\{u_n\}$ such that $u_n \longrightarrow u$ in $X_p^s(\Omega)$. Thus $\widetilde{\mathscr{F}}_\lambda$ satisfies $(PS)_c$ condition for $c < 0$. $\qquad\square$

We will use the genus of symmetric set in $X_p^s(\Omega)$, where the genus $\gamma(A)$ is the smallest integer m, such that there exists an odd map

$$\phi \in \mathscr{C}\left(A, \mathbb{R}^m \setminus \{0\}\right), \tag{43}$$

where A is a closed symmetric set in X that does not contain zero (see [19]).

It is possible to prove the existence of level sets of $\widetilde{I}_\lambda$ with arbitrarily large genus, more precisely,

Lemma 6. $\forall n \in \mathbb{N}$ $\exists \epsilon(n) > 0$ such that

$$\gamma\left(\left\{u \in X_p^s(\Omega) : \widetilde{\mathscr{F}}_\lambda(u) \le -\epsilon(n)\right\}\right) \ge n. \tag{44}$$

Proof. Let $n \in \mathbb{N}$. we consider E_n to be subspaces of $X_p^s(\Omega)$ with E_n being an n-dimensional subspace of $X_p^s(\Omega)$. Let $\{u_n\} \in E_n$ with norm $\|u_n\|_{s,p} = 1$. For $0 < \rho < \overline{R_0}$

$$\begin{aligned}
\widetilde{\mathscr{F}}_\lambda(\rho u_n) &\le \frac{1}{p}\rho^p - \frac{\rho^{p_s^*}}{p_s^*}\int_\Omega |u_n|^{p_s^*}\,dx \\
&\quad - \frac{\lambda}{r}\rho^r \int_\Omega |u_n|^r\,dx
\end{aligned} \tag{45}$$

we define

$$\alpha_n := \inf\left\{\int_\Omega |u_n|^{p_s^*}\,dx : u_n \in E_n, \|u_n\|_{s,p} = 1\right\} > 0, \tag{46}$$

and

$$\beta_n := \inf\left\{\int_\Omega |u_n|^r : u_n \in E_n, \|u_n\|_{s,p} = 1\right\} > 0. \tag{47}$$

By using the definitions of α_n, β_n, and inequality (45), we obtain

$$\widetilde{\mathscr{F}}_\lambda(\rho u_n) \le \frac{1}{p}\rho^p - \frac{\alpha_n}{p_s^*}\rho^{p_s^*} - \frac{\lambda\beta_n}{r}\rho^r. \tag{48}$$

Then, there exists $\epsilon(n) > 0$ and $0 < \rho < R_0$ such that

$$\widetilde{\mathscr{F}}_\lambda(\rho u) \le -\epsilon(n) \tag{49}$$

for $u \in E_n$ and $\|u_n\| = 1$.

Let $S_\eta = \{u \in X_p^s(\Omega)/\|u\| = \eta\}$, so

$$S_\eta \cap E_n \subset \left\{u \in X_p^s(\Omega)/\widetilde{\mathscr{F}}_\lambda(u) \le -\epsilon(n)\right\}, \tag{50}$$

therefore, by the properties of the genus (see [19])

$$\gamma\left(\left\{u \in X_p^s(\Omega)/\widetilde{\mathscr{F}}_\lambda(u) \le -\epsilon\right\}\right) \ge \gamma\left(S_\eta \cap E_n\right) \ge n. \tag{51}$$

$$\square$$

We are now in a position to prove our main result.

Proof of Theorem 1. For $n \in \mathbb{N}$, we define

$$\Gamma_n = \left\{A \subset X_p^s(\Omega) - \{0\}/A \text{ is close}, \ A = -A, \ \gamma(A) \right. \\ \left. \ge n\right\}. \tag{52}$$

Let us set

$$c_n = \min_{A \in \Gamma_n} \max_{u \in A} \widetilde{I}_\lambda(u), \tag{53}$$

and

$$K_c = \left\{u \in X_p^s(\Omega) : \widetilde{\mathscr{F}}_\lambda'(u) = 0, \ \widetilde{\mathscr{F}}_\lambda(u) = c\right\}, \tag{54}$$

and suppose $0 < \lambda < \lambda_o$, where λ_0 is the constant given by Lemma 5.

$$\widetilde{\mathscr{F}}_\lambda^{-\epsilon} = \left\{u \in X_p^s(\Omega)/\widetilde{\mathscr{F}}_\lambda(u) \le -\epsilon\right\}. \tag{55}$$

By Lemma 6 there exists $\epsilon(n) > 0$ such that $\gamma(\widetilde{\mathscr{F}}_\lambda^{-\epsilon}) \ge n$, for all $n \in \mathbb{N}$. Because $\widetilde{\mathscr{F}}_\lambda(u)$ is continuous and even, $\widetilde{\mathscr{F}}_\lambda^{-\epsilon} \in \Gamma_n$, then $c_n \le -\epsilon(n) < 0$ for all n in $\mathbb{N}$. But $\widetilde{\mathscr{F}}_\lambda$ is bounded from below; hence $c_n > -\infty$ for all n in $\mathbb{N}$.

Let us assume that $c = c_n = c_{n+1} = \cdots = c_{n+r}$. Note that $c < 0$; therefore, $\widetilde{\mathscr{F}}_\lambda$ verifies the Plais-Smale condition in c, and it is easy to see that K_c is a compact set.

If $\gamma(K_c) \le r$, there exists a closed and symmetric set U verifying $K_c \subset U$, such that $\gamma(U) \le r$. By the deformation lemma (see [20]), we have an odd homeomorphism $\eta : X \longrightarrow X$, such that $\eta(\widetilde{\mathscr{F}}_\lambda^{c+\delta} - U) \subset \widetilde{\mathscr{F}}_\lambda^{c-\delta}$, for some $\delta > 0$. By definition,

$$c = c_n = \inf_{A \in \Gamma_{n+r}} \sup_{u \in A} \widetilde{\mathscr{F}}_\lambda(u). \tag{56}$$

There exists then $A \in \Gamma_{n+r}$, such that $\sup_{u \in A} \widetilde{\mathscr{F}}_\lambda(u) < c + \delta$, i.e., $A \subset \widetilde{\mathscr{F}}_\lambda^{c+\delta}$,

$$\eta(A - U) \subset \eta\left(\widetilde{\mathscr{F}}_\lambda^{c+\delta} - U\right) \subset \widetilde{\mathscr{F}}_\lambda^{c-\delta}. \tag{57}$$

But $\gamma(\overline{A - U}) \ge \gamma(A) - \gamma(U) \ge n$, and $\gamma(\eta(\overline{A-U})) \ge \gamma(\overline{A - U})) \ge n$.

Then, $\eta(\overline{A - U}) \in \Gamma_n$. And this is a contradiction; in fact, $\eta(\overline{A-U}) \in \Gamma_n$ implies $\sup_{u \in \eta(\overline{A-U})}\widetilde{\mathscr{F}}_\lambda(u) \ge c_n = c$.

So we have proved that $\gamma(K_c) \ge r + 1$. We are now ready to show that $\mathscr{F}_\lambda$ has infinitely many critical point

solutions. Note that c_n is nondecreasing and strictly negative. We distinguish two cases.

Case 1. Suppose that there are $1 < n_1 < \cdots n_i < \cdots$, satisfying

$$c_{n_1} < \cdots < c_{n_i} < \cdots . \tag{58}$$

In this case, we have infinitely many distinct critical points.

Case 2. We assume in this case that, for some positive integer n_0, there is $r \geq 1$ such that $c = c_{n_0} = c_{n_0+1} = \cdots = c_{n_0+r}$; then $\gamma(K_{c_{n_0}}) \geq r + 1$ which shows that $K_{c_{n_0}}$ contains infinitely many distinct elements. Since $\widetilde{\mathscr{F}}_\lambda(u) = \mathscr{F}_\lambda(u)$ if $\widetilde{\mathscr{F}}_\lambda(u) < 0$, we see that there are infinitely many critical points of $\mathscr{F}_\lambda(u)$. The theorem is proved. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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