Hindawi
Abstract and Applied Analysis
Volume 2019, Article ID 6021293, 10 pages
http://dx.doi.org/10.1155/2019/6021293

Research Article

The Existence of Positive Solution for Semilinear Elliptic Equations with Multiple an Inverse Square Potential and Hardy-Sobolev Critical Exponents

M. Khiddi

E.G.A.L, Dépt. Maths, Fac. Sciences, Université Ibn Tofail, BP.133, Kénitra, Morocco

Correspondence should be addressed to M. Khiddi; mostapha-2@hotmail.com

Received 13 November 2018; Accepted 27 January 2019; Published 3 March 2019

Academic Editor: Aref Jeribi

Via the concentration compactness principle, delicate energy estimates, the strong maximum principle, and the Mountain Pass lemma, the existence of positive solutions for a nonlinear PDE with multi-singular inverse square potentials and critical Sobolev-Hardy exponent is proved. This result extends several recent results on the topic.

1. Introduction and Main Result

Let $\Omega \subset \mathbb{R}^N$ be smooth open bounded with $N > 2$. In this paper, we study the existence of solutions to the following nonlocal problem:

$$-\Delta u - \sum_{1}^{k} \mu_i \frac{u}{|x-a_i|^2} = \sum_{1}^{k} K_i(x) \frac{|u|^{2^*(s_i)-2} u}{|x-a_i|^{s_i}} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

for arbitrary k such that $1 \le k < \infty$, $0 < s_i < 2$, $\mu_i > 0$, $a_i \in \Omega$, $a_i \neq a_j$ if $i \neq j$ and $2^*_{s_i} = (2(N-s_i))/(N-2)$, is the critical Sobolev-Hardy exponent ($i = 1, 2, \cdots, k$).

We suppose the following:

($\mathscr{H}_1$) $0 < \mu_i < \overline{\mu}$, *for every* $i = 1, 2, \cdots, k$ and $\sum_{i=1}^{k} \mu_i < \overline{\mu} = ((N-2)/2)^2$.

($\mathscr{H}_2$) There is an i_0, $1 \le i_0 \le k$, such that

$$\min_{1\le i\le k} \left\{ \frac{2-s_i}{2(N-s_i)} \left(S^{a_i}_{\mu_i,s_i}\right)^{(N-s_i)/(2-s_i)}_{\mu_i,s_i} \cdot \left(K_i(a_i)\right)^{-(N-2)/(2-s_i)} \right\} = \frac{2-s_{i_0}}{2(N-s_{i_0})} \left(S^{a_{i_0}}_{\mu_{i_0},s_{i_0}}\right)^{(N-s_{i_0})/(2-s_{i_0})} \cdot \left(K_{i_0}(a_{i_0})\right)^{-(N-2)/(2-s_{i_0})}, \tag{2}$$

and

$$K_{i_0}(x) = K_{i_0}(a_{i_0}) + o\left(\left|x-a_{i_0}\right|^3\right) \quad \text{as } x \longrightarrow a_{i_0}. \tag{3}$$

($\mathscr{H}_3$) $0 < \mu_{i_0} \le \overline{\mu} - 1$ where i_0 is given in ($\mathscr{H}_2$). The function $K_i(x)$ is a positive bounded on $\overline{\Omega}$, for every $(1 \le i \le k)$. Furthermore,

$$0 < \overline{k}_i = \max_{x\in\overline{\Omega}} K(x). \tag{4}$$

The reason why we investigate (1) is the presence of the Hardy-Sobolev exponent and the so-called inverse square potential in the linear part, which cause the loss of compactness of embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$, $H^1_0(\Omega) \hookrightarrow L^2(|x|^{-2}, \Omega)$ and $H^1_0(\Omega) \hookrightarrow L^{2^*(s)}(|x|^{-s}, \Omega)$).

Hence, we face a type of triple loss of compactness whose interacting with each other will result in some new difficulties. In last two decades, loss of compactness leads to many interesting existence and nonexistence phenomena for elliptic equations. Many important results on the singular problems with Hardy-Sobolev critical exponents (the case that $s \neq 0$ and $\mu_i = \mu$ were obtained such as the existence and multiplicity of solutions in these works and these results give us very good insight into the problem; see, for example, [1–7] and references therein. In the present paper, we use a variational method to deal with problem (1) with general form and generalize the results in [8]. As $k \geq 2$ to our knowledge, there are no results on the existence of nontrivial solutions for (1). It is therefore significant for us to study the problem (1) deeply. However, because of the singularities caused by the terms $|x - ai|^{-sp}$ $(i = 1, 2, \ldots, k)$, our problem becomes more complicated to deal with than [8] and therefore we have to face more difficulties. Despite the multiple terms of hardy and the coefficients of the critical nonlinearity, but we will see how, they will play an important role in the search for the bubble whose energy is below the level of local compactness (PS). The existence result is obtained via constructing a minimax level within this range and the Mountain Pass Lemma due to A. Ambrosetti and P.H. Rabinowitz (see also[9]).

Our main result is the following.

Theorem 1. *Assume that conditions* $(\mathscr{H}_1)$, $(\mathscr{H}_2)$, $(\mathscr{H}_3)$ *hold. Then problem (1) has at least one positive solution.*

The paper is organized as follows: in Section 2, preliminary results about Palais-Smale condition for c in a suitable interval and construct some auxiliary functions and estimate their norms. In Section 3, fill the conditions of Mountain Pass Theorem and we establish our result.

2. Preliminary Results

Throughout this paper, C, $C_i (i = 1, 2, 3, \cdots)$ represent all kinds of positive constants. We denote the standard norm of the Sobolev space $H_0^1(\Omega)$ by

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}. \tag{5}$$

$B_r(a)$ is a ball centered at a with radius r. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)| < C\varepsilon^t$ and $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)/\varepsilon^t| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. We will look for solutions of (1) by finding critical points of the C^1–functional $I\colon H_0^1(\Omega) \longrightarrow \mathbb{R}^N$ given by

$$\begin{aligned} I(u) = {} & \frac{1}{2} \left(\int_\Omega |\nabla u|^2 \, dx - \sum_{i=1}^{k} \mu_i \int_\Omega \frac{|u|^2}{|x - a_i|^2} dx \right) \\ & - \sum_{i=1}^{k} \frac{1}{2^*(s_i)} \int_\Omega K_i(x) \frac{|u|^{2^*(s_i)}}{|x - a_i|^{s_i}} dx \end{aligned} \tag{6}$$

for all $H_0^1(\Omega)$. The function $u \in H_0^1(\Omega)$ is said to be a solution of problem (1) if u satisfies

$$\begin{aligned} \left\langle I'(u), u \right\rangle = {} & \int_\Omega \nabla u \nabla \varphi \, dx - \sum_{i=1}^{k} \mu_i \int_\Omega \frac{u\varphi}{|x - a_i|^2} dx \\ & - \sum_{i=1}^{k} \int_\Omega K_i(x) \frac{|u|^{2^*(s_i)}}{|x - a_i|^{s_i}} dx \end{aligned} \tag{7}$$

for all $\varphi \in H_0^1(\Omega)$.

Problem (1) is well defined by the both inequalities, Sobolev-Hardy inequalities which is essentially due to Caffarelli, Kohn, and Nirenberg (see [10]):

$$\left(\int_\Omega \frac{|u|^q}{|x - a|^s} dx \right)^{2/q} \leq C_{s,q} \int_\Omega |\nabla u|^2 \, dx, \qquad \forall u \in H_0^1(\Omega), \ \forall a \in \Omega, \tag{8}$$

where $2 < q < 2^*$, and the Hardy inequality (see [11, 12]), that is a special case ($q = s = 2$) of the above Sobolev-Hardy inequality.

$$\int_\Omega \frac{|u|^2}{|x - a|^2} dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\nabla u|^2 \, dx, \qquad \forall u \in H_0^1(\Omega), \ \forall a \in \Omega. \tag{9}$$

By (8) and (9), for $0 \leq \mu < \bar{\mu}$, $0 \leq s < 2$, $q = 2^*(s)$ and $a \in \Omega$ we can define the best Sobolev-Hardy constant:

$$S_{\mu,s}^a = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx - \mu \int_\Omega \left(|u|^2 / |x - a|^2 \right) dx}{\left(\int_\Omega \left(|u|^{2^*(s)} / |x|^s \right) dx \right)^{2/2^*(s)}}, \tag{10}$$

In the case where $s = 0$, then $(2^*(0) = 2^*)$; note $S_{\mu,0}^a$ is the best constant in the Sobolev inequality, i.e.,

$$S_{\mu,0}^a = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx - \mu \int_\Omega \left(|u|^2 / |x - a|^2 \right) dx}{\left(\int_\Omega |u|^{2^*} \, dx \right)^{2/2^*}}. \tag{11}$$

The best Sobolev-Hardy constant $S_{\mu,s}^a$ is achieved only when $\Omega = \mathbb{R}^N$ by a family of functions:

$$Y_{\mu,\varepsilon}^a(x) = \frac{\left(2\varepsilon (N - s)(\bar{\mu} - \mu) / \sqrt{\bar{\mu}} \right)^{\sqrt{\bar{\mu}}/(2-s)}}{\left(|x - a|^{a(\mu)} \left(\varepsilon + |x - a|^{(2(2-s)/(N-2))\sqrt{\bar{\mu} - \mu}} \right)^{(N-2)/(2-s)} \right)} \qquad \text{for } \mathbb{R}^N \setminus \{0\}. \tag{12}$$

Let

$$C_\varepsilon = \left(\frac{2\varepsilon (N - s)(\bar{\mu} - \mu)}{\sqrt{\bar{\mu}}} \right)^{\sqrt{\bar{\mu}}/(2-s)}, \qquad U_{\mu,\varepsilon}^a(x) = \frac{Y_{\mu,\varepsilon}^a(x)}{C_\varepsilon}, \tag{13}$$

where $b(\mu) = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$ and $a(\mu) = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$, (see [13] for details). Moreover,

$$0 \le a(\mu) < \frac{N-2}{2} < b(\mu) \le N-2. \tag{14}$$

We consider $\rho > 0$ such that $B(a, 2\rho) \subset \Omega$ and define a cut function $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi \le 1$, $|\nabla \varphi| \le C$, $\varphi = 1$ for $|x-a| \le \rho$ and $\varphi = 0$ for $|x-a| > 2\rho$. Set

$$u_{\mu,\varepsilon}^a(x) = \varphi(x) U_{\mu,\varepsilon}^a(x),$$
$$v_{\mu,\varepsilon}^a(x) = \frac{u_{\mu,\varepsilon}^a(x)}{\left(\int_\Omega \left(\left|u_{\mu,\varepsilon}^a(x)\right|^{2^*(s)} / |x-a|^s\right) dx\right)^{1/2^*(s)}}, \tag{15}$$

so that

$$\int_\Omega \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx = 1. \tag{16}$$

Then we have the following estimates.

Lemma 2. *For any* $0 < \mu < \overline{\mu}$ *and* $a \in \Omega$,

$$\int_\Omega \left|\nabla v_{\mu,\varepsilon}^a\right|^2 dx - \mu \int_\Omega \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^2}{|x-a|^2} dx = S_{\mu,s}^a + O\left(\varepsilon^{(N-2)/(2-s)}\right), \tag{17}$$

$$\int_\Omega |x-a|^k \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx = O\left(\varepsilon^{k(N-s)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right), \tag{18}$$

$$\int_\Omega K_i(x) \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx = K_i(a) + O\left(\varepsilon^{3(N-s)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right). \tag{19}$$

For $a \ne b$, *we have*

$$\int_\Omega \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^2}{|x-b|^2} dx = \begin{cases} O\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) & \text{if } \mu < \overline{\mu} - 1, \\ O\left(\varepsilon^{(N-2)/(2-s)} |\ln \varepsilon|\right) & \text{if } \mu = \overline{\mu} - 1, \end{cases} \tag{20}$$

Proof. It is easy to get the following results (17),(18) (see [14].) We show (19) and (20) and for the proof (19). By using (18) and assumption $(\mathscr{H}_3)$ we have

$$\begin{aligned} &\left| \int_\Omega K_i(x) \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx - \int_\Omega K_i(a) \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx \right| \\ &\le \int_\Omega \left|K_i(x) - K_i(a)\right| \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx \\ &\le C_1 \int_\Omega |x-a|^3 \frac{\left|v_{\mu,\varepsilon}^a(x)\right|^{2^*(s)}}{|x-a|^s} dx \\ &= C_2 \varepsilon^{3(N-s)/2(2-s)\sqrt{\overline{\mu}-\mu}}. \end{aligned} \tag{21}$$

Now we show (20). Let $a \ne b$ and $\varepsilon_0 = \varepsilon^{(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}$. We have

$$\begin{aligned} \int_\Omega \frac{\left|v_{\mu,s,\varepsilon}^a(x)\right|^2}{|x-b|^2} dx &= \int_{B_\rho(a)} \frac{\left|W_{\mu,s,\varepsilon}^a(x)\right|^2}{|x-b|^2} dx \\ &+ \int_{B_\rho(a)} \left(\varphi^2(x) - 1\right) \frac{\left|W_{\mu,s,\varepsilon}^a(x)\right|^2}{|x-b|^2} dx = \frac{1}{|a-b|^2} \\ &\cdot \int_{B_\rho(a)} \left|W_{\mu,s,\varepsilon}^a(x)\right|^2 dx \\ &+ \int_{B_\rho(a)} \left(\varphi^2(x) - 1\right) \frac{\left|W_{\mu,s,\varepsilon}^a(x)\right|^2}{|x-b|^2} dx \\ &\cdot \int_{B_\rho(a)} \left(\frac{1}{|x-b|^2} - \frac{1}{|a-b|^2}\right) \left|W_{\mu,s,\varepsilon}^a(x)\right|^2 dx \\ &= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon). \end{aligned} \tag{22}$$

For

$$\begin{aligned} I_1(\varepsilon) &= \frac{1}{|a-b|^2} \int_{B_\rho(a)} \left|W_{\mu,s,\varepsilon}^a(x)\right|^2 dx = C_1 \varepsilon^{(N-2)/(2-s)} \int_{|x| \le \rho} \frac{1}{\varepsilon^{2(N-2)/(2-s)} |x|^{2a(\mu)} \left(1 + |x|^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} dx \\ &= C_1 \varepsilon^{-(N-2)/(2-s) + N(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu} - (N-2)a(\mu)/(2-s)\sqrt{\overline{\mu}-\mu}} \int_0^{\rho/\varepsilon_0} \frac{r^{N-1}}{r^{2a(\mu)} \left(1 + r^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} dr, \end{aligned} \tag{23}$$

we know that

$$\frac{r^{N-1}}{r^{2a(\mu)}\left(1+r^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} \sim \frac{1}{r^{1-N+2a(\mu)}}, \quad \text{in the neighbourhood of } 0$$
$$\frac{r^{N-1}}{r^{2a(\mu)}\left(1+r^{((2-s)/2(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} \sim \frac{1}{r^{1-N+2b(\mu)}}, \quad \text{in the neighbourhood of } +\infty, \tag{24}$$

and since $a(\mu) \le (N-2)/2 < N/2$ we deduct that

$$\int_0^1 \frac{1}{r^{1-N+2a(\mu)}}dr < \infty, \quad \text{for every } 0<\mu<\overline{\mu} \tag{25}$$

$$\int_1^{\rho/\varepsilon_0} \frac{1}{r^{1-N+2b(\mu)}}dr \le \begin{cases} C_1, & \text{if } 0<\mu<\overline{\mu}-1, \\ C_2\left|\ln\varepsilon_0\right|, & \text{if } \mu=\overline{\mu}-1. \end{cases} \tag{26}$$

By using (23) and (26), we obtain

$$\left|I_1(\varepsilon)\right| \le \begin{cases} C_1\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right), & \text{if } 0<\mu<\overline{\mu}-1, \\ C_2\left(\varepsilon^{(N-2)/(2-s)}\left|\ln\varepsilon\right|\right), & \text{if } \mu=\overline{\mu}-1, \end{cases} \tag{27}$$

where $C_1, C_2 > 0$ are constant.
For the second integral,

$$I_2(\varepsilon) = \int_{B_\rho(a)}\left(\frac{1}{|x-b|^2}-\frac{1}{|a-b|^2}\right)\frac{\left|W^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2}dx = C_1\varepsilon^{-(N-2)/(2-s)}\int_{|x-a|\le\rho}\left(\frac{\left(|b-a|^2-|x-b|^2\right)}{|x-b|^2|a-b|^2}\right)$$
$$\cdot\frac{1}{\varepsilon_0^{2a(\mu)}\left|(x-a)/\varepsilon_0\right|^{2a(\mu)}\left(1+\left|(x-a)/\varepsilon_0\right|^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}}dx$$
$$= C_1\varepsilon^{-(N-2)/(2-s)}\varepsilon_0^{N-2a(\mu)}\int_{|y|\le\rho/\varepsilon_0}\frac{\left(-\varepsilon_0^2|y|^2-2\varepsilon_0\langle y,a-b\rangle\right)}{|y|^{2a(\mu)}\left(\left|\varepsilon_0y+a-b\right|^2|a-b|^2\right)\left(1+|y|^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}}dy. \tag{28}$$

For $|y| \le \rho/\varepsilon_0$, we have $|(-\varepsilon_0|y|^2 - 2\langle y,(a-b)\rangle)/|\varepsilon_0 y + (a-b)|^2|a-b|^2| \le (\rho + 2|a-b|/(|a-b|-\rho)^2|a-b|^2)$.

$$\left|I_2(\varepsilon)\right| \le C_2\varepsilon^{-(N-2)/(2-s)-(N-2)a(\mu)/(2-s)\sqrt{\overline{\mu}-\mu}+(N-2)(N+1)/2(2-s)\sqrt{\overline{\mu}-\mu}}\int_{|y|\le\rho/\varepsilon_0}\frac{|y|}{|y|^{2a(\mu)}\left(1+|y|^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}}dy$$
$$= C_2\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}\int_0^{\rho/\varepsilon_0}\frac{r^N}{r^{2a(\mu)}\left(1+r^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}}dr. \tag{29}$$

Since $2a(\mu) - N < 1$, this implies that

$$\int_0^1\frac{r^N}{r^{2a(\mu)}\left(1+r^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}}dr < \infty. \tag{30}$$

Also,

$$\frac{r^N}{r^{2a(\mu)}\left(1+r^{((2-s)/2(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} \le \frac{1}{r^{2b(\mu)-N}}, \tag{31}$$

and

$$2b(\mu) - N > 1 \Longleftrightarrow 0 \le \mu < \overline{\mu} - \frac{9}{4}. \tag{32}$$

Then, if $0 \le \mu < \overline{\mu} - 9/4$, we have

$$\left|I_2(\varepsilon)\right| = \left|\int_{B_\rho(a)}\left(\frac{1}{|x-b|^2}-\frac{1}{|a-b|^2}\right)\frac{\left|W^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2}dx\right| = O\left(\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right). \tag{33}$$

And, if $\mu = \overline{\mu} - 9/4$, we have

$$\begin{aligned} &\left|I_2(\varepsilon)\right| \\ &= \left|\int_{B_\rho(a)} \left(\frac{1}{|x-b|^2} - \frac{1}{|a-b|^2}\right) \frac{\left|W^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2} dx\right| \\ &\le C_1 \varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} \left\{\int_0^1 \frac{1}{r^{2a(\mu)-N}} dr + \int_1^{\rho/\varepsilon_0} \frac{1}{r} dr\right\} \\ &= C_2 \varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} + C_3 \varepsilon^{2(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} |\ln \varepsilon| \\ &= O\left(\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} |\ln \varepsilon|\right). \end{aligned} \tag{34}$$

If $\overline{\mu} - 9/4 < \mu < \overline{\mu}$, this implies that

$$2b(\mu) - N < 1, \tag{35}$$

since $2a(\mu) - N < 1$, we deduct that

$$\int_0^1 \frac{1}{r^{2a(\mu)-N}} dr < \infty \tag{36}$$

$$\begin{aligned} &\left|\int_{B_\rho(a)} \left(\frac{1}{|x-b|^2} - \frac{1}{|a-b|^2}\right) \frac{\left|W^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2} dx\right| \\ &\le C_2 \varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} \left\{\int_0^1 \frac{1}{r^{-N+2a(\mu)}} dr\right. \\ &\left. + \int_1^{\rho/\varepsilon_0} \frac{1}{r^{-N+2b(\mu)}} dr\right\} = C_3 \varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} \\ &+ C_4 \varepsilon^{(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}+(N-2)/(2-s)}. \end{aligned} \tag{37}$$

We deduct that

$$\left|I_2(\varepsilon)\right| \le \begin{cases} C\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} & \text{if } 0 < \mu < \overline{\mu} - \dfrac{9}{4}, \\ C\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} |\ln \varepsilon| & \text{if } \mu = \overline{\mu} - \dfrac{9}{4}, \\ C\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}} + C\varepsilon^{(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}+(N-2)/(2-s)} & \text{if } \overline{\mu} - \dfrac{9}{4} < \mu < \overline{\mu}. \end{cases} \tag{38}$$

So, if $0 < \mu < \overline{\mu} - 1$, we have

$$\begin{aligned} &\left|I_2(\varepsilon)\right| \\ &= \left|\int_{B_\rho(a)} \left(\frac{1}{|x-b|^2} - \frac{1}{|a-b|^2}\right) \frac{\left|W^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2} dx\right| \\ &= o\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right). \end{aligned} \tag{39}$$

Let $\rho < |a-b|$, and $\rho/2 < |x-a| < \rho$; we have $(|a-b| - \rho) < |x-b|$.

$$\begin{aligned} \left|I_3(\varepsilon)\right| &= \left|\int_{\Omega\setminus B_\rho(a)} \left(\varphi^2(x) - 1\right) \frac{\left|W^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2} dx\right| \\ &\le \frac{C_1 \varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}}{\left(|a-b| - \rho\right)^2} \\ &\cdot \int_{\rho/2\varepsilon_0}^{\rho/\varepsilon_0} \frac{r^{N-1}}{r^{2a(\mu)} \left(1 + r^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} dr \\ &\le C_2 \varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}} \int_{\rho/2\varepsilon_0}^{\rho/\varepsilon_0} r^{N-1-2a(\mu)-4\sqrt{\overline{\mu}-\mu}} dr \\ &\le C_2 \varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}} \int_{\rho/2\varepsilon_0}^{\rho/\varepsilon_0} \varepsilon_0^{-1+2\sqrt{\overline{\mu}-\mu}} dr \\ &\le C_3 \varepsilon^{(N-2)/(2-s)}. \end{aligned} \tag{40}$$

Then

$$\left|I_3(\varepsilon)\right| = O\left(\varepsilon^{(N-2)/(2-s)}\right). \tag{41}$$

So, for $0 < \mu < \overline{\mu} - 1$ and taking (27),(39), and (41) in (22), we get

$$\begin{aligned} \int_\Omega \frac{\left|v^a_{\mu,s,\varepsilon}(x)\right|^2}{|x-b|^2} dx &= O\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) \\ &+ o\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) \\ &+ O\left(\varepsilon^{(N-2)/(2-s)}\right) \\ &= O\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right). \end{aligned} \tag{42}$$

For the case $\mu = \overline{\mu} - 1$. Using (29) we have

$$\begin{aligned} \left|I_2(\varepsilon)\right| &\le C_2 \varepsilon^{(N-2)/(2-s)} \int_{|y|\le \rho/\varepsilon_0} \frac{\left|-\varepsilon_0^2 |y|^2 - 2\varepsilon_0 \langle y, a-b\rangle\right|}{|y|^{2a(\mu)} \left(\left|\varepsilon_0 y + a - b\right|^2 |a-b|^2\right) \left(1 + |y|^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} dy \\ &\le C_3 \varepsilon^{(N-2)/(2-s)} \int_{|y|\le \rho/\varepsilon_0} \frac{\varepsilon_0^2 |y|^2 + 2\varepsilon_0 |y| |a-b|}{|y|^{2a(\mu)} \left(1 + |y|^{(2(2-s)/(N-2))\sqrt{\overline{\mu}-\mu}}\right)^{2(N-2)/(2-s)}} \\ &\le C_4 \varepsilon^{(N-2)/(2-s)} \int_0^{\rho/\varepsilon_0} \frac{\left(\varepsilon_0^2 r^2 + 2\varepsilon_0 r |a-b|\right) r^{N-1}}{r^{N-4} \left(1 + r^{2(2-s)/(N-2)}\right)^{2(N-2)/(2-s)}} dr \le C_5 \varepsilon^{(N-2)/(2-s)} \int_0^{\rho/\varepsilon_0} \varepsilon_0^2 r + 2\varepsilon_0 |a-b|\, dr \\ &\le C_6 \varepsilon^{(N-2)/(2-s)}. \end{aligned} \tag{43}$$

Then if $\mu = \overline{\mu} - 1$ we get

$$\left|I_2(\varepsilon)\right| = O\left(\varepsilon^{(N-2)/(2-s)}\right). \tag{44}$$

$$\left|I_1(\varepsilon)\right| = C_2 \left(\varepsilon^{(N-2)/(2-s)} |\ln \varepsilon|\right), \quad \text{if } \mu = \overline{\mu} - 1, \tag{45}$$

$$\left|I_3(\varepsilon)\right| = O\left(\varepsilon^{(N-2)/(2-s)}\right). \tag{46}$$

By (44), (45), and (46), we derive (20) for $\mu = \overline{\mu} - 1$. □

Let X be a Banach space and X^{-1} be the dual space of X. The functional $I \in C^1(X, \mathbb{R})$ is said to satisfy the Palais–Smale condition at level c ($(PS)_c$ in short), if any sequence $\{u_n\} \subset X$ satisfying $I(u_n) \longrightarrow c$, $I'(un) \longrightarrow 0$ strongly in X^{-1} as $n \longrightarrow +\infty$ contains a subsequence converging in X to a critical point of the functional I.

In our case, $X = H_0^1(\Omega)$ and $X^{-1} = H^{-1}(\Omega)$.

Lemma 3. *The functional I satisfies $(PS)_c$ condition for any*

$$c < \min_{1\le i\le k} \left\{ \frac{2-s_i}{2(N-s_i)} \frac{\left(S_{\mu_i,s_i}^{a_i}\right)^{(N-s_i)/(2-s_i)}}{\left(K_i(a_i)\right)^{(N-2)/(2-s_i)}} \right\} = c_*. \tag{47}$$

Proof. Suppose $\{u_n\}$ is a $(PS)_c$ sequence for I with $c < c_*$. Then,

$$\begin{gathered} I(u_n) \longrightarrow c, \\ I'(u_n) \longrightarrow 0 \\ \text{as } n \longrightarrow +\infty. \end{gathered} \tag{48}$$

First, we show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Let $2 < \theta < 2^*(s_i)$ for all $1 \le i \le k$. So, $1/2 - 1/\theta > 0$ and $1/\theta - 1/2^*(s_i) > 0$. By Hardy and Sobolev-Hardy inequality we have

$$\begin{aligned} (c + \|u_n\|) &\ge I(u_n) - \frac{1}{\theta} I'(u_n) u_n \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(\|u_n\|^2 - \sum_{i=1}^k \mu_i \int_\Omega \frac{|u_n|^2}{|x-a_i|^2} dx\right) \\ &\quad + \sum_{i=1}^k \left(\frac{1}{\theta} - \frac{1}{2^*(s_i)}\right) \int_\Omega K_i(x) \frac{|u_n|^{2^*(s_i)}}{|x-a_i|^{s_i}} dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(1 - \sum_{i=1}^k \frac{\mu_i}{\overline{\mu}}\right) \|u_n\|^2 \\ &\quad + \sum_{i=1}^k \left(\frac{1}{\theta} - \frac{1}{2^*(s_i)}\right) \int_\Omega K_i(x) \frac{|u_n|^{2^*(s_i)}}{|x-a_i|^{s_i}} dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(1 - \sum_{i=1}^k \frac{\mu_i}{\overline{\mu}}\right) \|u_n\|^2. \end{aligned} \tag{49}$$

Therefore, up to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ &u_n \rightharpoonup u \\ &\qquad \text{weakly in } L^2\left(\Omega, |x-a_i|^{-2} dx\right) \text{ for } 1 \le i \le k, \\ &u_n \rightharpoonup u \\ &\qquad \text{weakly in } L^{2^*(s_i)}\left(\Omega, |x-a_i|^{-s_i} dx\right) \text{ for } 1 \le i \le k, \\ &u_n \longrightarrow u \quad \text{a.e. on } \Omega. \end{aligned} \tag{50}$$

Then $u \in H_0^1(\Omega)$ is a weak solution of problem (1). We may suppose that

$$\begin{gathered} |\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \nu, \\ (\text{weak}^* - \text{sense of measures}). \end{gathered} \tag{51}$$

Using the concentration-compactness principle due to Lions (cf. [[15], Lemma 1.2]), we obtain an atmost countable set Λ, a set of different points $\{xj\}_{j\in\Lambda} \subset \Omega \setminus \{a_1, a_1, \cdots, a_k\}$, real numbers $\nu_{x_j}, \nu_{a_j}, \tau_{a_j}$ and σ_{a_i} for $1 \le j \le k$ such that

$$\nu \ge \sum_{j\in\Lambda} \nu_{x_j}\delta_{x_j} + \sum_{j=1}^{k} \nu_{a_j}\delta_{a_j}, \tag{52}$$

and since $2 < 2^*(s_i) < 2^* (1 \le i \le k)$ we have

$$\begin{aligned} \frac{|u_n|^{2^*(s_i)}}{|x-a_i|^{s_i}} &\rightharpoonup \frac{|u|^{2^*(s_i)}}{|x-a_i|^{s_i}} + \tau_{a_i}\delta_{a_i}, \\ \frac{|u_n|^2}{|x-a_i|^2} &\rightharpoonup \frac{|u|^2}{|x-a_i|^2} + \sigma_{a_i}\delta_{a_i}, \end{aligned} \tag{53}$$

where δ_{aj} is the Dirac mass at $a_j \in \mathbb{R}^N$.

Let $\varepsilon > 0$ such that for any $j \in \Lambda$, $x_j \notin B_\varepsilon(a_i)$ $(1 \le i \le k)$. Choose a smooth cut-off function $\varphi_{i,\varepsilon}$ centered at the point a_i satisfying $0 \le \varphi_{i,\varepsilon} \le 1$, $\varphi_{i,\varepsilon} = 1$ for $|x - a_i| \le \varepsilon/2$, $\varphi_{i,\varepsilon} = 0$ for $|x - a_i| \ge \varepsilon$ and $|\nabla\varphi_{i,\varepsilon}| \le 4/\varepsilon$. Since $\{u_n\varphi_{i,\varepsilon}\}$ is bounded, $I'(u_n)u_n\varphi_{i,\varepsilon} = o(1)$, that is,

$$\begin{aligned} I'(u_n)u_n\varphi_{i,\varepsilon} &= \int_\Omega |\nabla u_n|^2 \varphi_{i,\varepsilon}dx + \int_\Omega u_n\nabla u_n\nabla\varphi_{i,\varepsilon}dx \\ &\quad - \sum_{i=1}^{k}\mu_i \int_\Omega \frac{|u_n|^2}{|x-a_i|^2}\varphi_{i,\varepsilon}dx \\ &\quad - \sum_{i=1}^{k}\int_\Omega K_i(x)\frac{|u_n|^{2^*(s_i)}}{|x-a_i|^{s_i}}\varphi_{i,\varepsilon}dx. \end{aligned} \tag{54}$$

Moreover, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2\varphi_{i,\varepsilon}dx\right] \\ &\quad \ge \lim_{\varepsilon\to 0}\left[\int_\Omega |\nabla u|^2\varphi_{i,\varepsilon}dx + \nu_{a_i}\right] = \nu_{a_i} \\ &\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_\Omega \frac{|u_n|^2}{|x-a_i|^2}\varphi_{i,\varepsilon}dx\right] \\ &\quad = \lim_{\varepsilon\to 0}\left[\int_\Omega \frac{|u|^2}{|x-a_i|^2}\varphi_{i,\varepsilon}dx + \sigma_{a_i}\right] = \sigma_{a_i} \\ &\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_\Omega K_i(x)\frac{|u_n|^{2^*(s_i)}}{|x-a_i|^{s_i}}\varphi_{i,\varepsilon}dx\right] \\ &\quad = \lim_{\varepsilon\to 0}\left[\int_\Omega K_i(x)\frac{|u|^{2^*(s_i)}}{|x-a_i|^{s_i}}\varphi_{i,\varepsilon}dx + \tau_{a_i}\right] = \tau_{a_i}. \end{aligned} \tag{55}$$

Arguing as in [3], we can prove that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_\Omega u_n\nabla u_n\nabla\varphi_{i,\varepsilon}dx\right] = 0, \\ &\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_\Omega \frac{|u_n|^2}{|x-a_j|^2}\varphi_{i,\varepsilon}dx\right] = 0, \\ &\qquad \forall j \ne i, \\ &\lim_{\varepsilon\to 0}\left[\lim_{n\to+\infty}\int_\Omega K_i(x)\frac{|u_n|^{2^*(s_i)}}{|x-a_j|^{s_i}}\varphi_{i,\varepsilon}dx\right] = 0, \\ &\qquad \forall j \ne i. \end{aligned} \tag{56}$$

From (55)-(56), let $n \longrightarrow +\infty$ and $\varepsilon \longrightarrow 0$ in expression (54), we obtain

$$0 \ge \nu_{a_i} - \mu_i\sigma_{a_i} - K_i(a_i)\tau_{a_i}. \tag{57}$$

By the definition of $S^{a_i}_{\mu_i,s_i}$, we deduce that

$$S^{a_i}_{\mu_i,s_i}\left(\tau_{a_i}\right)^{2/2^*(s_i)} \le \nu_{a_i} - \mu_i\sigma_{a_i}. \tag{58}$$

Combining (57) with (58), we get

$$S^{a_i}_{\mu_i,s_i}\left(\tau_{a_i}\right)^{2/2^*(s_i)} \le K_i(a_i)\tau_{a_i}, \tag{59}$$

which implies that

$$\tau_{a_i} = 0 \text{ or } \tau_{a_i} \ge \left(S^{a_i}_{\mu_i,s_i}\right)^{(N-s_i)/(2-s_i)} \text{ for every } 1 \le i \le k. \tag{60}$$

Arguing by contradiction, let us suppose that there exist i_0 such that

$$\tau_{a_{i_0}} \ge \left(S^{a_{i_0}}_{\mu_{i_0},s_{i_0}}\right)^{(N-s_{i_0})/(2-s_{i_0})}. \tag{61}$$

Thus,

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{2}I'(u_n)u_n = \sum_{i=1}^{k}\left(\frac{1}{2} - \frac{1}{2^*(s_i)}\right) \\ &\quad \cdot \int_\Omega K_i(x)\frac{|u_n|^{2^*(s_i)}}{|x-a_i|^{s_i}}dx = \sum_{i=1}^{k}\left(\frac{1}{2} - \frac{1}{2^*(s_i)}\right) \\ &\quad \cdot \left(\int_\Omega K_i(x)\frac{|u|^{2^*(s_i)}}{|x-a_i|^{s_i}}dx + K_i(a_i)\tau_i\right). \end{aligned} \tag{62}$$

Letting $n \longrightarrow +\infty$, we get

$$c \ge \sum_{i=1}^{k}\left(\frac{1}{2} - \frac{1}{2^*(s_i)}\right)K_i(a_i)\tau_i, \tag{63}$$

so, by (61), we obtain

$$c \ge \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)}\frac{\left(S^{a_{i_0}}_{\mu_{i_0},s_{i_0}}\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(K_{i_0}\left(a_{i_0}\right)\right)^{(N-2)/(2-s_{i_0})}}, \tag{64}$$

which contradicts the assumption that

$$c < \min_{1\le i\le k}\left\{\frac{2-s_i}{2\left(N-s_i\right)}\frac{\left(S^{a_i}_{\mu_i,s_i}\right)^{(N-s_i)/(2-s_i)}}{\left(K_i(a_i)\right)^{(N-2)/(2-s_i)}}\right\} = c_*. \tag{65}$$

Hence, up to a subsequence, we obtain that $u_n \longrightarrow u$ strongly in $H_0^1(\Omega)$. □

Lemma 4. *Under the assumptions of* ($\mathscr{H}_1$), ($\mathscr{H}_2$), *and* ($\mathscr{H}_3$), *there is a nonnegative function* $v_0 \in H_0^1(\Omega)$, *such that* $\sup_{t\geq 0} I(tv_0) < c_*$.

Proof. Let us prove only for the following case $0 < \mu_{i_0} < \overline{\mu}$, for the other case the proof is the same. We consider the following functions on the interval $[0, +\infty($

$$\overline{g}(t) = \frac{t^2}{2}\int_\Omega \left(\left|\nabla v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^2 - \mu_{i_0} \frac{\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^2}{\left|x-a_{i_0}\right|^2} - \sum_{i\neq i_0} \mu_i \int_\Omega \frac{\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^2}{\left|x-a_i\right|^2} \right) dx - \frac{t^{2^*(s_{i_0})}}{2^*\left(s_{i_0}\right)} \cdot \int_\Omega K_{i_0}(x) \frac{\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}(x)\right|^{2^*(s_{i_0})}}{\left|x-a_{i_0}\right|^{s_{i_0}}} dx, \tag{66}$$

and

$$I\left(tv_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right) = \overline{g}(t) - \sum_{i\neq i_0}^{k} \frac{t^{2^*(s_i)}}{2^*\left(s_i\right)} \int_\Omega K_i(x) \frac{\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^{2^*(s_i)}}{\left|x-a_i\right|^{s_i}} dx. \tag{67}$$

Using the following formula,

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A - \frac{t^{2^*(s_{i_0})}}{2^*\left(s_{i_0}\right)}B\right) = \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)} A^{(N-s_{i_0})/(2-s_{i_0})} B^{(N-2)/(s_{i_0}-2)}, \tag{68}$$

and using (17), (19), and (20), we have

$$\begin{aligned}
\sup_{t\geq 0}\overline{g}(t) &= \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)} \frac{\left(\int_\Omega \left(\left|\nabla v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^2 - \mu_{i_0}\left(\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^2 / \left|x-a_{i_0}\right|^2\right)\right) dx - \sum_{i\neq i_0}\mu_i \int_\Omega \left(\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right|^2 / \left|x-a_i\right|^2\right) dx\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(\int_\Omega K_{i_0}(x)\left(\left|v_{\mu_{i_0},\varepsilon}^{a_{i_0}}(x)\right|^{2^*(s_{i_0})} / \left|x-a_{i_0}\right|^{s_{i_0}}\right) dx\right)^{(N-2)/(s_{i_0}-2)}} \\
&= \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)} \frac{\left(S_{\mu_{i_0},s_{i_0}}^{a_{i_0}} - C_1 \sum_{i\neq i_0}\left(\mu_i / \left|a_i - a_{i_0}\right|\right) \varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}} - o\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) + O\left(\varepsilon^{(N-2)/(2-s)}\right)\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(K_{i_0}\left(a_{i_0}\right) + O\left(\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right)\right)^{(N-2)/(s_{i_0}-20)}} \\
&= \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)} \frac{\left(S_{\mu_{i_0},s_{i_0}}^{a_{i_0}}\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(K_{i_0}\left(a_{i_0}\right)\right)^{(N-2)/(2-s_{i_0})}} \frac{\left(1 - O\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) - o\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) + O\left(\varepsilon^{(N-2)/(2-s)}\right)\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(1 + O\left(\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right)\right)^{(N-2)/(s_{i_0}-2)}} \\
&= \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)} \frac{\left(S_{\mu_{i_0},s_{i_0}}^{a_{i_0}}\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(K_{i_0}\left(a_{i_0}\right)\right)^{(N-2)/(2-s_{i_0})}} \\
&\quad \times \left(1 - O\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) - o\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right)\right)\left(1 - O\left(\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right) - o\left(\varepsilon^{3(N-2)/2(2-s)\sqrt{\overline{\mu}-\mu}}\right)\right) \\
&< \frac{2-s_{i_0}}{2\left(N-s_{i_0}\right)} \frac{\left(S_{\mu_{i_0},s_{i_0}}^{a_{i_0}}\right)^{(N-s_{i_0})/(2-s_{i_0})}}{\left(K_{i_0}\left(a_{i_0}\right)\right)^{(N-2)/(2-s_{i_0})}} - O\left(\varepsilon^{(N-2)/(2-s)\sqrt{\overline{\mu}-\mu}}\right) = c_*,
\end{aligned} \tag{69}$$

for $0 < \varepsilon$ sufficiently small. And since, for all $1 \leq i \leq k$, the function K_i is a positive on Ω, we have

$$\sup_{t\geq 0} I\left(tv_{\mu_{i_0},\varepsilon}^{a_{i_0}}\right) \leq \sup_{t\geq 0}\overline{g}(t) < c_*, \tag{70}$$

for $0 < \varepsilon$ sufficiently small. □

3. Proof of Main Result 1

We verify that the functional I satisfies the mountain pass geometry. To this end, we consider the energy level.

$$c_1 = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I\left(\gamma(t)\right), \tag{71}$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1]; H_0^1(\Omega)\right) : \gamma(0) = 0, I(\gamma(1)) < 0\right\}. \quad (72)$$

For any $u \in H_0^1(\Omega) \setminus \{0\}$, by Hardy and Sobolev-Hardy inequality, (8) and (9) (take $q = 2^*(s_i)$), we get that

$$\begin{aligned} I(u) &= \frac{1}{2}\left(\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^{k}\mu_i \int_\Omega \frac{|u|^2}{|x-a_i|^2}\right)dx\right) \\ &\quad - \frac{1}{2^*(s_i)}\int_\Omega K_i(x)\frac{|u|^{2^*(s_i)}}{|x-a_i|^{s_i}}dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\overline{\mu}}\sum_{i=1}^{k}\mu_i\right)\int_\Omega |\nabla u|^2 dx \\ &\quad - \sum_{i=1}^{k}\frac{\overline{k}_i\left(C_{s_i,2^*(s_i)}\right)^{-2^*(s_i)/2}}{2^*(s_i)}\left(\int_\Omega |\nabla u|^2 dx\right)^{2^*(s_i)/2} \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\overline{\mu}}\sum_{i=1}^{k}\mu_i\right)\|u\|^2 \\ &\quad - \sum_{i=1}^{k}\frac{\overline{k}_i\left(C_{s_i,2^*(s_i)}\right)^{-2^*(s_i)/2}}{2^*(s_i)}\|u\|^{2^*(s_i)}. \end{aligned} \quad (73)$$

Hence, there exists $\rho_0 > 0$ small enough such that

$$r = \inf_{\|u\|=\rho_0} I(u) > 0. \quad (74)$$

Then

$$I(u) \geq r, \quad (75)$$

for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho_0$. Let $v \in H_0^1(\Omega)$ given in Lemma 4. Since $\lim_{t\longrightarrow+\infty} I(tv) = -\infty$, hence there exists $t_0 > 0$ such that $\|t_0 v\| \geq \rho_0$ and $I(t_0 v) < 0$, by Lemma 4, we obtain

$$c_1 \leq \sup_{t\in[0,1]} I(tt_0 v) \leq \sup_{t\geq 0} I(tv) < c_*. \quad (76)$$

Moreover, by the Mountain Pass Theorem [9] and Lemma 3, we obtain that c_1 is critical value of I at point u and thus is a solution of problem (1). Then the rest of the proof follows exactly the same lines as that in [3]. In order to find the positive solution of (1), we replace $I(u)$ with $I^+(u)$ defined as follows:

$$\begin{aligned} I^+(u) &= \frac{1}{2}\left(\int_\Omega |\nabla u|^2 dx - \sum_{i=1}^{k}\mu_i \int_\Omega \frac{u^2}{|x-a_i|^2}dx\right) \\ &\quad - \sum_{i=1}^{k}\frac{1}{2^*(s_i)}\int_\Omega K_i(x)\frac{(u^+)^{2^*(s_i)}}{|x-a_i|^{s_i}}dx \end{aligned} \quad (77)$$

where $u^+ = \max\{u, 0\}$. Repeating the above arguments, we find a critical point of I^+ and by applying the maximum principle we obtain a positive solution. So, the proof of Theorem 1 is therefore completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] B. Abdellaoui, V. Felli, and I. Peral, "Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian," *Bollettino della Unione Matematica Italiana-B*, vol. 9, no. 2, pp. 445–484, 2006.

[2] Y. Cao and D. Kang, "Solutions of a quasilinear elliptic problem involving a critical Sobolev exponent and multiple Hardy-type terms," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 2, pp. 889–903, 2007.

[3] D. Cao and P. Han, "Solutions to critical elliptic equations with multi-singular inverse square potentials," *Journal of Differential Equations*, vol. 224, no. 2, pp. 332–372, 2006.

[4] L. Ding and C.-L. Tang, "Existence and multiplicity of solutions for semilinear elliptic equations with hardy terms and hardy–sobolev critical exponents," *Applied Mathematics Letters*, vol. 20, no. 12, pp. 1175–1183, 2007.

[5] V. Felli and S. Terracini, "Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity," *Communications in Partial Differential Equations*, vol. 31, no. 3, pp. 469–495, 2006.

[6] P. Han, "Quasilinear elliptic problems with critical exponents and Hardy terms," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 5, pp. 735–758, 2005.

[7] L. Huang, X.-P. Wu, and C.-L. Tang, "Existence and multiplicity of solutions for semilinear elliptic equations with critical weighted hardy–sobolev exponents," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1916–1924, 2009.

[8] W. Gao and S. Peng, "An elliptic equation with combined critical sobolev–hardy terms," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 8, pp. 1595–1612, 2006.

[9] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.

[10] L. Caffarelli, R. Kohn, and L. Nirenberg, "First order interpolation inequalities with weights," *Compositio Mathematica*, vol. 53, no. 3, pp. 259–275, 1984.

[11] J. P. G. Azorero and I. P. Alonso, "Hardy inequalities and some critical elliptic and parabolic problems," *Journal of Differential Equations*, vol. 144, no. 2, pp. 441–476, 1998.

[12] N. Ghoussoub and C. Yuan, "Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents," *Transactions of the American Mathematical Society*, vol. 352, no. 12, pp. 5703–5743, 2000.

[13] D. Kang and S. Peng, "Positive solutions for singular critical elliptic problems," *Applied Mathematics Letters*, vol. 17, no. 4, pp. 411–416, 2004.

[14] J. Chen, “Existence of solutions for a nonlinear pde with an inverse square potential,” *Journal of Differential Equations*, vol. 195, no. 2, pp. 497–519, 2003.

[15] P. L. Lions, “The concentration-compactness principle in the calculus of variations.(the limit case, part i.),” *Revista Matemática Iberoamericana*, vol. 1, no. 1, pp. 145–201, 1985.