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Research Article

The Maximal Length of 2-Path in Random Critical Graphs

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Given a graph, its 2-*core* is the maximal subgraph of G without vertices of degree 1. A 2-path in a connected graph is a simple path in its 2-core such that all vertices in the path have degree 2, except the endpoints which have degree $\geqslant 3$. Consider the Erdős-Rényi random graph $\mathbb{G}(n, M)$ built with n vertices and M edges uniformly randomly chosen from the set of $\binom{n}{2}$ edges. Let $\xi_{n,M}$ be the maximum 2-path length of $\mathbb{G}(n, M)$. In this paper, we determine that there exists a constant $c(\lambda)$ such that $\mathbb{E}(\xi_{n,(n/2)(1+\lambda n^{-1/3})}) \sim c(\lambda)n^{1/3}$, for any real λ. This parameter is studied through the use of generating functions and complex analysis.

1. Preliminaries

Let us recall that an *undirected graphG* is a couple (V, E), where V is the set of vertices and E the set of edges, and an edge is an unordered pair of vertices. If we allow an edge between a vertex and itself (loop) or multiple edges between two vertices, we obtain a *multigraph*. An undirected graph without loops or multiple edges is known as a *simple graph*. A *path* in graph $G = (V, E)$ is sequence of vertices $\langle v_0, v_1, \ldots, v_k\rangle$, where $\{v_i, v_{i+1}\} \in E$ for $i \in [\![0, k-1]\!]$ and $v_i \neq v_j$ for $i \neq j$ except that its first vertex v_0 might be the same as its last v_k. When any two vertices of G are connected by a path G is called *connected*.

A connected graph has *excess* if it has more edges than vertices. A connected component of excess ℓ is also called ℓ-component. A tree or acyclic component is a connected component of excess -1, an unicyclic component in a connected component of excess 0. If $\ell \geqslant 1$, ℓ-components are called complex. A graph (not necessarily connected) is called *complex* when all its components are complex. The *total excessr* of a graph is the number of edges plus the number of acyclic components, minus the number of vertices. In other words, the total excess of a graph is the sum of the excess of its complex components. Note that the total excess of a tree component is equal to 0 whereas its excess is equal to -1 and the total excess of a graph is nonnegative.

Given a graph G, its 2-core is obtained by deleting recursively all nodes of degree 1. A 3-*core* or *kernel* of a complex graph is the graph obtained from its 2-core by repeating the following process on any vertex of degree two: for a vertex of degree two, we can remove it and splice together the two edges that it formerly touched. We observe that G, its 2-core, and its kernel have the same excess. A graph is said *cubic* or 3-*regular* if all of its vertices are of degree 3. A graph is called *clean* if its 3-core is 3-regular (see [1]).

A random graph $\mathbb{G}(n, M = cn)$ is called *critical* if the density $c = 1/2 \pm \mathcal{O}(n^{-1/3})$. Such a graph contains a complex component with nonzero probability [2, 3]. Janson et al. [1] proved these graphs are clean (its complex components are clean) with high probability when the size of graph goes to infinity.

Theorem 1. *The maximum 2-path length $\xi_{n,M}$ of $\mathbb{G}(n, M)$ satisfies*

$$\mathbb{E}\left(\xi_{n,(n/2)(1+\lambda n^{-1/3})}\right) \sim c(\lambda)\, n^{1/3}, \tag{1}$$

where

$$c(\lambda) = \frac{1}{\alpha}\int_0^{+\infty}\left(1 - \sum_{r\geqslant 0}\sqrt{2\pi}e_r A(3r+1/2,\lambda)\left(1-e^{-x}\right)^{3r}\right)dx, \tag{2}$$

where α is the positive solution of

$$\lambda = \alpha^{-1} - \alpha, \tag{3}$$

e_r is given by

$$e_r = \frac{(6r)!}{2^{5r}3^{2r}(3r)!\,(2r)!}, \tag{4}$$

and the function A is defined by

$$A(y,\lambda) = \frac{e^{-\lambda^3/6}}{3^{(y+1)/3}}\sum_{k\geqslant 0}\frac{\left((1/2)\,3^{2/3}\lambda\right)^k}{k!\Gamma\left(\left(y+1-2k\right)/3\right)}. \tag{5}$$

We remark that for Erdős-Rényi random graph $\mathbb{G}(n, p = (1+\varepsilon)/n)$, Ding et al. [4] and Ding et al. [5] provided a complete characterisation of the structure of the giant component when $\varepsilon = o(1)$ but $\varepsilon^3 n \to \infty$. Using our notation, $\varepsilon = \lambda n^{-1/3}$ but $\lambda \to \infty$ as $n \to \infty$. They describe that the 2-core of a graph is obtained by "stretching" the edges into paths of lengths i.i.d. geometric with mean $1/\varepsilon = \lambda^{-1}n^{1/3}$. Next, in order to reconstruct the graph, they attached trees to vertices i.i.d. Poisson$(1-\varepsilon)$-Galton-Watson.

2. Enumerative Tools

As shown in [1, 3], exponential generating functions (EGFs) can lead to stringent results about the main characteristics of random graphs when they apply. Let us recall briefly the main EGFs involved in our proofs. We refer the reader to Harary and Palmer [6] for EGFs related to graphical enumeration.

For $n \geqslant 0$ and $-1 \leqslant \ell \leqslant \binom{n}{2}$, let $c(n, n+\ell)$ be the number of connected graphs of excess ℓ and

$$W_\ell(z) = \sum_{n=0}^{+\infty} c(n, n+\ell)\frac{z^n}{n!}, \tag{6}$$

the associated EGF. We know from [7] that $c(n, n-1) = n^{n-2}$ and

$$W_{-1}(z) = \sum_{n=1}^{+\infty} n^{n-2}\frac{z^n}{n!} = T(z)^2 - \frac{1}{2}T(z)^2, \tag{7}$$

where $T(z)$ is the EGF of rooted Cayley trees given by

$$T(z) = ze^{T(z)} = \sum_{n=1}^{+\infty} n^{n-1}\frac{z^n}{n!}. \tag{8}$$

We also have (see, e.g., [1, Equation (3.5)])

$$W_0(z) = \frac{1}{2}\log\frac{1}{1-T(z)} - \frac{T(z)}{2} - \frac{T(z)^2}{4}. \tag{9}$$

Wright [8] has shown that the EGFs $(W_\ell)_{\ell\geqslant 1}$ can be expressed in terms of $T(z)$. More precisely, Wright proved that for each $\ell \geqslant 1$ there exist rational coefficients $w_{\ell,d}, d \in \{0,\ldots,3\ell+2\}$ such that

$$W_\ell(z) = \sum_{d=0}^{3\ell+2}\frac{w_{\ell,d}}{(1-T(z))^{3\ell-d}}. \tag{10}$$

The coefficients $b_\ell := w_{\ell,0}$ are known as Wright's constants (see [9]). For complex graphs, denote by $E_r(z)$ the EGF of these graphs of excess r. Then we have $E_0(z) = 1$ (empty graphs) and $E_1(z) = W_1(z)$. More generally, as detailed in [1, Section 8], the EGF $E_r(z)$ satisfies

$$\sum_{r=0}^{+\infty} E_r(z) = \exp\left(\sum_{r=1}^{+\infty} W_\ell(z)\right). \tag{11}$$

Following [1], the EGF $E_r(z)$ can also be expressed as a rational function of $T(z)$

$$E_r(z) = \sum_{d\geqslant 0} e_{r,d}\frac{T(z)^{5r-d}}{(1-T(z))^{3r-d}} = \sum_{d\geqslant 0}\frac{e'_{r,d}}{(1-T(z))^{3r-d}}, \tag{12}$$

where $e_r := e_{r,0} = e'_{r,0}$. The coefficients (e_r) and (b_r) are related by

$$e_0 = 1, \qquad re_r = rb_r + \sum_{j=1}^{r-1} jb_j e_{r-j} \quad \text{as } r \geqslant 1. \tag{13}$$

As shown in [1, 10], we remark that the dominant asymptotic behavior of $[z^n]W_\ell(z)$ and $[z^n]E_r(z)$ (for any power series $A(z) = \sum a_n z^n$, $[z^n]A(z)$ denotes the nth coefficient of $A(z)$, namely, $[z^n]A(z) = a_n$.) is governed by the leading coefficients b_ℓ and e_r. In particular, if ℓ and r are about $o(n^{1/3})$, these EGFs satisfy

$$W_\ell(z) \asymp_\ell \frac{b_\ell}{(1-T(z))^{3\ell}}, \qquad E_r(z) \asymp_r \frac{e_r}{(1-T(z))^{3r}}, \tag{14}$$

where $A(z) \asymp_\ell B(z)$ if and only if $[z^n]A(z) \sim [z^n]B(z)$ as $n \to +\infty$ and $\ell = o(n^{1/3})$.

The EGF $b_\ell/(1-T(z))^{3\ell}$ (resp., $e_r/(1-T(z))^{3r}$) can be interpreted as EGF of connected graphs (resp., complex graphs) whose kernels are 3-regular. Such a graph has exactly 3ℓ (resp., $3r$) 2-paths. A 2-path in the 2-core is enumerated by $1/(1-z)$. Substituting z by $T(z)$ to obtain $1/(1-T(z))$ means attaching tree to each vertex of the 2-core. Since a 3-regular graph of excess r has exactly $3r$ edges, the associated graph has $3r$ 2-paths. Note that, in the stated range, complex graphs and multigraphs of excess r are both enumerated by $e_r/(1-T(z))^{3r}$ and for large r, $e_r \sim b_r$ (see [1, Equation (7.16)] and [9, Section 7]).

In our case, we need to control the length of each 2-path to a graph. We restrict our attention to complex graphs whose 2-paths are of length at most k. So, instead of allowing 2-path of

any length $(1/(1-T(z)) = 1+T(z)+T(z)^2+\cdots+T(z)^k+\cdots)$, we use a 2-path of length at most k $(1+T(z)+T(z)^2+\cdots+T(z)^k)$. The associated EGF is

$$e_r\left(1+T(z)+T(z)^2+\cdots+T(z)^k\right)^{3r} = e_r\frac{\left(1-T(z)^{k+1}\right)^{3r}}{(1-T(z))^{3r}}. \tag{15}$$

3. Proof of Theorem 1

Consider a graph with n vertices, M edges, and a total excess r. Such a graph contains exactly $n-M+r$ tree components. They are enumerated by the following EGF:

$$\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\exp\left(W_0(z)\right)E_r(z). \tag{16}$$

Since the total number of graphs with n vertices and M edges is $\binom{\binom{n}{2}}{M}$, the probability that a random (n,M)-graph (graphs with n vertices and M edges) is of total excess r is

$$\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\exp\left(W_0(z)\right)E_r(z). \tag{17}$$

Similarly, the probability that a random (n,M)-graph is of total excess r and has no 2-path of length greater than k is

$$\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\exp\left(W_0(z)\right)E_r^{[k]}(z), \tag{18}$$

where $E_r^{[k]}(z)$ denotes the EGF of all complex components of total excess r whose 2-paths are of length at most k, $k \geqslant 1$. Then summing over r, we get that the probability that a random (n,M)-graph has no 2-path of length greater than k is

$$\mathbb{P}\left(\xi_{n,M} \leqslant k\right) = \sum_{r\geqslant 0}\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\exp\left(W_0(z)\right)E_r^{[k]}(z). \tag{19}$$

Following discussion in the previous section and using (15), the probability of a random $\mathbb{P}(\xi_{n,M} \leqslant k)$ is asymptotically equivalent to

$$\mathbb{P}\left(\xi_{n,M} \leqslant k\right) \sim \sum_{r\geqslant 0}\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\cdot\exp\left(W_0(z)\right)\frac{e_r\left(1-T(z)^{k+1}\right)^{3r}}{(1-T(z))^{3r}}. \tag{20}$$

As in [11, Section 4] where Flajolet et al. described generating functions based methods to study extremal statistics on random mappings, we characterize the expectation of $\xi_{n,M}$ by means of truncated generating functions aforementioned. In fact, the mean value of $\xi_{n,M}$ is given by

$$\mathbb{E}\left(\xi_{n,M}\right) = \sum_{k\geqslant 0}k\mathbb{P}\left[\xi_{n,M}=k\right] = \sum_{k\geqslant 0}\left(1-\mathbb{P}\left[X\leqslant k\right]\right). \tag{21}$$

Then, combining (19), (20), and (21), we have

$$\mathbb{E}\left(\xi_{n,M}\right) \sim \sum_{k\geqslant 0}\left(1-\sum_{r\geqslant 0}\frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\cdot\exp\left(W_0(z)\right)\frac{e_r\left(1-T(z)^{k+1}\right)^{3r}}{(1-T(z))^{3r}}\right). \tag{22}$$

To compute $E(\xi_{n,M})$, we use the following lemma.

Lemma 2. *Let $M = (n/2)(1+\lambda n^{-1/3})$. For any natural integers r and k, one has*

$$p_{r,k}(n,M) := \frac{n!}{\binom{\binom{n}{2}}{M}}[z^n]\frac{W_{-1}(z)^{n-M+r}}{(n-M+r)!}\exp\left(W_0(z)\right)\cdot\frac{\left(1-T(z)^k\right)^{3r}}{(1-T(z))^{3r}} = \sqrt{2\pi}A\left(3r+\frac{1}{2},\lambda\right)\cdot\left(1-e^{k\alpha n^{-1/3}}\right)^{3r}\left(1+\mathcal{O}\left(\frac{\lambda^4}{n^{1/3}}\right)\right). \tag{23}$$

Proof. We set

$$\begin{aligned}&\mathrm{St}(n,M,r) = \frac{n!}{\binom{\binom{n}{2}}{M}(n-M+r)!},\\ &\mathrm{Ca}(n,M,r) = [z^n]W_{-1}(z)^{n-M+r}e^{W_0(z)}\frac{\left(1-T(z)^k\right)^{3r}}{(1-T(z))^{3r}}.\end{aligned} \tag{24}$$

First, using Stirling's formula, we obtain

$$\mathrm{St}(M,n) = \sqrt{2\pi n}\frac{2^{n-M+r}}{n^r}\exp\left(-\frac{\lambda^3}{6}+\frac{3}{4}-n\right)\cdot\left(1+O\left(\frac{\lambda^4}{n^{1/3}}\right)\right). \tag{25}$$

Next, using Cauchy integral's formula and substituting z by ze^{-z}, we obtain

$$\mathrm{Ca}(M,n) = \frac{1}{2\pi i}\oint\left(T(z)-\frac{T(z)^2}{2}\right)^{n-M+r}\cdot e^{-T(z)/2-T(z)^2/4}\frac{\left(1-T(z)^k\right)^{3r}}{(1-T(z))^{3r+1/2}}\frac{dz}{z^{n+1}} = \frac{2^{M-n-r}e^n}{2\pi i}\cdot\oint g(z)\exp\left(nh(z)\right)\frac{dz}{z}, \tag{26}$$

where

$$g(z) = \frac{\left(2z - z^2\right)^r e^{-z/2 - z^2/4} \left(1 - z^k\right)^{3r}}{(1-z)^{3r-1/2}},$$
$$h(z) = z - 1 - \log z - \left(1 - \frac{M}{n}\right) \log \frac{1}{1-(z-1)^2}. \tag{27}$$

The contour in (26) should keep $|z| < 1$. At the critical value $M = n/2$, we also have $h(1) = h'(1) = h''(1) = 0$. This triple zero occurs in the procedure Janson et al. [1] used when investigating the value of the integral for large n. Let $\nu = n^{-1/3}$, and let α be the positive solution of (3). Following the proof of [1, Lemma 3], we will evaluate (26) on the path $z = e^{-(\alpha+it)\nu}$, where t runs from $-\pi n^{1/3}$ to $\pi n^{1/3}$:

$$\oint f(z) \frac{dz}{z} = i\nu \int_{-\pi n^{1/3}}^{\pi n^{1/3}} f\left(e^{-(\alpha+it)\nu}\right) dt. \tag{28}$$

The main contribution to the value of this integral comes from the vicinity of $t = 0$. The magnitude of $e^{h(z)}$ depends on the real part of $h(z)$, namely, Re $h(z)$. Re $h(e^{-(\alpha+it)\nu})$ decreases as $|t|$ increases and $|e^{nh(z)}|$ has its maximum on the circle $z = e^{-(\alpha+it)\nu}$ when $t = 0$.

We have for $nh(e^{-s\nu})$

$$nh\left(e^{-s\nu}\right) = \frac{1}{3}s^3 + \frac{1}{2}\lambda s^2 + O\left(\left(\lambda^2 s^2 + s^4\right)\nu\right), \tag{29}$$

uniformly in any region such that $|s\nu| < \log 2$. In [1, equation (10.7)], the authors define

$$A(y,\mu) = \frac{1}{2\pi i} \int_{\Pi(1)} s^{1-y} e^{K(\mu,s)} ds, \tag{30}$$

where $K(\mu, s)$ is the polynomial

$$K(\mu,s) = \frac{(s+\mu)^2(2s-\mu)}{6} = \frac{s^3}{3} + \frac{\mu s^2}{2} - \frac{\mu^3}{6} \tag{31}$$

and $\Pi(\alpha)$ is a path in the complex plane that consists of the following three straight line segments:

$$s(t) = \begin{cases} -e^{-\pi i/3} t, & \text{for } -\infty < t \leqslant -2\alpha; \\ \alpha + it \sin\dfrac{\pi}{3}, & \text{for } -2\alpha \leqslant t \leqslant +2\alpha; \\ e^{+\pi i/3} t, & \text{for } +2\alpha \leqslant t < +\infty. \end{cases} \tag{32}$$

In particular, they proved that $A(y, \mu)$ can be expressed as (5).

For the function $g(z)$, we have

$$g\left(e^{-s\nu}\right) = \frac{\left(2e^{-s\nu} - e^{-2s\nu}\right)^r}{\left(1 - e^{-s\nu}\right)^{3r-1/2}} e^{-e^{-s\nu}/2 - e^{-2s\nu}/4} \left(1 - e^{-ks\nu}\right)^{3r}$$
$$= (s\nu)^{1/2-3r} e^{-3/4} \left(1 - e^{-ks\nu}\right)^{3r} (1 + O(s\nu)). \tag{33}$$

For $g(z)e^{nh(z)}$ in the integrand of (26), we have

$$e^{-\lambda^3/6} f\left(e^{-s\nu}\right) = e^{-3/4} \nu^{1/2-3r} \left(1 - e^{-ks\nu}\right)^{3r} \cdot s^{1-(3r+1/2)} e^{K(\lambda,s)} \left(1 + O(s\nu) + O\left(\lambda^2 s^2 \nu\right) + O\left(s^4 \nu\right)\right) \tag{34}$$

when $s = O(n^{1/12})$. Next,

$$\frac{e^{-\lambda^3/6}}{2\pi i} \oint g(z) e^{nh(z)} \frac{dz}{z} = e^{-3/4} \left(1 - e^{-k\alpha\nu}\right)^{3r} \nu^{3/2-3r} A\left(3r + \frac{1}{2}, \lambda\right) + O\left(\nu^{5/2-3r} e^{-\lambda^3/6} \lambda^{3r/2+1/4}\right), \tag{35}$$

where the error term has been derived from those already in [1]. The proof of the lemma is completed by multiplying (25), (26), and e_r. □

Now, to complete the proof of the theorem, we use first Lemma 2 to get

$$\mathbb{P}(\xi_{n,M} \leqslant k) \sim \sum_{r \geqslant 0} \sqrt{2\pi} e_r A\left(3r + \frac{1}{2}, \lambda\right) \left(1 - e^{-(k+1)\alpha n^{-1/3}}\right)^{3r}. \tag{36}$$

Next, using Euler-Maclaurin summation, and after a change of variable ($x = (k+1)\alpha n^{-1/3}$ so $dx/dk = \alpha n^{-1/3}$ and $dk = \alpha^{-1} n^{1/3} dx$), we get

$$\mathbb{E}(\xi_{n,M}) = \sum_{k=0}^{+\infty} \left(1 - \mathbb{P}(\xi_{n,M} \leqslant k)\right),$$
$$\sim \sum_{k=0}^{+\infty} \left(1 - \sum_{r \geqslant 0} \sqrt{2\pi} e_r A\left(3r + \frac{1}{2}, \lambda\right) \left(1 - e^{-(k+1)\alpha n^{-1/3}}\right)^{3r}\right)$$
$$\sim \int_0^{+\infty} \left(1 - \sum_{r \geqslant 0} \sqrt{2\pi} e_r A\left(3r + \frac{1}{2}, \lambda\right) \left(1 - e^{-x}\right)^{3r}\right) dk$$
$$\sim \alpha^{-1} n^{1/3} \int_0^{+\infty} \left(1 - \sum_{r \geqslant 0} \sqrt{2\pi} e_r A\left(3r + \frac{1}{2}, \lambda\right) \left(1 - e^{-x}\right)^{3r}\right) dx. \tag{37}$$

4. Conclusion

In this paper, we have studied the expectation of the maximal length of 2-path in random critical graph by means of

enumerative and analytic combinatorics approaches when the size of the graph goes to infinity. Our analysis gives a precise description of the parameter near the critical point.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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