Research Article

Global Stability of an Economic Model with a Continuous Delay of Kaldor Type Modified

Aka Fulgence Nindjin, Tetchi Albin N'guessan ⓘ, Sahoua Hypolithe Okou A Kpetihi ⓘ, and Kessé Tiban Tia

UFR de Mathématiques et Informatique Université Félix Houphouët Boigny d'Abidjan Cocody 22 BP 582 Abidjan 22, Côte d'Ivoire

Correspondence should be addressed to Tetchi Albin N'guessan; albintetchi@gmail.com and
Sahoua Hypolithe Okou A Kpetihi; okouakpetihi@hotmail.com

This paper studies continuous nonlinear economic dynamics with a continuous delay of a Kaldor type modified in dimension two. The important results are, on the one hand, the boundedness of solutions, the existence of an attractive set, and the permanence of the system and, on the other hand, the local and global stability of equilibrium points.

1. Model

The complex nonlinear dynamic has been introduced into the analysis of economic phenomena to explain not only the fluctuations observed in the series studies but also the economic crisis in the capitalist system. Thus, economists such as Goodwin (1967) and Kaldor (1955-1956) have employed dynamic models to explain that the cyclic and chaotic growth curves are the economic phenomena endogenous to the capitalist system itself. Within the framework of structural reforms for economic dynamics, NINDJIN et al. in [1] suggest a proven model originating from ecology which could have a certain range in analysis and regulation of financial systems. This model of dimension two describes how the GDP and an economic capital interact in accordance with the model of Kaldor modified in order to increase the resilience of these dynamics against possible disruptions. Mathematical analysis of this model (see [1]) demonstrates that it is bounded and permanent and admits under certain circumstances an attractive set. On the one hand, this permanence manifests itself in the form of stationary growth of capital stock and product (stable interior equilibrium point). On the other hand, it appears in the capital cyclic growth and the product (limit cyclic). So, a capital stock rupture or a long-term production is prevented. It is also shown that the financial system stability (relative to the capital and the product) is overall; i.e., it does not depend on either stock level or production at the initial time. Facing a possible disruption of one of the control parameters of the economic system, we have analyzed, in [2], the model bifurcations. We have shown that the model admits a transcritical bifurcation, a pitchfork bifurcation or a Hopf bifurcation. In the last one, when the model is disrupted, it changes from a stationary balanced growth to a cyclic growth by preventing a GDP or capital crisis. Interactions between the GDP and the economy's capital could not be clearly and definitively understood or explained regardless of past situations which may affect the present or the future. In this paper, we are going to focus on the way in which investments are evolved and savings are established. Indeed, economies finance their investments through their own savings or those of other economies via financial structures with an interest rate. Regarding the savings, it is known that they are made up of a portion of profits or wages. Thus, when the economy is able to be self-funding by its savings, then the net profit increases $\pi_{net} = \pi - rD$ (i.e., profit, π deprived of the portion set aside to pay interest, r debt, D). So, saving at a time t depends on the net profit; consequently, the GDP from a time $t_0 = t - \overline{T}$ where $\overline{T} > 0$. Let us suppose that $\overline{T}$ is the deadline needed by this saving to reach a certain threshold likely to ensure the self-financing of the investments of the economy.

Let us consider the dynamics with no delay of the modified Kaldor type and the following assumption:

$$\dot{Y} = \left[a_0 Y \left(1 - \frac{Y}{C_0} \right) \right]$$

$$+ \alpha \left(a_1 K + b_1 \right) Y \left(1 - \frac{m_2 K}{Y + C_2} \right)$$

$$- \left[\alpha \frac{m_1 Y}{Y + C_1} \right] K, \tag{1.1}$$

$$\dot{K} = \left[\left(a_2 Y + b_2 \right) \left(1 - \frac{m_2 K}{Y + C_2} \right) \right] K - \varrho K,$$

$$Y(0) > 0, \quad K(0) > 0, \quad Y, K \in C^1 \left([0; +\infty[\ ; \mathbb{R}_+ \right),$$

with $(a_0, C_0, C_1, C_2, m_1, m_2, \alpha) \in (\mathbb{R}_+^*)^7$ and $(a_1, a_2, b_1, b_2, \varrho) \in (\mathbb{R}_+)^5$.

Y denotes the product, K denotes the stock of capital, and $\dot{Y}$ and $\dot{K}$ indicate, respectively, the growth rate of the product and the stock of capital depending on the following economic parameters:

(i) a_0 the trend of rate of increase in GDP for a given (future) period in absence (or in neglect) of the losses,

(ii) C_0 the maximum (monetary) value of GDP we can get from this economy for that given period,

(iii) α the currency adjustment factor,

(iv) m_1 the maximum (monetary) value of the genuine saving of this economy for the given period,

(v) C_1 the maximum (monetary) value of the saving supported by the economy in the given period,

(vi) m_2 the maximum value of the investment rates losses for the given period,

(vii) C_2 the maximum capital stock for the given period,

(viii) a_1 the derivative relative to the capital of the investment rate in the absence (or in the neglect) of the losses (when $m_2 = 0$) for the given period,

(ix) b_1 the investment rate when the capital is null ($K = 0$) and this rate has suffered no loss ($m_2 = 0$) for the given period,

(x) a_2 the share of the GDP converted into stock of capital for the given period,

(xi) b_2 the accumulation rate of capital when the product is null ($Y = 0$) and that the investment rate has suffered no loss ($m_2 = 0$) for the given period.

(see [1], page 4).

Assumption 1. The ratio saving-capital $g(Y, K) = m_1 Y/(Y + C_1)$ at a given time t depends on the GDP produced since $t_0 = t - \overline{T}$ with a probability of exponential lag (see, [3], appendix, page 272).

So, the ratio saving-capital with delay is $\tilde{g}(t, \overline{T}) = m_1 x(t)/(x(t) + C_1)$ with $x(t) = \int_{t-\overline{T}}^{t} \mu e^{-\mu(t-\xi)} Y(\xi) d\xi$ so that $\int_{-\infty}^{t} \mu e^{-\mu(t-\xi)} d\xi = 1, \mu \in \mathbb{R}_+^*$. Then,

$$\dot{x} = -\mu (x - Y) - \mu e^{-\mu \overline{T}} Y \left(t - \overline{T} \right). \tag{1}$$

By replacing $g(Y, K)$ by $\tilde{g}(t, \overline{T})$, one obtains continuous nonlinear economic dynamics with a continuous delay of a Kaldor type modified in dimension two. Enclosing (1) to the system with a continuous delay, one gets the following system with discrete delay in dimension three:

$$\dot{Y} = a_0 \left[1 - \frac{Y}{C_0} \right] Y$$

$$+ \alpha \left(a_1 K + b_1 \right) \left[\left(1 - \frac{m_2 K}{Y + C_2} \right) \right] Y$$

$$- \alpha \left[\frac{m_1 x}{x + C_1} \right] K,$$

$$\dot{K} = \left(a_2 Y + b_2 \right) \left[\left(1 - \frac{m_2 K}{Y + C_2} \right) \right] K - \varrho K, \tag{1.2}$$

$$\dot{x} = -\mu (x - Y) - \mu e^{-\mu \overline{T}} Y \left(t - \overline{T} \right),$$

$$Y, K, x \in C^1 \left([0; +\infty[\ ; \ \mathbb{R}_+ \right), \quad Y(0) > 0, \quad K(0) > 0,$$

with $(a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, m_1, m_2, \varrho, \alpha, \overline{T}, \mu) \in (\mathbb{R}_+^*)^{14}$, $Y_t(\theta) = Y(t + \theta), \forall \theta \in [-\overline{T}; 0]$, and the feedback function $Y_0 = \varphi_1 \in C([-\overline{T}; 0]; \mathbb{R}_+)$ is consequently $Y_0(0) = Y(0)$.

To facilitate the qualitative study of system (1.2) which possesses 14 parameters $(a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, m_1, m_2, \varrho, \alpha, \overline{T}, \mu) \in (\mathbb{R}_+^*)^{14}$, let us change the variables by reducing the number of parameters.

Let us define the new variables:

$$\tau = a_0 t,$$

$$u(\tau) = \frac{Y(t)}{c_0},$$

$$v(\tau) = \frac{m_2}{c_0} K(t), \tag{2}$$

$$w(t) = \frac{x}{c_0} = \int_{t-T}^{t} \lambda e^{-\lambda(t-\xi)} u(\xi) d\xi.$$

Let us define the new parameters of control:

$$\beta_1 = \frac{\alpha a_1 C_0}{a_0 m_2},$$

$$\beta_2 = \frac{a_2 C_0}{a_0},$$

$$\alpha_1 = \frac{\alpha b_1}{a_0},$$

$$\alpha_2 = \frac{b_2}{a_0},$$

$$d_1 = \frac{C_1}{C_0},$$

$$d_2 = \frac{C_2}{C_0},$$

$$\gamma = \alpha \frac{m_1}{a_0 m_2},$$

$$\delta = \frac{\varrho}{a_0},$$

$$(3)$$

$T = a_0 \overline{T}$, and $\lambda = \mu/a_0$. Then, system (1.2) becomes

$$\dot{u}(\tau) = [1 - u(\tau)] u(\tau)$$

$$+ (\beta_1 v(\tau) + \alpha_1) \left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right) u(\tau)$$

$$- \frac{\gamma w(\tau)}{w(\tau) + d_1} v(\tau),$$

$$\dot{v}(\tau) = (\beta_2 u(\tau) + \alpha_2) \left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right) v(\tau) - \delta v(\tau),$$

$$\dot{w}(\tau) = -\lambda (w(\tau) - u(\tau)) - \lambda e^{-\lambda T} u(\tau - T),$$

$$u, v, w \in C^1([0; +\infty[; \mathbb{R}_+) \ u(0) > 0, \ v(0) > 0, \ w(0) > 0,$$

$$(1.3)$$

with $(d_1, d_2, \gamma, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta, T, \lambda) \in (\mathbb{R}_+^*)^{10}$ and $u_0 = \varphi \in C([-T; 0]; \mathbb{R}_+)$ so that $u_0(0) = u(0)$.

In the long run, we shall adopt the following notations:

$$\|\varphi\| = \sup_{\theta \in [-T; 0]} \{|\varphi(\theta)|\},$$

$$k = 1 - e^{-\lambda T}, \qquad (4)$$

and $w(0) = \int_{-T}^{0} \lambda \varphi(\theta) e^{\lambda \theta} d\theta.$

2. Boundedness and Equilibria Points of (1.3)

2.1. Boundedness of the Solutions of Model (1.3)

Lemma 2. *The interior* $\text{int}(\mathbb{R}_+^3)$ *and the boundary* $\partial(\mathbb{R}_+^3)$ *of a positive cone* $\mathbb{R}_+^3$ *are invariant for model* (1.3).

Proof. Given $0 \le T_0 < +\infty$. Let τ be in $[-T; T_0[$. One knows that

$$u(\tau) = u(0) \exp \left\{ \int_0^\tau \left[1 - u(t) + (\beta_1 v(t) + \alpha_1) \left(1 - \frac{v(t)}{u(t) + d_2}\right) - \frac{\gamma v(t) w(t)}{u(t)(w(t) + d_1)} \right] dt \right\},$$

$$v(\tau) = v(0) \exp \left\{ \int_0^\tau \left[(\beta_2 u(t) + \alpha_2) \left(1 - \frac{v(t)}{u(t) + d_2}\right) - \delta \right] dt \right\}, \qquad (5)$$

$$w(\tau) = w(0) \exp \left\{ \int_0^\tau \frac{-\lambda (w(t) - u(t)) - \lambda e^{-\lambda T} u(t - T)}{w(t)} dt \right\}.$$

Thus, on the one hand, $(u(0), v(0), w(0)) \in \partial(\mathbb{R}_+^3) \implies (u(\tau), v(\tau), w(\tau)) \in \partial(\mathbb{R}_+^3)$, and, on the other hand, $(u(0), v(0), w(0)) \in \text{int}(\mathbb{R}_+^3) \implies (u(\tau), v(\tau), w(\tau)) \in \text{int}(\mathbb{R}_+^3)$. □

Lemma 3 (see [4]). *Given* $(A, B) \in \mathbb{R}_+^2$ *and* ϕ *a continuous and derivable function as there is* $t_0 \ge 0$ *verifying* $\phi(t_0) > 0$. *Then,* $\forall t \ge t_0$,

$$\frac{d\phi}{dt} \le \phi(B - A\phi) \implies$$

$$\limsup_{t \to +\infty} \phi(t) \le \frac{B}{A},$$

$$\frac{d\phi}{dt} \ge \phi(B - A\phi) \implies$$

$$(7)$$

$$\liminf_{t \to +\infty} \phi(t) \ge \frac{B}{A}.$$

Theorem 4. *Let us assume that* $\alpha_2 > \delta$. *Let us pose*

$$M_u = \frac{4 d_2 \beta_1 + (\beta_1 d_2 + 2\alpha_1) \beta_1 d_2 + \alpha_1^2}{\beta_1 d_2 (4 - \beta_1)} \quad \text{if } 0 < \beta_1 < 4$$

and $M_u = 1 + \alpha_1 \quad \text{if } \beta_1 = 0,$

$$M_v = \frac{(M_u + d_2)(\beta_2 M_u + \alpha_2 - \delta)}{\alpha_2}$$

and $m_v = \frac{(\alpha_2 - \delta) d_2}{\beta_2 d_2 + \alpha_2},$

$$m_u^{(0)} = m^u + \beta_1 m_v,$$

$$m^u = -\frac{\beta_1}{d_2} M_v^2 - \frac{\alpha_1}{d_2} M_v + 1 + \alpha_1,$$

$$m_u^\varepsilon = \frac{m_u^{(0)} + \sqrt{\left[m_u^{(0)} - \varepsilon\right]^2 - 4\gamma k M_u M_v / d_1}}{2},$$

$$0 \le \varepsilon \le m_u^{(0)} - 2\sqrt{\frac{\gamma k M_u M_v}{d_1}}$$

$$or \ m_u^{(0)} + 2\sqrt{\frac{\gamma k M_u M_v}{d_1}} \le \varepsilon,$$

$$M_w = k M_u$$

$$and \ m_w = k m_u^\varepsilon. \tag{8}$$

$$If \ m_u^{(0)} \ge 2\sqrt{\frac{\gamma k M_u M_v}{d_1}}, \tag{9}$$

and, then, model (1.3) *is permanent. Otherwise, the set defined by*

$$\mathscr{A}^+ = \Big\{ (u, v, w) \in \mathbb{R}_+^3 : m_u^\varepsilon \le u \le M_u, \ m_v \le v \\ \le M_v \ and \ m_w \le w \le M_w \Big\} \tag{10}$$

Which is a bounded set, positively invariant for model (1.3).

Proof. Let us consider system (1.3) and Lemma 3.

(1) One has the following: $\dot{v}(\tau)$ does not depend on the variable w. So, one obtains the same results like those of the model with no delay, i.e.,

$\liminf_{t \to +\infty} [v(t)] \ge m_v$ and $\limsup_{t \to +\infty} [v(t)] \le M_v$.

(2) One has the following: $du/dt \le (1-u)u + (\beta_1 v + \alpha_1)(1 - v/(u + d_2))u$ because $-\gamma w v/(w + d_1) < 0$.

Then, $du/dt \le (B_1 - A_1 u)u$ with $(A_1, B_1) = (1; 1 + \alpha_1)$ if $\beta_1 = 0$

and $(A_1, B_1) = ((4 - \beta_1)/4; (4d_2\beta_1 + (\beta_1 d_2 + 2\alpha_1)\beta_1 d_2 + \alpha_1^2)/4\beta_1 d_2)$ if $0 < \beta_1 < 4$ and, in that case, $\max_v\{(\beta_1 v + \alpha_1)(1 - v/(u + d_2))u\} = u[\beta_1(u + d_2) + \alpha_1]^2/4\beta_1(u + d_2)$.

By property (6) of Lemma 3, one obtains $\limsup_{t \to +\infty} [u(t)] \le M_u$.

(3) One knows that $0 \le u(\tau)$. Let us suppose that $u(0) > 0$ and, then, there exist t_0 so that, $\forall \tau \ge t_0, 0 < u(\tau) \le M_u$. Let us suppose that $l = \liminf_{t \to +\infty} [u(t)]$. Thus, for any ε very tiny, there is $t_1 > 0$ so that, $\forall \tau > t_1$, $u(\tau) \ge l - \varepsilon > 0$.

Let us pose $t_2 = \max(t_0, t_1)$ and $m = l - \varepsilon$. Then, $\forall \tau > t_2, 0 < m < u(\tau) \le M_u$. Thus

$$\forall \tau > t_2,$$
$$0 < w(\tau) < \frac{u(\tau)w(\tau)}{m} \le \frac{M_u w(\tau)}{m}. \tag{11}$$

Hence, $\forall \tau > t_2, -\gamma w(\tau)/(w(\tau) + d_1) > -\gamma(u(\tau)w(\tau)/m)/(u(\tau)w(\tau)/m + d_1)$ for $g(x) = -\gamma x/(x + d_1)$ is decreasing upon $\mathbb{R}_+$. On the one hand,

it is known that $w(t) = \int_{t-T}^t \lambda e^{-\lambda(t-\xi)} u(\xi) d\xi$ and $\int_{-\infty}^t \lambda e^{-\lambda(t-\xi)} d\xi = 1$.

Meanwhile, $\limsup_{t \to +\infty} \{\int_{t-T}^t \lambda e^{-\lambda(t-\xi)} u(\xi) d\xi\} \le \int_{t-T}^t \lambda e^{-\lambda(t-\xi)} d\xi \times \limsup_{t \to +\infty} [u(t)]$ and, then,

$\limsup_{\tau \to +\infty} [w(\tau)] \le M_u(1 - e^{-\lambda T}) = k M_u = M_w$. On the other hand,

$\dot{u}/u = [1 - u(\tau)] + (\beta_1 v(\tau) + \alpha_1)(1 - v(\tau)/(u(\tau) + d_2)) - (\gamma w(\tau)/(w(\tau) + d_1))(v(\tau)/u(\tau))$ and $1 + (\beta_1 v(\tau) + \alpha_1)(1 - v(\tau)/(u(\tau) + d_2)) \ge m_u^{(0)} = -(\beta_1/d_2)M_v^2 - (\alpha_1/d_2)M_v + 1 + \alpha_1 + \beta_1 m_v$. Then, $\tau > t_2$, and $\dot{u} \ge [m_u^{(0)} - \gamma w(\tau)v(\tau)/(u(\tau)w(\tau) + md_1) - u]u(\tau)$. So, $du/dt \ge (B_2 - A_2 u)u$ with $A_2 = 1$ and $B_2 = m_u^{(0)} - \gamma M_v M_w/md_1$. Therefore, applying property (7) of Lemma 3 one obtains $\liminf_{t \to +\infty} [u(t)] \ge m_u^{(0)} - \gamma M_v M_w/md_1$.

Consequently, $l \ge m_u^{(0)} - \gamma M_v M_w/(l - \varepsilon)d_1$ for $m = l - \varepsilon$. So, one obtains

$$l^2 - \left(m_u^{(0)} + \varepsilon\right)l + \frac{\gamma M_u M_w}{d_1} + m_u^{(0)}\varepsilon \ge 0. \tag{12}$$

If $m_u^{(0)} \ge 2\sqrt{\gamma k M_u M_v/d_1}$ then $[m_u^{(0)} - \varepsilon]^2 - 4\gamma M_w M_v/d_1 \ge 0$.

Thus one obtains $\liminf_{t \to +\infty} [u(t)] \ge m_u^\varepsilon = (m_u^{(0)} + \sqrt{[m_u^{(0)} - \varepsilon]^2 - 4\gamma M_w M_v/d_1})/2$

with $0 \le \varepsilon \le m_u^{(0)} - 2\sqrt{\gamma k M_u M_v/d_1}$ or $m_u^{(0)} + 2\sqrt{\gamma k M_u M_v/d_1} \le \varepsilon$.

(4) It is known that $\liminf_{t \to +\infty} \{\int_{t-T}^t \lambda e^{-\lambda(t-\xi)} u(\xi) d\xi\} \ge \int_{t-T}^t \lambda e^{-\lambda(t-\xi)} d\xi \times \liminf_{t \to +\infty} [u(t)]$. So, $\liminf_{t \to +\infty} [w(t)] \ge m_w = k m_u^\varepsilon$.

□

Remark 5. Let us consider the notations of Theorem 4.

(1) Let us consider the following assumptions:

$$\alpha_2 > \delta,$$

$$m_u^{(0)} > 0,$$

$$\frac{d_1 \left[m_u^{(0)}\right]^2}{4\gamma M_u M_v} < 1 \tag{13}$$

$$\text{and } T < -\frac{1}{\lambda} \ln \left[1 - \frac{d_1 \left[m_u^{(0)}\right]^2}{4\gamma M_u M_v} \right].$$

$$\alpha_2 > \delta,$$

$$m_u^{(0)} > 0 \tag{14}$$

$$\text{and } 0 \le \gamma < \frac{d_1 \left[m_u^{(0)}\right]^2}{4k M_u M_v}.$$

One remarks that each of assumptions (13) and (14) implies, respectively, condition (9).

(2) If $m_u^{(0)} \ge 2\sqrt{\gamma k M_u M_v / d_1}$, one has $0 < m_u^{(0)}/2 \le m_u^\varepsilon \le \liminf_{t \to +\infty}[u(t)]$. So, we can adjust the minimum value of $\liminf_{t \to +\infty}[u(t)]$ by choosing the value of ε.

2.2. The Equilibria Points of Model (1.3)

2.2.1. Points of Trivial Equilibria.
If $0 < \alpha_2 \le \delta$ then system (1.3) admits two points of trivial equilibria: $E_0^* = (0; 0; 0)$, $E_1^* = (1 + \alpha_1; 0; k(1 + \alpha_1))$.

If $\alpha_2 > \delta \ge 0$ then system (1.3) admits three points of trivial equilibria: $E_0^* = (0; 0; 0)$, $E_1^* = (1 + \alpha_1; 0; k(1 + \alpha_1))$, and $E_2^* = (0; (\alpha_2 - \delta)d_2/\alpha_2; 0)$.

2.2.2. Points of Interior Equilibria.
Given $p_1(x) = a_{14}x^4 + a_{13}x^3 + a_{12}x^2 + a_{11}x + a_{10}$ such as

$$a_{14} = -k\beta_2^2,$$

$$a_{13} = (k - d_1)\beta_2^2 - 2k\beta_2\alpha_2 + k\beta_2(\beta_1\delta - \gamma\beta_2),$$

$$a_{12} = 2\beta_2\alpha_2(k - d_1) - k\alpha_2^2 + d_1\beta_2^2 + k\alpha_1\delta\beta_2$$
$$\quad - k\gamma\beta_2\alpha_2 + \beta_1\beta_2\delta d_1$$
$$\quad + k(\beta_2 d_2 + \alpha_2 - \delta)(\beta_1\delta - \gamma\beta_2),$$

$$a_{11} = \alpha_1\delta(\beta_2 d_1 + k\alpha_2) + (k - d_1)\alpha_2^2 + 2\beta_2\alpha_2 d_1$$
$$\quad + k(\beta_1\delta - \gamma\beta_2)d_2(\alpha_2 - \delta)$$
$$\quad + (\beta_1\delta d_1 - k\gamma\alpha_2)(\beta_2 d_2 + \alpha_2 - \delta),$$

$$a_{10} = d_1\alpha_2^2 + \alpha_1\alpha_2\delta d_1 + (\beta_1\delta d_1 - k\gamma\alpha_2)d_2(\alpha_2 - \delta). \tag{15}$$

(1) System (1.3) does not admit any point of interior equilibria if $\beta_2 M_u + \alpha_2 - \delta < 0$.

(2) Any interior equilibrium point $E_3^* = (u^*; v^*; w^*)$ of system (1.3) verifies the following relations:

$$p_1(u^*) = 0,$$

$$v^* = \frac{(\beta_2 u^* + \alpha_2 - \delta)}{\beta_2 u^* + \alpha_2}(u^* + d_2)$$
$$\text{with } \beta_2 u^* + \alpha_2 - \delta > 0, \tag{16}$$

$$w^* = ku^*.$$

Designating P_{xy} the projection over the plan xy, we study the border dynamics of the model.

3. Border Dynamics on the Plan uv

On uv plan, model (1.3) becomes

$$\dot{u}(\tau) = [1 - u(\tau)]u(\tau) + (\beta_1 v(\tau) + \alpha_1)\left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right)u(\tau),$$

$$\dot{v}(\tau) = (\beta_2 u(\tau) + \alpha_2)\left(1 - \frac{v(\tau)}{u(\tau) + d_2}\right)v(\tau) - \delta v, \tag{3.1}$$

$$u, v, w \in C^1\left([0; +\infty[; \ \mathbb{R}_+\right), \ u(0) > 0, \ v(0) > 0, \ u_0 = \varphi \in C\left([-T; 0]; \mathbb{R}_+\right),$$

with $(d_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta, T, \lambda) \in (\mathbb{R}_+^*)^7$.
The equilibria points of model (3.1) are

$$U_0^{(1)} = P_{uv}(E_0^*) = (0, 0),$$

$$U_1^{(1)} = P_{uv}(E_1^*) = (1 + \alpha_1; 0), \tag{17}$$

$$U_2^{(1)} = P_{uv}(E_2^*) = \left(0; \frac{(\alpha_2 - \delta)d_2}{\alpha_2}\right), \tag{18}$$

$$U_3^{(1)} = \left(\bar{u}; \frac{(\beta_2\bar{u} + \alpha_2 - \delta)(\bar{u} + d_2)}{\beta_2\bar{u} + \alpha_2}\right) \tag{19}$$
$$\text{if } \bar{u} > 0 \text{ and } (\beta_2\bar{u} + \alpha_2 - \delta) > 0,$$

where $p_0(\overline{u}) = a_{03}\overline{u}^3 + a_{02}\overline{u}^2 + a_{01}\overline{u} + a_{00} = 0.$ (20)

With $a_{03} = -\beta_2^2$,

$$a_{02} = (\beta_2 - 2\alpha_2 + \delta\beta_1)\beta_2,$$

$$a_{01} = \alpha_2(2\beta_2 - \alpha_2) + \beta_1\delta(\beta_2 d_2 + \alpha_2 - \delta) \quad (21)$$

$$+ \delta\beta_2\alpha_1,$$

$$a_{00} = \alpha_2^2 + \delta\beta_1 d_2(\alpha_2 - \delta) + \alpha_1\alpha_2\delta.$$

Let us give below the results on the permanence of the model. Therefore, let us consider the notations of Theorem 4. If $\alpha_2 > \delta$ and $m_u^{(0)} > 0$ then model (3.1) is permanent. Among others, the set defined by

$$\mathscr{A}^{(0)}$$

$$\quad (22)$$

$$= \left\{ (u, v) \in \mathbb{R}_+^2 : m_u^{(0)} \le u \le M_u, \text{ and } m_v \le v \le M_v \right\}$$

which is a bounded set, positively invariant for model (3.1).

3.1. Local Stability of the Equilibria Points of Model (3.1). Performing the spectral study of Jacobian matrix of the system linearized around each of the points of equilibrium, one obtains, classically, the following conclusions:

(1) Stability of $U_0^{(1)} = (0; 0)$:

    (a) $U_0^{(1)}$ is an unstable node if $\alpha_2 > \delta$.

    (b) $U_0^{(1)}$ is a point unstable saddle repulsive following direction u and attractive following direction v if $\alpha_2 < \delta$.

(2) Stability $U_1^{(1)} = (1 + \alpha_1; 0)$:

    (a) $U_1^{(1)}$ is stable if $\alpha_2 + (1 + \alpha_1)\beta_2 < \delta$.

    (b) $U_1^{(1)}$ is a point unstable saddle attractive following direction u and repulsive following direction w_1 if $\alpha_2 + (1 + \alpha_1)\beta_2 > \delta$.

(3) Stability of $U_2^{(1)} = (0; (\alpha_2 - \delta)d_2/\alpha_2)$ for $\alpha_2 > \delta$:

    (a) $U_2^{(1)}$ is stable if $\psi(\delta) < 0$.

    (b) $U_2^{(1)}$ is a point of unstable saddle if $\psi(\delta) > 0$, repulsive following w_2, and attractive following v. Consider

$$\psi(x) = -d_2\beta_1 x^2 + [d_2\beta_1\alpha_2 + \alpha_1\alpha_2]x + \alpha_2^2. \quad (23)$$

(4) Given $U_3^{(1)} = (\overline{u}; \overline{v})$ interior equilibrium point of (3.1) testifying the system (19)-(20) and J its associated Jacobian matrix.

    (a) $U_3^{(1)}$ is stable (stable node or stable focus) if $\det(J) > 0$ and $Tr(J) < 0$.

    (b) $U_3^{(1)}$ is marginal or center if $\det(J) \ge 0$ and $Tr(J) = 0$.

    (c) $U_3^{(1)}$ is unstable if $\det(J) < 0$ or ($\det(J) > 0$ and $Tr(J) > 0$), precisely:

        (i) $U_3^{(1)}$ is a node or a focus if $\det(J) > 0$ and $Tr(J) > 0$,

        (ii) $U_3^{(1)}$ is a point unstable saddle if $\det(J) < 0$.

With $Tr(J)$ the trace and $\det(J)$ determinant of J, vectors

$$w_1 = \left((1 + \alpha_1)\left[\beta_1 - \frac{\alpha_1}{1 + \alpha_1 + d_2} \right]; (1 + \alpha_1)(\beta_2 + 1) \right.$$

$$\quad (24)$$

$$\left. + \alpha_2 - \delta \right).$$

and $w_2 = (\psi(\delta)/\alpha_2^2 + \alpha_2 - \delta; [\beta_2 d_2\delta + \alpha_2(\alpha_2 - \delta)](\alpha_2 - \delta)/\alpha_2^2).$

3.2. Global Stability of $U_3^{(1)} = (\overline{u}; \overline{v})$. Now let us define the conditions for which the stability of the product and the stock of capital of the economy is global; i.e., it does not depend on the produced quantities and level of stock at the initial moment. For this study, we define an appropriate Lyapunov function.

Theorem 6. *Let us consider the following assumptions:*

(3.1) admits a unique interior equilibrium point $U_3^{(1)}$

$$\quad (25)$$

$$= (\overline{u}; \overline{v}),$$

$$0 < \beta_1 < 4,$$

$$\alpha_2 > \delta,$$

$$\beta_2 d_2 < 2\alpha_2,$$

$$\quad (26)$$

$$d_2 < d_1,$$

$$\frac{2\delta\beta_1}{\alpha_2} < \frac{\alpha_1}{M_u + d_2},$$

$$\frac{\beta_1 M_v}{d_2} + \frac{\alpha_1}{d_2} < \frac{\beta_1 d_2(2\alpha_2 - \delta)}{\alpha_2(M_u + d_2)} + \frac{\delta\beta_1(1 + d_2\beta_2/2\alpha_2)}{\alpha_2}, \quad (27)$$

$$\frac{1 + \alpha_1}{d_2} - 1 + \frac{2\beta_1 d_2^2}{\alpha_1^2} < 0. \quad (28)$$

where M_u and M_v are defined in Theorem 4.

If assumptions (25), (26), (27), (28), (29), (30), (31), (35), (36), and (37) are verified, then, the unique point of the interior equilibrium of model (3.1) is globally and asymptotically stable.

Proof. Let us consider system (3.1). Let us suppose that assumption (35) is verified and, then, model (3.1) admits a unique interior equilibrium point $U_3^{(1)} = (\bar{u}; \bar{v})$ verifying

$$1 = \bar{u} - \alpha_1 \left(1 - \frac{\bar{v}}{\bar{u} + d_2} \right),$$

$$\alpha_2 - \delta = -\beta_2 \bar{u} + \frac{(\beta_2 \bar{u} + \alpha_2) \bar{v}}{\bar{u} + d_2}. \tag{29}$$

Let us note that $\lambda = \beta_1 d_2 / 2\alpha_2$. Given $V_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $V_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ one has

$$V_1(u, v) = \left[u - \bar{u} - \bar{u} \ln \left(\frac{u}{\bar{u}} \right) \right] = \int_{\bar{u}}^{u} \left[1 - \frac{\bar{u}}{x} \right] dx, \tag{30}$$

$$V_2(u, v) = \lambda \left[v - \bar{v} - \bar{v} \ln \left(\frac{v}{\bar{v}} \right) \right] = \lambda \int_{\bar{v}}^{v} \left[1 - \frac{\bar{v}}{x} \right] dx. \tag{31}$$

Considering the Lyapunov functional $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$ one has

$$V(u, v) = V_1(u, v) + V_2(u, v). \tag{32}$$

Let us pose

$$g(u, v) = -1 + \frac{\beta_1 v^2 + \alpha_1 v}{(\bar{u} + d_2)(u + d_2)};$$

$$h(u, v) = \frac{1}{2} \left[\beta_1 - \frac{\beta_1 (v + \bar{v}) + \alpha_1}{(\bar{u} + d_2)} \right] \tag{33}$$

$$+ \frac{\lambda}{2} \left[\beta_2 - \frac{\beta_2 \bar{v}}{\bar{u} + d_2} + \frac{(\beta_2 u + \alpha_2) v}{(\bar{u} + d_2)(u + d_2)} \right].$$

By using relation (29) and model (3.1), one obtains

$$\frac{dV}{dt} \leq [\phi_1(u, v)] (u - \bar{u})^2 + [\phi_2(u, v)] (v - \bar{v})^2 \tag{34}$$

with $\phi_1 = g(u, v) + |h(u, v)|$ and $\phi_2 = |h(u, v)| - \lambda(\beta_2 u + \alpha_2)/(\bar{u} + d_2)$, $\forall (u, v) \in \mathbb{R}^2$.

Let us overestimate $g(u, v)$ and $h(u, v)$ and, then, $\phi_1(u, v)$ and $\phi_2(u, v)$.

(1) One has $g(u, v) < R_0 < 0$, $\forall (u, v) \in \mathbb{R}^2$, if $R_0 = -1 + (\beta_1 M_v^2 + \alpha_1 M_v)/d_2^2 < 0$.

(2) if $\beta_2 < 2\alpha_2/d_2$ then $\partial h/\partial v < 0$. So, $h(M_u, M_v) \leq h(u, v) \leq h(0, 0)$. Therefore, $h(0, 0) < 0$ if $\beta_2 d_2 < 2\alpha_2$, $d_2 < d_1$, $2\delta\beta_1/\alpha_2 < \alpha_1/(M_u + d_2)$. Consequently, $|h(u, v)| = -h(u, v) < -h(M_u, M_v) < R_1 = \beta_1 M_v/2d_2 - (d_2\beta_1 M_v/2\alpha_2)(\beta_2 M_u + \alpha_2)/(M_u + d_2)^2 + (1/2)[-(\beta_1\delta/\alpha_2)(1 + \beta_2 d_2/2\alpha_2) + \alpha_1/d_2]$. Thus, $\phi_1(u, v) < R_0 + R_1$ and $\phi_2(u, v) < R_1 - \beta_1 d_2/2(M_u + d_2)$ if $\beta_2 d_2 < 2\alpha_2$, $d_2 < d_1$, and $2\delta\beta_1/\alpha_2 < \alpha_1/(M_u + d_2)$.

Hence, $dV/dt < 0$, $\forall (u, v) \in \mathbb{R}^2$, if

$$0 < \beta_1 < 4,$$

$$\alpha_2 > \delta,$$

$$\beta_2 < \frac{\alpha_2}{d_2}, \tag{35}$$

$$\frac{2\delta\beta_1}{\alpha_2} < \frac{\alpha_1}{M_u + d_2},$$

$$\frac{\beta_1 M_v}{2d_2} + \frac{\alpha_1}{2d_2}$$

$$< \frac{\beta_1 d_2}{2(M_u + d_2)} + \frac{d_2\beta_1 M_v}{2\alpha_2} \frac{(\beta_2 M_u + \alpha_2)}{(M_u + d_2)^2} \tag{36}$$

$$+ \frac{\beta_1\delta}{2\alpha_2} \left(1 + \frac{\beta_2 d_2}{2\alpha_2} \right),$$

and $\dfrac{\beta_1 M_v^2 + \alpha_1 M_v}{d_2^2} - 1 + \dfrac{\beta_1 d_2}{2(M_u + d_2)} < 0.$ (37)

The model is permanent if $m^u = -(\beta_1/d_2)M_v^2 - (\alpha_1/d_2)M_v + 1 + \alpha_1 > 0$. Then, $(\beta_1/d_2^2)M_v^2 + (\alpha_1/d_2^2)M_v < (1 + \alpha_1)/d_2$. Moreover, $1/(M_u + d_2) < 4d_2/\alpha_1^2$.

So, from formulas (35), one obtains (27) $\Longrightarrow$ (36) and (28) $\Longrightarrow$ (37).

Hence, $dV/dt < 0$, $\forall (u, v) \in \mathbb{R}^2$, if assumptions (25)–(28) are verified. $\square$

Remark 7. As $\beta_1 = 0$, the unique interior equilibrium point of model (3.1) is globally and asymptotically stable if the following assumptions are verified:

$$\alpha_2 > \delta,$$

$$\beta_2 < \frac{\alpha_2}{d_2}, \tag{38}$$

$$\frac{\delta\beta_2}{\alpha_2} < \frac{\alpha_1}{M_u + d_2},$$

$$-1 + \frac{1 + \alpha_1}{d_2} + \frac{4\alpha_2 d_2}{\alpha_1^2} < 0, \tag{39}$$

$$\frac{\alpha_1}{2d_2} + \frac{\delta\beta_2}{\alpha_2} < \frac{3(\alpha_2 - \delta)}{2(M_u + d_2)}, \tag{40}$$

with constant M_u and M_v defined in Theorem 4.

In fact, it is worth taking the Lyapunov functional $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$ so that $V(u, v) = V_1(u, v) + V_2(u, v)$ with $V_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $V_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ so that

$$V_1(u, v) = \left[u - \bar{u} - \bar{u} \ln \left(\frac{u}{\bar{u}} \right) \right] = \int_{\bar{u}}^{u} \left[1 - \frac{\bar{u}}{x} \right] dx, \tag{41}$$

$$V_2(u, v) = \left[v - \bar{v} - \bar{v} \ln \left(\frac{v}{\bar{v}} \right) \right] = \int_{\bar{v}}^{v} \left[1 - \frac{\bar{v}}{x} \right] dx. \tag{42}$$

Considering (38)–(40), one shows, similarly in $\beta_1 \neq 0$, that $dV/dt < 0$, $\forall (u, v) \in \mathbb{R}^2$.

4. Border Dynamics of Plan uw

On the plan uw model (1.3) becomes

$$
\begin{aligned}
&\dot{u}(\tau) = [1 + \alpha_1 - u(\tau)] u(\tau), \\
&\dot{w} = -\lambda (w - u) - \lambda e^{-\lambda T} u(\tau - T), \\
&u, w \in C^1 ([0; +\infty[; \mathbb{R}_+), \ u(0) > 0, \ u_0 = \varphi \in C([-T; 0]; \mathbb{R}_+)
\end{aligned}
\tag{4.1}
$$

with $(\alpha_1, T, \lambda) \in (\mathbb{R}_+^*)^3$. The equilibria of model (4.1) are

$$
\begin{aligned}
U_0^{(2)} &= P_{uw}(E_0^*) = (0, 0), \\
U_1^{(2)} &= P_{uw}(E_1^*) = (1 + \alpha_1; k(1 + \alpha_1)).
\end{aligned}
\tag{43}
$$

Let $M_u^{(2)} = 1 + \alpha_1$, $0 < m_u^{(2)} < 1 + \alpha_1$, $M_w^{(2)} = k(1 + \alpha_1)$, and $m_w^{(2)} = k m_u^{(2)}$. The model is permanent and the set defined by

$$
\begin{aligned}
\mathscr{A}^{(2)} = &\left\{ (u, w) \in \mathbb{R}_+^2 : m_u^{(2)} \leq u \leq M_u^{(2)}, \text{ and } m_w^{(2)} \leq w \right. \\
&\left. \leq M_w^{(2)} \right\}
\end{aligned}
\tag{44}
$$

is limited, positively invariant.

Concerning the local stability, one obtains, for any delay T, the following results:

(1) $U_0^{(2)} = (0; 0)$ is an unstable saddle point, repulsive along the direction u, and attractive along the direction w.

(2) $U_1^{(2)} = (1 + \alpha_1; k(1 + \alpha_1))$ is stable.

The global stability of equilibrium $U_1^{(2)}$ is given in the following theorem.

Theorem 8.

$$
\begin{aligned}
&\text{If } T < \frac{\ln 2}{\lambda} \\
&\text{and } \lambda < (1 + \alpha_1)(1 - \ln 2)
\end{aligned}
\tag{45}
$$

and, then, the unique interior equilibrium point $U_1^{(2)} = (1 + \alpha_1; k(1 + \alpha_1))$ of model (4.1) is globally and asymptotically stable.

Proof. Let us consider $U_1^{(2)} = (\overline{u}, \overline{w})$ with $\overline{u} = 1 + \alpha_1$ and $\overline{w} = k(1 + \alpha_1)$. Let us pose
$U = \ln(u/\overline{u})$ and $W = \ln(w/\overline{w})$. Then, one obtains the following system:

$$
\begin{aligned}
\dot{U}(\tau) &= -\overline{u} \left[e^{U(\tau)} - 1 \right], \\
\dot{W}(\tau) &= \frac{\lambda e^{-w(\tau)}}{k} \left[\left(e^{U(\tau)} - 1 \right) - k \left(e^{W(\tau)} - 1 \right) \right. \\
&\quad \left. - e^{-\lambda T} \left(e^{U(\tau - T)} - 1 \right) \right].
\end{aligned}
\tag{46}
$$

Let us note that $V_1(\tau) = |U(\tau)|$.

Then, the superior derivative $D^+ V_1(\tau)$ of $V_1(\tau)$ in relation to time, alongside the solutions of (46), gives

$$
D^+ V_1(\tau) \leq -\overline{u} \left| e^{U(\tau)} - 1 \right| = -|u - \overline{u}|.
\tag{47}
$$

Let us note that $V_{21}(\tau) = |W(\tau)|$. Considering $(u, w) \in \mathscr{A}^{(2)}$, one has $m_u^{(2)}/\overline{u} \leq e^{U(\tau)} \leq M_u^{(2)}/\overline{u}$ and $m_w^{(2)}/\overline{w} \leq e^{W(\tau)} \leq M_w^{(2)}/\overline{w}$. Hence, the superior derivative of $V_{21}(\tau)$ in relation to time, alongside solutions (46), gives

$$
\begin{aligned}
D^+ V_{21}(\tau) \leq &\ \lambda \left| e^{U(\tau)} - 1 \right| - \lambda \frac{w^*}{m_w^{(2)}} \left| e^{W(\tau)} - 1 \right| \\
&+ \frac{M_u^{(2)} \lambda e^{-\lambda T}}{k} \int_{\tau-T}^{\tau} \left| e^{U(s)} - 1 \right| ds.
\end{aligned}
\tag{48}
$$

Considering the functional: $V_{22}(\tau) = (M_u^{(2)} \lambda e^{-\lambda T}/k) \int_{\tau-T}^{\tau} \int_v^{\tau} |e^{U(s)} - 1| ds dv$ then

$$
\begin{aligned}
D^+ V_{22}(\tau) \leq &\ \frac{T M_u^{(2)} \lambda e^{-\lambda T}}{k} \left| e^{U(\tau)} - 1 \right| \\
&- \frac{M_u^{(2)} \lambda e^{-\lambda T}}{k} \int_{\tau-T}^{\tau} \left| e^{U(s)} - 1 \right| ds.
\end{aligned}
\tag{49}
$$

Let us pose $V_2(\tau) = V_{21}(\tau) + V_{22}(\tau)$. Thus,

$$
\begin{aligned}
D^+ V_2(\tau) \leq &\ \lambda \left(1 + \frac{T M_u^{(2)} e^{-\lambda T}}{k} \right) \left| e^{U(\tau)} - 1 \right| \\
&- \lambda \frac{w^*}{m_w^{(2)}} \left| e^{W(\tau)} - 1 \right|.
\end{aligned}
\tag{50}
$$

Then, $D^+ V_2(\tau) \leq \dfrac{\lambda}{\overline{u}} \left(1 + \dfrac{T M_u^{(2)} e^{-\lambda T}}{k} \right) |u - \overline{u}|$

$$
- \frac{\lambda}{m_w^{(2)}} |w - \overline{w}|.
\tag{51}
$$

Let $V(\tau) = V_1(\tau) + V_2(\tau)$. Then,

$$
\begin{aligned}
D^+ V(\tau) \leq &- \left[1 - \frac{\lambda}{\overline{u}} \left(1 + \frac{T M_u^{(2)} e^{-\lambda T}}{k} \right) \right] |u - \overline{u}| \\
&- \frac{\lambda}{m_w^{(2)}} |w - \overline{w}|.
\end{aligned}
\tag{52}
$$

So, $D^+ V(\tau) < 0$ if $(1 + (Te^{-\lambda T}/(1 - e^{-\lambda T})) M_u^{(2)} < (1 + \alpha_1)/\lambda$. Consequently, $D^+ V(\tau) < 0$ if $(T < (\ln 2)/\lambda$ and $\lambda < (1 + \alpha_1)(1 - \ln 2)$. □

5. Border Dynamics of the Plan vw

On the vw plan model (1.3) becomes

$$
\begin{aligned}
&\dot{v}(\tau) = \left[\alpha_2 - \delta - \frac{\alpha_2 v(\tau)}{d_2} \right] v(\tau), \\
&\dot{w} = -\lambda w - \lambda e^{-\lambda T} 1_{[0;T]}(\tau) u(\tau - T), \\
&v, w \in C^1 ([0; +\infty[; \mathbb{R}_+) \ u(0) > 0, \ u_0 = \varphi \in C([-T; 0]; \mathbb{R}_+)
\end{aligned}
\tag{5.1}
$$

with $(d_2, \alpha_2, \delta, T, \lambda) \in (\mathbb{R}_+^*)^4$ and $1_{[0;T]}(\tau) = 1$ if $0 \leq \tau \leq T$, $1_{[0;T]}(\tau) = 0$ if $\tau > T$.

The equilibria of model (5.1) are as follows: for $\tau \geq T$,

$$U_0^{(3)} = P_{vw}(U_0) = (0, 0),$$

$$U_2^{(3)} = P_{vw}(U_2) = \left(\frac{(\alpha_2 - \delta) d_2}{\alpha_2}; 0 \right). \tag{53}$$

Model (4.1) is nonpermanent. In fact, $\lim_{t \to +\infty}[w(t)] = 0$.

Let us now give the results upon stability of the equilibria of model (5.1).

(1) The equilibrium $U_0^{(3)} = (0; 0)$ is *stable* if $\alpha_2 < \delta$ and a point *unstable saddle* if $\alpha_2 > \delta$.

(2) The equilibrium $U_2^{(3)} = ((\alpha_2 - \delta)d_2/\alpha_2; 0)$ for $\alpha_2 > \delta$ is stable.

Moreover, the equilibrium $U_2^{(3)} = ((\alpha_2 - \delta)d_2/\alpha_2; 0)$ is globally and asymptotically stable. In effect, $\lim_{t \to +\infty}[v(t)] = (\alpha_2 - \delta)d_2/\alpha_2$ and $\lim_{t \to +\infty}[w(t)] = 0$.

6. Local Stability of Model (1.3)

Let us consider model (1.3).

Let us pose that $U_i(\tau) = u(\tau) - u_i^*$, $V_i(\tau) = v(\tau) - v_i^*$, and $W_i(\tau) = w(\tau) - w_i^*$ for $i = 0, 1, 2, 3$ with

$$u_0^* = 0,$$

$$u_1^* = 1 + \alpha_1,$$

$$u_2^* = 0, \tag{54}$$

$$u_3^* = u^*$$

$$v_0^* = 0,$$

$$v_1^* = 0,$$

$$v_2^* = \frac{(\alpha_2 - \delta) d_2}{\alpha_2}, \tag{55}$$

$$v_3^* = v^*$$

$$w_0^* = 0,$$

$$w_1^* = k(1 + \alpha_1),$$

$$w_2^* = 0, \tag{56}$$

$$w_3^* = ku^*.$$

The system linearized around the point of equilibrium $E_i^* = (u_i^*, v_i^*, w_i^*), i = 0, 1, 2, 3$, is

$$\dot{U}_i(\tau) = A_{11}^{(i)} U_i(\tau) + A_{12}^{(i)} V_i(\tau) + A_{13}^{(i)} W_i(\tau),$$

$$\dot{V}_i(\tau) = A_{21}^{(i)} U_i(\tau) + A_{22}^{(i)} V_i(\tau), \tag{57}$$

$$\dot{W}_i(\tau) = \lambda U_i(\tau) - \lambda W_i(\tau) - \lambda e^{-\lambda T} U_i(\tau - T),$$

with

$$A_{11}^{(i)} = 1 - 2u_i^* + (\beta_1 v_i^* + \alpha_1) \left(1 - \frac{v_i^*}{u_i^* + d_2} \right)$$

$$+ \frac{(\beta_1 v_i^* + \alpha_1) u_i^* v_i^*}{(u_i^* + d_2)^2}, \tag{58}$$

$$A_{12}^{(i)} = \left[\beta_1 - \frac{2\beta_1}{u_i^* + d_2} v_i^* - \frac{\alpha_1}{u_i^* + d_2} \right] u_i^* - \frac{\gamma w_i^*}{w_i^* + d_1}, \tag{59}$$

$$A_{13}^{(i)} = \frac{-\gamma d_1 v_i^*}{(w_i^* + d_1)^2}, \tag{60}$$

$$A_{21}^{(i)} = \left[\beta_2 \left(1 - \frac{v_i^*}{u_i^* + d_2} \right) + \frac{(\beta_2 u_i^* + \alpha_2) v_i^*}{(u_i^* + d_2)^2} \right] v_i^*,$$

$$A_{22}^{(i)} = (\beta_2 u_i^* + \alpha_2) \left(1 - \frac{2v_i^*}{u_i^* + d_2} \right) - \delta, \tag{61}$$

$$A_{23}^{(i)} = 0,$$

$$A_{31}^{(i)} = \lambda,$$

$$A_{32}^{(i)} = 0, \tag{62}$$

$$A_{33}^{(i)} = -\lambda,$$

$$A_{34}^{(i)} = -\lambda e^{-\lambda T}.$$

The characteristic equation of (57) is

$$\Delta_i(x, T) = P_i(x, T) + e^{-xT} Q_i(x, T) = 0 \tag{63}$$

with $P_i(x, T) = -\left[x^3 + a^{(i)} x^2 + b^{(i)} x + c^{(i)} \right]$,

$$Q_i(x, T) = e^{-\lambda T} \left[D^{(i)} x + E^{(i)} \right] \tag{64}$$

where, $D^{(i)} = -\lambda A_{13}^{(i)}$,

$$E^{(i)} = \lambda A_{13}^{(i)} A_{22}^{(i)}, \tag{65}$$

$$a^{(i)} = -\left(A_{11}^{(i)} + A_{22}^{(i)} - \lambda \right),$$

$$b^{(i)} = -\left[\lambda \left(A_{11}^{(i)} + A_{22}^{(i)} + A_{13}^{(i)} \right) + A_{21}^{(i)} A_{12}^{(i)} - A_{11}^{(i)} A_{22}^{(i)} \right], \tag{66}$$

$$c^{(i)} = \lambda \left[A_{11}^{(i)} A_{22}^{(i)} - A_{21}^{(i)} A_{12}^{(i)} + A_{13}^{(i)} A_{22}^{(i)} \right]. \tag{67}$$

6.1. Local Stability of E_0^* and E_1^*. We are in situations where $Q_i(x, T) = 0$. Let us use the following criterion Routh-Hurwitz: the interior equilibrium E_i^* is stable if $a^{(i)} > 0$, $b^{(i)} > 0$, $c^{(i)} > 0$, and $a^{(i)} b^{(i)} - c^{(i)} > 0$ and unstable if not.

Theorem 9. *Let us suppose that $\alpha_2 > \delta$. Then, the equilibria E_0^* and E_1^* of model (1.3) are all unstable.*

Proof. Let us consider formula (63)–(67).

(1) The equation of E_0^* is $\Delta_0(x, T) = x^3 + a^{(0)}x^2 + b^{(0)}x + c^{(0)} = 0$.

One has the following: $a^{(0)} = \lambda - (1 + \alpha_1 + \alpha_2 - \delta)$ and $b^{(0)} = (1+\alpha_1)(\alpha_2-\delta)-\lambda(1+\alpha_1+\alpha_2-\delta)$. Let us suppose that $(a^{(0)} > 0$ and $b^{(0)} > 0)$. Then, one would have $(1+\alpha_1+\alpha_2-\delta) < \lambda < (1+\alpha_1)(\alpha_2-\delta)/(1+\alpha_1+\alpha_2-\delta)$. Consequently, $(1+\alpha_1)^2 + (\alpha_2-\delta)^2 + (1+\alpha_1)(\alpha_2-\delta) < 0$ which is absurd. Hence, the equilibrium E_0^* of model (1.3) is unstable by application of the Routh-Hurwitz criterion.

(2) One has $\Delta_1(x, T) = x^3 + a^{(1)}x^2 + b^{(1)}x + c^{(1)} = 0$. Meanwhile, $\alpha_2 > \delta$ and, then, $c^{(1)} = -\lambda(1+\alpha_1)[\beta_2(1+\alpha_1) + \alpha_2 - \delta] < 0$ so the equilibrium E_0^* of model (1.3) is unstable by application of Routh-Hurwitz criterion. $\square$

6.2. Local Stability of E_2^* and E_3^*. We are in the situation where $Q_i(x, T) \neq 0$. In order to assess the influence of delay T over the local stability of model (1.3), we use the results upon the local stability of model obtained taking $T = 0$ into model (1.3) and the results of [5] upon the local stability of the delay systems. Let us consider the characteristic equation (63).

(i) One has the following: $Q_i(z, T) = 0 \implies z = -E^{(i)}/D^{(i)} \in \mathbb{R}$. Then, $P_i(x, T)$ and $Q_i(x, T)$ do not have any common imaginary roots.

(ii) One has the following: $P_i(x, T)$ and $Q_i(x, T)$ are polynomials with real coefficients

so $\overline{P_i(-i\omega, T)} = P_i(i\omega, T)$ and $\overline{Q_i(-i\omega, T)} = Q_i(i\omega, T)$.

(iii) If $-(1/\lambda)\ln(c^{(i)}/E^{(i)}) \neq T$ so $P_i(0, T) + Q_i(0, T) = E^{(i)}e^{-\lambda T} - c^{(i)} \neq 0$.

(iv) Let us pose $x = \rho e^{i\varphi}$. Then, $|P_i(x, T)/Q_i(x, T)| \sim 1/\rho^4$ when $\rho \longrightarrow +\infty$.

$$\text{Thus } \limsup_{|x| \longrightarrow +\infty} \left\{ \left| \frac{Q_i(x, T)}{P_i(x, T)} \right| \right\} = 0 < 1. \tag{68}$$

Let us consider the function defined on $\mathbb{R}$ by the following: $F_i(\omega) = |P_i(i\omega, T)|^2 - |Q_i(i\omega, T)|^2$.

$$\text{Then, } F(\omega) = \left[b^{(i)}\omega - \omega^3\right]^2 + \left[c^{(i)} - a^{(i)}\omega^2\right]^2 - e^{-2\lambda T}\left(D^{(i)}\omega^2 + E^{(i)}\right) \tag{69}$$

$$\text{So } F_i(\omega) = \omega^6 + m_2\omega^4 + m_1\omega^2 + m_0, \text{ where} \tag{70}$$

$$m_2 = \left(\left[a^{(i)}\right]^2 - 2b^{(i)}\right),$$

$$m_1 = \left(\left[b^{(i)}\right]^2 - 2a^{(i)}c^{(i)} - e^{-2T\lambda}D^{(i)}\right), \tag{71}$$

$$m_0 = \left[c^{(i)}\right]^2 - E^{(i)}e^{-2\lambda T}.$$

Let us determine $\delta(T_i^*) = \text{sgn}((d\,\text{Re}(x)/dT)|_{x=i\omega(T_i^*)})$ the sign of the real part of a solution x of the characteristic equation $\Delta_i(x, T) = 0$.

Lemma 10. *Let us consider x, a solution of the characteristic equation $\Delta_i(x, T) = P_i(x, T) + e^{-xT}Q_i(x, T) = 0$, with*

$$P_i(x, T) = -\left[x^3 + a^{(i)}x^2 + b^{(i)}x + c^{(i)}\right]$$

$$and \ Q_i(x, T) = e^{-\lambda T}\left\{D^{(i)}x + E^{(i)}\right\}. \tag{72}$$

Given $\omega_i = \omega(T_i^*)$ positive root of $F_i(\omega)$ and T_i^* the associated delay certifying the following relation:

$$T_i^* = \frac{1}{\omega_i}$$

$$\cdot \arctan\left\{\frac{\left(a^{(i)}D^{(i)} - E^{(i)}\right)\omega^3 + \left(b^{(i)}E^{(i)} - c^{(i)}D^{(i)}\right)\omega}{-D^{(i)}\omega^4 + \left(-a^{(i)}E^{(i)} + b^{(i)}D^{(i)}\right)\omega^2 + c^{(i)}E^{(i)}}\right\} \tag{73}$$

$$+ \frac{2n\pi}{\omega_i}, \quad n \in \mathbb{N}.$$

Let us pose $\delta(T_i^*) = \text{sgn}\{(d\,\text{Re}(x)/dT)|_{x=i\omega(T_i^*)}\}$. Then,

$$\delta(T_i^*) = \text{sgn}\left\{\Lambda_{1i}H_{3i} + \Lambda_{2i}H_{1i}H_{2i}\right\} \tag{74}$$

with

$$H_{1i} = \lambda^2 + \omega^2,$$

$$H_{2i} = \left(\lambda E^{(i)} - D^{(i)}\omega^2\right)^2 + \left(E^{(i)} + \lambda D^{(i)}\right)^2\omega^2,$$

$$H_{3i} = \left(c^{(i)} - a^{(i)}\omega^2\right)^2 + \left(b^{(i)} - \omega\right)^2\omega^2,$$

$$\Lambda_{1i} = \left(-T_i^*\left[D^{(i)}\right]^2 - \lambda^2\left[D^{(i)}\right]^2\right)\omega^4 + \left\{\lambda D^{(i)}E^{(i)} - T_i^*\left[\left(E^{(i)} - \lambda D^{(i)}\right)^2 - 2\lambda D^{(i)}E^{(i)}\right] - \lambda^2\left[D^{(i)}\right]^2\right\}$$

$$\cdot \omega^2 + \left(\lambda^3 D^{(i)}E^{(i)} - \lambda^2\left[E^{(i)}\right]^2 T_i^*\right),$$

$$\Lambda_{2i} = -3a^{(i)}\omega^4 + 2\omega^3 + \left(3c^{(i)} + a^{(i)}b^{(i)} - 2b^{(i)}\right)\omega^2$$

$$- b^{(i)}c^{(i)}. \tag{75}$$

Proof. Given $\omega_i = \omega(T_i^*)$ positive root of $F_i(\omega)$. Let us assume that there is a delay T_i^* so that

$$T_i^* = \frac{1}{\omega(T_i^*)}\arctan\left\{-\frac{\text{Im}\left[P_i\left(i\omega, T_i^*\right)/Q_i\left(i\omega, T_i^*\right)\right]}{\text{Re}\left[P_i\left(i\omega, T_i^*\right)/Q_i\left(i\omega, T_i^*\right)\right]}\right\}$$

$$+ \frac{2n\pi}{\omega\left(T_i^*\right)}, \quad n \in \mathbb{N}. \tag{76}$$

Then, $T_i^* = \frac{1}{\omega_i}$

$$\cdot \arctan\left\{\frac{\left(a^{(i)}D^{(i)} - E^{(i)}\right)\omega^3 + \left(b^{(i)}E^{(i)} - c^{(i)}D^{(i)}\right)\omega}{-D^{(i)}\omega^4 + \left(-a^{(i)}E^{(i)} + b^{(i)}D^{(i)}\right)\omega^2 + c^{(i)}E^{(i)}}\right\} \tag{77}$$

$$+ \frac{2n\pi}{\omega_i}, \quad n \in \mathbb{N}.$$

Let us pose

$$\delta\left(T_i^*\right) = \text{sgn}\left(\left.\frac{d\,\text{Re}\,(x)}{dT}\right|_{x=i\omega(T_i^*)}\right)$$

$$= \left[\text{sgn}\left(\left.\frac{d\,\text{Re}\,(x)}{dT}\right|_{x=i\omega(T_i^*)}\right)\right]^{-1}. \tag{78}$$

One has

$$\Delta_i(x,T) = P_i(x,T) + e^{-xT}Q_i(x,T) = 0 \implies$$

$$\frac{dP_i(x,T)}{dT} - \left[T\frac{d(x)}{dT} + x\right]e^{-xT}Q_i(x,T) \tag{79}$$

$$+ e^{-xT}\frac{dQ_i(x,T)}{dT} = 0.$$

Therefore, $P_i(x,T) = -[x^3 + a^{(i)}x^2 + b^{(i)}x + c^{(i)}]$ and $Q_i(x,T) = e^{-\lambda T}\{D^{(i)}x + E^{(i)}\}$. Thus, $\{e^{-xT}[D^{(i)}e^{-\lambda T} - TQ_i(x,T)] - (3x^2 + 2a^{(i)}x + b^{(i)})\}(d(x)/dT) - e^{-xT}\{xQ_i(x,T) + \lambda e^{-\lambda T}(D^{(i)}x + E^{(i)})\} = 0$.

Hence, $\left[\dfrac{d(x)}{dT}\right]^{-1}$

$$= \frac{\left\{e^{-xT}\left[D^{(i)}e^{-\lambda T} - TQ_i(x,T)\right] - \left(3x^2 + 2a^{(i)}x + b^{(i)}\right)\right\}}{e^{-xT}\left\{xQ_i(x,T) + \lambda e^{-\lambda T}\left(D^{(i)}x + E^{(i)}\right)\right\}}.$$

Thus, $\left[\dfrac{d(x)}{dT}\right]^{-1}$ \hfill (80)

$$= -\frac{T}{x+\lambda} + \frac{D^{(i)}}{(x+\lambda)\left(D^{(i)}x + E^{(i)}\right)}$$

$$- \frac{\left(3x^2 + 2a^{(i)}x + b^{(i)}\right)}{x^3 + a^{(i)}x^2 + b^{(i)}x + c^{(i)}}$$

and, meanwhile, $\delta(T_i^*) = [\text{sgn}((d\,\text{Re}(x)/dT)|_{x=i\omega(T_i^*)})]^{-1}$. Then, $\delta(T_i^*) = \text{sgn}\{\Lambda_{1i}H_{3i} + \Lambda_{2i}H_{1i}H_{2i}\}$ with $H_{1i} = \lambda^2 + \omega^2$, $H_{2i} = (\lambda E^{(i)} - D^{(i)}\omega^2)^2 + (E^{(i)} + \lambda D^{(i)})^2\omega^2$, $H_{3i} = (c^{(i)} - a^{(i)}\omega^2)^2 + (b^{(i)} - \omega)^2\omega^2$, $\Lambda_{1i} = (-T_i^*[D^{(i)}]^2 - \lambda^2[D^{(i)}]^2)\omega^4 + \{\lambda D^{(i)}E^{(i)} - T_i^*[(E^{(i)} - \lambda D^{(i)})^2, -2\lambda D^{(i)}E^{(i)}] - \lambda^2[D^{(i)}]^2\}\omega^2 + (\lambda^3 D^{(i)}E^{(i)} - \lambda^2[E^{(i)}]^2 T_i^*)$, and $\Lambda_{2i} = -3a^{(i)}\omega^4 + 2\omega^3 + (3c^{(i)} + a^{(i)}b^{(i)} - 2b^{(i)})\omega^2 - b^{(i)}c^{(i)}$. $\qquad\square$

It is noticed that the analysis of the local stability of the system with delay when $Q_i(x,T) \neq 0$ depends on the existence of positive root for function $F_i(\omega)$. The coefficients m_0 and m_1, of $F_i(\omega)$, depend on T. Therefore, its positive root ω depends on T. Using the Cartan method and Viet

formulas, one proves on the one hand that $F_i(\omega)$ admits at least a positive root if $m_0 < 0$. On the other hand, if $m_0 \geq 0$, then, $F_i(\omega)$ admits either of two positive roots whereas $F_i(\omega)$ does not admit any positive root.

The condition $m_0 < 0$ is equivalent to the following assumption:

$$0 \leq T \leq T_s = -\frac{1}{2\lambda}\ln\frac{\left[c^{(i)}\right]^2}{E^{(i)}} \tag{81}$$

with $E^{(i)} > 0$ and $\left[c^{(i)}\right]^2 < E^{(i)}$.

Let us denote $\det[J^{(i)}] = A_{11}^{(i)}A_{22}^{(i)} - A_{21}^{(i)}A_{12}^{(i)}$, $p = -m_1m_3 + m_2^2/3$, and $q = -m_0m_3^2 - 2(m_2/3)^3 + m_1m_2m_3/3$. Then, one obtains the following proposition.

Proposition 11. *Let us suppose that $A_{13}^{(i)}A_{22}^{(i)} > 0$.*

Let us pose $\lambda_{max}^{(i)} = A_{13}^{(i)}A_{22}^{(i)}/[\det[J^{(i)}] + A_{13}^{(i)}A_{22}^{(i)}]^2$ and $T_s^{(i)} = -(1/2\lambda)\ln([c^{(i)}]^2/E^{(i)})$, $i = 2, 3$.

(1) $F_i(\omega)$ admits at least a positive root if

$$0 \leq \lambda < \lambda_{max}^{(i)}$$

$$\text{and } 0 \leq T < T_s^{(i)}. \tag{82}$$

(2) $F_i(\omega)$ does not admit any positive root if one of the following conditions is verified:

$$T_s^{(i)} \leq T$$

$$\text{and } 27q^2 - 4p^3 > 0. \tag{83}$$

$$T_s^{(i)} \leq T,$$

$$27q^2 - 4p^3 = 0 \tag{84}$$

$$\text{and } \left[m_1m_2 - m_0\right] > 0.$$

$$T_s^{(i)} \leq T,$$

$$27q^2 - 4p^3 < 0,$$

$$m_2 < 0, \tag{85}$$

$$m_1 < 0$$

$$\text{and } \left[m_1m_2 - m_0\right] > 0.$$

Now, let us use the results of Theorem 4 from [5] to study the stability of these points of equilibriums E_2^* and E_3^*.

Theorem 12. *One poses*

$$x^- = \frac{\beta_1 d_2 + \alpha_1 + \gamma d_2/d_1^2 - \sqrt{\left(\beta_1 d_2 + \alpha_1 + \gamma d_2/d_1^2\right)^2 - 4\beta_1 d_2\left(\gamma d_2/d_1^2 - 1\right)}}{2\beta_1 d_2}. \tag{86}$$

Let us consider one of these assumptions (82)–(85) and one of the following conditions:

$$\frac{\gamma d_2}{d_1^2} < 1, \tag{87}$$

$$\frac{\gamma d_2}{d_1^2} \geq 1 \tag{88}$$

$$and \ \max\left(0; x^-\right) \leq \frac{\delta}{\alpha_2} < 1,$$

$$\frac{\gamma d_2}{d_1^2} \geq 1,$$

$$0 \leq \frac{\delta}{\alpha_2} < \min\left(1; x^-\right) \tag{89}$$

$$and \ T \neq -\frac{1}{\lambda} \ln\left(1 - \frac{d_1^2 \psi(\delta)}{\gamma \alpha_2 (\alpha_2 - \delta)}\right),$$

$$\det(J) = A_{11}^{(3)} A_{22}^{(3)} - A_{21}^{(3)} A_{12}^{(3)} > 0$$

$$and \ Tr(J) = A_{11}^{(3)} + A_{22}^{(3)} < 0. \tag{90}$$

With $\psi(\delta)$ defined in formula (23), consider the following.

(1) *Let us suppose that one of assumptions (83)–(85) is verified for i=2,3. Then, there is no change of stability for equilibrium $E_i^*, i = 2, 3$.*

(2) *Let us suppose that assumptions (82) and one of these assumptions (87)–(89) are verified for i=2.*

    (a) *If $\delta(T_2^*) > 0$ then the equilibrium E_2^* is unstable for any $T \in [0; T_s[$.*

    (b) *If $\delta(T_2^*) < 0$ then the equilibrium E_2^* is unstable if $0 \leq T < T_2^*$ and stable if $T_2^* \leq T$.*

(3) *Let us suppose that assumption (82) is verified for i=3.*

    (a) *In case $\delta(T_3^*) > 0$,*

        (i) *If assumption (90) is verified then the equilibrium E_3^* is stable if $0 \leq T < T_3^*$ and unstable if $T_3^* \leq T$.*

        (ii) *If assumption (90) is not verified then the equilibrium E_3^* is unstable for any $T \in [0; T_s[$.*

    (b) *Considering $\delta(T_3^*) < 0$,*

        (i) *If assumption (90) is verified then the equilibrium E_3^* is stable for every $T \in [0; T_s[$.*

        (ii) *If assumption (90) is not verified then the equilibrium E_3^* is unstable if $0 \leq T < T_3^*$ and stable if $T_3^* \leq T$.*

With T_i^* and $\delta(T_i^*)$, $i = 2, 3$ are the denotations defined in Lemma 10.

Proof. Let us consider formula (63)–(67). Let us consider the function $F_i(\omega)$ defined in relation to the formula (70)-(71) for any point of equilibrium E_i^*, $i = 2, 3$. Let us

consider assumptions (82). Then, $T_s^{(i)}$ exists so that, for any $T \leq T_s^{(i)}$, $F_i(\omega)$ admits $\omega_i = \omega(T_i^*)$ a positive root (see Proposition 11). Considering T_i^* the associated delay determined from Lemma 10, consider the following:

(1) Let us suppose that one of assumptions (83)–(85) is verified in i=2,3. Then, $\forall T > T_s^{(i)}$, $F_i(\omega)$ does not admit any positive root. Thus, there is no change of stability of the equilibrium $E_i^*, i = 2, 3$.

(2) Stability of E_2^*: let us consider that one of assumptions (87)-(88) is verified. Then, $c^{(2)} < 0$. If assumption (89) is verified, then, $T \neq T_l = -(1/\lambda) \ln(c^{(2)}/E)$. Hence, $P_2(0, T) + Q_2(0, T) = E^{(2)} e^{-\lambda T} - c^{(2)} \neq 0$ if assumptions (87)-(88) are verified. Then, let us apply Theorem 4 from [5]. So, let us suppose that $0 \leq T < T_s^{(2)}$. It is known that $T = 0$ and $E_2^* = U_2^{(1)}$ so E_2^* is unstable because $\psi(\delta) > 0$ for $\delta < \alpha_2$. Taking into account the conclusions of Theorem 4 from [5], one obtains the stability of E_2^*.

(3) Let us suppose that the assumption $\det(J) = A_{11}^{(3)} A_{22}^{(3)} - A_{21}^{(3)} A_{12}^{(3)} > 0$ of (90) is verified. Then, $c > B$ so $T \geq 0 > T_l = -(1/\lambda) \ln(c/B)$. Thus, let us apply Theorem 4 from [5]. It is known that when $T = 0$ and $E_3^{(3)} = U_3^{(1)}$ then $E_3^{(3)}$ is stable if assumption (90) is verified and unstable if not. Taking into account the conclusions of Theorem 4 from [5], one gets the stability of E_3^*.

□

7. Global Stability of Model (1.3)

Theorem 13. *Let us suppose that model (1.3) is permanent and that it admits a unique interior equilibrium point. If*

$$\delta < \alpha_2,$$

$$\beta_2 d_2 - \alpha_2 < 0,$$

$$d_2 < d_1,$$

$$\gamma < \alpha_2, \tag{91}$$

$$T < \frac{\ln 2}{\lambda},$$

$$\frac{1}{2} < \lambda,$$

$$\frac{1 + \alpha_1}{d_2^2} + \frac{\alpha_2 M_v}{d_2^2} + \frac{4 d_2}{\alpha_1^2}\left(\frac{1 + \alpha_1}{d_1} + \lambda\right) + \frac{\beta_2 \delta}{\alpha_2} + \ln 2$$
$$- 1 < 0, \tag{92}$$

$$\beta_1 \frac{\delta}{\alpha_2} + \frac{\gamma}{M_u + d_1} < \frac{\alpha_2}{M_u + d_2}, \tag{93}$$

then the unique interior equilibrium point $E_3^* = (u^*; v^*, k u^*)$ of model (1.3) is globally and asymptotically stable.

Proof. Let us consider model (1.3) and $E_3^* = (u^*; v^*, ku^*)$ a unique interior equilibrium point. Let us pose $U = \ln(u/u^*)$, $V = \ln(v/v^*)$, and $W = \ln(w/w^*)$.

Then, $u - u^* = u^*[e^U - 1]$, $v - v^* = v^*[e^V - 1]$, and $w - w^* = w^*[e^W - 1]$. Thus,

$$\dot{U} = u^* \left[-1 + \frac{\beta_1 v^2 + \alpha_1 v}{(u^* + d_2)(u + d_2)} \right.$$

$$+ \left. \frac{\gamma k v}{(ku^* + d_1)(w + d_1)u} \right] \{e^U - 1\} + v^* \left[\beta_1 \right.$$

$$- \frac{(\beta_1(v + v^*) + \alpha_1)}{(u^* + d_2)} - \left. \frac{\gamma k}{ku^* + d_1} \right] \{e^V - 1\}$$

$$+ w^* \left[\frac{-d_1}{(ku^* + d_1)u} \right] \{e^W - 1\},$$

$$\dot{V} = u^* \left[\beta_2 - \frac{\beta_2 v^*}{u^* + d_2} + \frac{(\beta_2 u + \alpha_2)v}{(u^* + d_2)(u + d_2)} \right] \{e^U \quad (7.1)$$

$$- 1\} - \frac{(\beta_2 u + \alpha_2)}{u^* + d_2} v^* \{e^V - 1\},$$

$$\dot{W}(\tau) = \frac{\lambda e^{-w(\tau)}}{k} \left[\{e^{U(\tau)} - 1\} - k \{e^{W(\tau)} - 1\} \right.$$

$$- \left. e^{-\lambda T} \{e^{U(\tau - T)} - 1\} \right].$$

Let us denote $V_1(\tau) = |U(\tau)|$. Consider

$$(u, v, w) \in \mathscr{A}^+ = \left\{ (u, v, w) \in \mathbb{R}_+^3 : m_u^\varepsilon \le u \le M_u, \ m_v \right.$$

$$\le v \le M_v \text{ and } m_w \le w \le M_w \}. \quad (94)$$

Then, $dV_1(\tau)/d\tau = \text{sgn}(U(\tau))\dot{U}(\tau)$. Meanwhile, $[\text{sgn}(e^{U(\tau)} - 1)](e^{U(\tau)} - 1) = |e^{U(\tau)} - 1|$ so the superior derivative of $V_1(\tau)$ in relation to the time, alongside the solutions of system (7.1), gives

$$D^+ V_1(\tau) \le \left\{ -1 + \frac{\beta_1 M_v^2 + \alpha_1 M_v}{d_2^2} + \frac{\gamma k M_v}{m_u^\varepsilon d_1^2} \right\} |u - u^*|$$

$$+ \left| \frac{\beta_1 \delta}{\alpha_2} - \frac{\gamma k}{k M_u + d_1} \right| |v - v^*| \quad (95)$$

$$+ \frac{d_1}{(k M_u + d_1) M_u} |w - w^*|.$$

Let us denote $V_2(\tau) = |V(\tau)|$. However, $(\beta_2 u + \alpha_2)v/(u^* + d_2)(u + d_2) \le \alpha_2 v/(u^* + d_2)d_2$ if $\beta_2 d_2 - \alpha_2 < 0$. Then,

$$D^+ V_2(\tau) \le \left| \frac{\beta_2 \delta}{\alpha_2} + \frac{\alpha_2}{d_2^2} M_v \right| |u - u^*|$$

$$- \frac{\alpha_2}{M_u + d_2} |v - v^*|. \quad (96)$$

Let us note $V_{31}(\tau) = |W(\tau)|$. One has $m_u^\varepsilon/u^* \le e^{U(\tau)} \le M_u/u^*$ and $m_w/w^* \le e^{W(\tau)} \le M_w/w^*$. Then, superior derivative of $V_{31}(\tau)$ in relation to the time, alongside the solutions of (7.1), results in

$$D^+ V_{31}(\tau) \le \lambda \left| e^{U(\tau)} - 1 \right| - \frac{\lambda}{m_w} w^* \left| e^{W(\tau)} - 1 \right|$$

$$+ \frac{M_u \lambda e^{-\lambda T}}{k} \int_{\tau - T}^{\tau} \left| e^{U(s)} - 1 \right| ds. \quad (97)$$

Let us consider the functional $V_{32}(\tau) = (M_u \lambda e^{-\lambda T}/k) \int_{\tau - T}^{\tau} \int_v^{\tau} |e^{U(s)} - 1| ds \, dv$; then,

$$D^+ V_{32}(\tau) \le \frac{T M_u \lambda e^{-\lambda T}}{k} \left| e^{U(\tau)} - 1 \right|$$

$$- \frac{M_u \lambda e^{-\lambda T}}{k} \int_{\tau - T}^{\tau} \left| e^{U(s)} - 1 \right| ds. \quad (98)$$

Let us pose that $V_3(\tau) = V_{31}(\tau) + V_{32}(\tau)$. Thus,

$$D^+ V_3(\tau) \le \lambda \left(1 + \frac{T M_u e^{-\lambda T}}{k} \right) \left| e^{U(\tau)} - 1 \right|$$

$$- \lambda \left| e^{W(\tau)} - 1 \right|. \quad (99)$$

Then, $D^+ V_3(\tau) \le \frac{\lambda}{M_u} \left(1 + \frac{T M_u e^{-\lambda T}}{k} \right) |u - u^*|$

$$- \frac{2\lambda}{m_w} |w - w^*|. \quad (100)$$

Let us pose $V(\tau) = V_1(\tau) + V_2(\tau) + V_3(\tau)$. Then, from formulas (95), (96), and (100), one obtains

$$D^+ V(\tau) \le h_1 |u - u^*| + h_2 |v - v^*|$$

$$+ h_3 |w - w^*| \quad (101)$$

$$\text{if } \beta_2 d_2 - \alpha_2 < 0$$

with $h_1 = -1 + \dfrac{\beta_1 M_v^2 + \alpha_1 M_v}{d_2^2} + \dfrac{\gamma k M_v}{m_u^\varepsilon d_1^2}$

$$+ \frac{\beta_2 \delta}{\alpha_2} + \frac{\alpha_2}{d_2^2} M_v$$

$$+ \frac{\lambda}{M_u} \left(1 + \frac{T e^{-\lambda T}}{k} M_u \right),$$

$$h_2 = -\frac{\alpha_2}{M_u + d_2} + \left| \frac{\beta_1 \delta}{\alpha_2} - \frac{\gamma k}{k M_u + d_1} \right|$$

and $h_3 = -\dfrac{2\lambda}{k m_u^\varepsilon} + \dfrac{d_1}{(k M_u + d_1) M_u}$.

We have: $T < \dfrac{\ln 2}{\lambda} \implies$

$$\dfrac{e^{-\lambda T}}{k} < 1,$$

$$\beta_1 \dfrac{\delta}{\alpha_2} + \dfrac{\gamma}{M_u + d_1} < \dfrac{\alpha_2}{M_u + d_2} \implies$$

$$h_2 < 0,$$

and $\dfrac{1}{2} < \lambda \implies$

$$h_3 < 0.$$

(102)

One has $\alpha_1^2/4d_2 < M_u \implies 1/M_u < 4d_2/\alpha_1^2$. Model (1.3) is permanent. So, on the one hand, one has $m_u^{(0)} \geq m^u > 0$ if $(\beta_1/d_2^2)M_v^2 + (\alpha_1/d_2^2)M_v < (1 + \alpha_1)/d_2^2$. In addition, $m_u^{(0)}/2 \leq m_u^\varepsilon \leq m_u^{(0)}$. Moreover, $m_u^{(0)} \geq 2\sqrt{\gamma k M_u M_v/d_1}$.

Then, $\gamma k M_v/m_u^\varepsilon d_1^2 \leq 2\gamma M_v/m_u^{(0)} d_1^2 \leq m_u^{(0)}/2k M_u d_1 < m_u^{(0)}/M_u d_1$ if $T < (\ln 2)/\lambda$.

So, $D^+V(\tau) < 0$ if assumptions (91)–(93) are checked. $\qquad\square$

Corollary 14. *For $r \geq 14$, if the following assumptions are verified:*

$$\gamma \leq 3 + \left[1 + 2\left(r + \sqrt{r^2 + r}\right)\right]^2,$$

(103)

$$d_2 \geq \dfrac{1 - 8r\gamma - 8r + 2r\sqrt{(1/2r - 4\gamma - 4)^2 - (8/r)\left\{\gamma - 3 - \left[1 + 2\left(r + \sqrt{r^2 + r}\right)\right]^2\right\}}}{8},$$

(104)

$$\dfrac{d_2}{2r} \geq \alpha_2 \geq \max\left\{\dfrac{4d_2 + 4}{4d_2 - 1}; \dfrac{4d_2 + 3 + \left[1 + 2\left(r + \sqrt{r^2 + r}\right)\right]^2}{4d_2 - 1}\right\} + \gamma.$$

(105)

$$2\left(r + \sqrt{r^2 + r}\right) \leq \alpha_1 \leq -1 + \sqrt{(\alpha_2 - \gamma)(4d_2 - 1) - 3 - 4d_2}$$

(106)

$$0 < \beta_1 < \dfrac{1}{d_2},$$

$$\beta_2 < \dfrac{\alpha_2}{d_2},$$

(107)

$$\dfrac{1}{2} < \lambda < 1 + \alpha_1,$$

$$1 - \dfrac{d_2}{M_u + d_2} < \dfrac{\delta}{\alpha_2} < 1,$$

(108)

$$\beta_2 < \beta_2^{max} = \dfrac{\alpha_2}{M_u}\left[\dfrac{d_2}{M_u + d_2} + \dfrac{\delta}{\alpha_2} - 1\right],$$

$$d_2 < d_1,$$

$$\gamma < \alpha_2,$$

(109)

$$T < \dfrac{\ln 2}{\lambda},$$

then, the unique interior equilibrium point $E_3^* = (u^*; v^*, ku^*)$ of model (1.3) is globally and asymptotically stable.

Proof. Let us consider the assumptions of Theorem 13.

$$\text{Posing: } \chi(\beta_1) = \begin{cases} d_2 & \text{si } 0 < \beta_1 < \dfrac{1}{d_2}, \\[2ex] \sqrt{\dfrac{d_2}{\beta_1}} & \text{si } \dfrac{1}{d_2} < \beta_1 < 4. \end{cases}$$

(110)

The model is permanent. Then

$$m^{(0)} \geq m^u > 0 \Longleftrightarrow$$

$$\left(1 - \dfrac{\chi(\beta_1)}{M_u + d_2} < \dfrac{\delta}{\alpha_2} < 1, \ \beta_2 < \beta_2^{max}\right.$$

(111)

$$\left. = \dfrac{\alpha_2}{M_u}\left[\dfrac{\chi(\beta_1)}{M_u + d_2} + \dfrac{\delta}{\alpha_2} - 1\right]\right).$$

Assumptions (91) and (93) give the following assumption:

$$\beta_1 < \frac{\alpha_2 - \gamma}{M_u + d_2}. \tag{112}$$

Moreover, assumptions (91) and (92) give the following assumption:

$$\frac{1 + \alpha_1}{d_2^2} + \frac{2\alpha_2}{d_2} + \frac{4d_2}{\alpha_1^2}\left(\frac{1 + \alpha_1}{d_1} + \lambda\right) + \ln 2 - 1 < 0. \tag{113}$$

Posing $(1 + \alpha_1)/d_2^2 = 1/r_1$, $2\alpha_2/d_2 = 1/r_2$, $(4d_2/\alpha_1^2)((1 + \alpha_1)/d_1) = 1/r_3$, and $4d_2\lambda/\alpha_1^2 = 1/r_4$.

Then, condition (113) becomes

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} < 1 - \ln 2. \tag{114}$$

By imposing the condition, $r_1 = r_2 = r_3 = r_4 = r$, one obtains $r \geq 14$ and assumptions (103)–(109). According to Theorem 13, the unique interior equilibrium point $E_3^* = (u^*; v^*, ku^*)$ of model (1.3) is globally and asymptotically stable. □

8. Numerical Simulations

8.1. Limit Cycle. Let us consider the period $p = 46$ and the following control parameters: $(d_1, d_2, \gamma, \alpha_1, \alpha_2, \delta, \beta_1, \beta_2) = (1.75; 0.25; 0.5; 3.5; 1.75; 0.4375; 3.617; 1.1)$, $\lambda = 0.0000123$, and $T = 45$.

For the graphic illustration, let us consider the initial conditions $(u_0, v_0, w_0) = (6; 3; 0.00331962170895946)$, the step $h = 10^{-5}$, and the numbers iterations of time $N = 4600000$. One obtains the maximum values of u, v, and w:

$$\begin{aligned} u_{max} &= 66.4527115210953, \\ v_{max} &= 2836.21769136928, \\ w_{max} &= 0.0367713984036395. \end{aligned} \tag{115}$$

The trivial equilibrium points of the model are $E_0 = (0; 0.0)$,

$E_1 = (4.5; 0; 0.00249006081209868)$, and $E_2 = (0; 0.000476921218349683; 0)$.

In addition, $E^* = (2.19899664564055; 2.19198952607149; 0.00121680786072133)$ is the unique interior equilibrium. So, the illustrating Figure 1 is for the initial time $t_0 = 0$.

(i) Interpretation. The trajectories of GDP u and of capital v and w (the effect of delay) are periodic. The orbit starting from the initial condition (u_0, v_0, w_0) gravitates around the equilibrium point E^* without never reaching it. There is, therefore, a limit cycle around this equilibrium point E^*. The orbit revolves around the equilibrium point. They are moving away from trivial equilibrium points and converge towards the limit cycle around E^*. Indeed, one has $\lambda = 0.0000123 <$

$\lambda_{max}^{(3)} = 0.2302$ and $T = 45 < T_s^{(3)} = 305733.267595647$. According to Proposition 11, $\omega = 0.000686044679105366$ is the positive root of $F(\omega)$. One has $T_3^* = 2383.38276771532$ and $\delta(T_3^*) < 0$. So, E^* is unstable according to Theorem 12 because $Tr = 3.255 > 0$ for $T = 0$. The trivial equilibrium points of the model are all unstable for $T < T_3^* < T_2^* = 4437.41337922542$. Hence, we have the graphic illustration.

8.2. Global Stability. Let us consider the period $p = 25$ and the following control parameters: $(d_1, d_2, \gamma, \alpha_1, \alpha_2, \delta, \beta_1, \beta_2) = (101390.241; 50695.121; 0.5; 1822.498; 1747.022; 1721.154; 9.863.10^{-6}; 7.784.10^{-6})$ and $\lambda = 0.75$ and $T = 0.462$.

For the graphic illustration, let us consider the initial conditions $(u_0, v_0, w_0) = (5000; 8000; 1344.89438670147)$, the step $h = 10^{-4}$, and the numbers iterations of time $N = 250000$. From condition (9) of Theorem 4, the system is permanent if $\alpha_2 > \delta, 0 \leq \beta_1 < 4$, and $m_u^{(0)} \geq 2\sqrt{\gamma k M_u M_v / d_1}$. One obtains the maximum and minimum values of u, v, and w for $\varepsilon = m_u^{(0)} - 2\sqrt{\gamma k M_u M_v / d_1}$:

$$\begin{aligned} u_{max} &= 1661666.077, \\ v_{max} &= 38031.766, \\ w_{max} &= 486690.726. \\ u_{min} &= m_u^\varepsilon = 1139.233, \\ v_{min} &= 750.460, \\ w_{min} &= 333.6737. \end{aligned} \tag{116}$$

The trivial equilibrium points of the model are $E_0 = (0; 0; 0)$, $E_1 = (1823.498; 0; 534.090)$, and $E_2 = (0; 750.630; 0)$.

In addition, $E^* = (1796.505; 777.6445; 526.184)$ is the unique interior equilibrium point verifying condition (91)–(93) of Theorem 13. Then, E^* is globally stable. Hence, the illustrating Figure 2 is for the initial time $t_0 = 0$:

(ii) Interpretation. The trajectories of GDP u and capital v and w (the effect of delay) stabilize, respectively, around $u^* = 1796.505$, $v^* = 777.6445$, and $w^* = 526.184$ when t is greater than T. The border dynamics show that when the orbit arrives in a border plan, it remains and converges towards a corresponding equilibrium point (see Orbit.2 to Orbit.4 of Figure 2). Moreover, the trivial equilibrium points of the model are all unstable. So, the orbits, are moving away from the borders plans and converge towards E^*. Hence, E^* is globally stable, which is illustrated by the figure "phase portrait" of Figure 2.

9. Conclusion

The economic basic model of our work was one of dynamics Kaldor at an effective growth rate whose rate of saving is the Holling-II type and investment rate is from Leslie and Gower type modified in dimension two. Taking into account the time required for the saving to ensure its self-financing of all the investments, one obtains a model with a delay of the Kaldor type modified. This model is bounded and
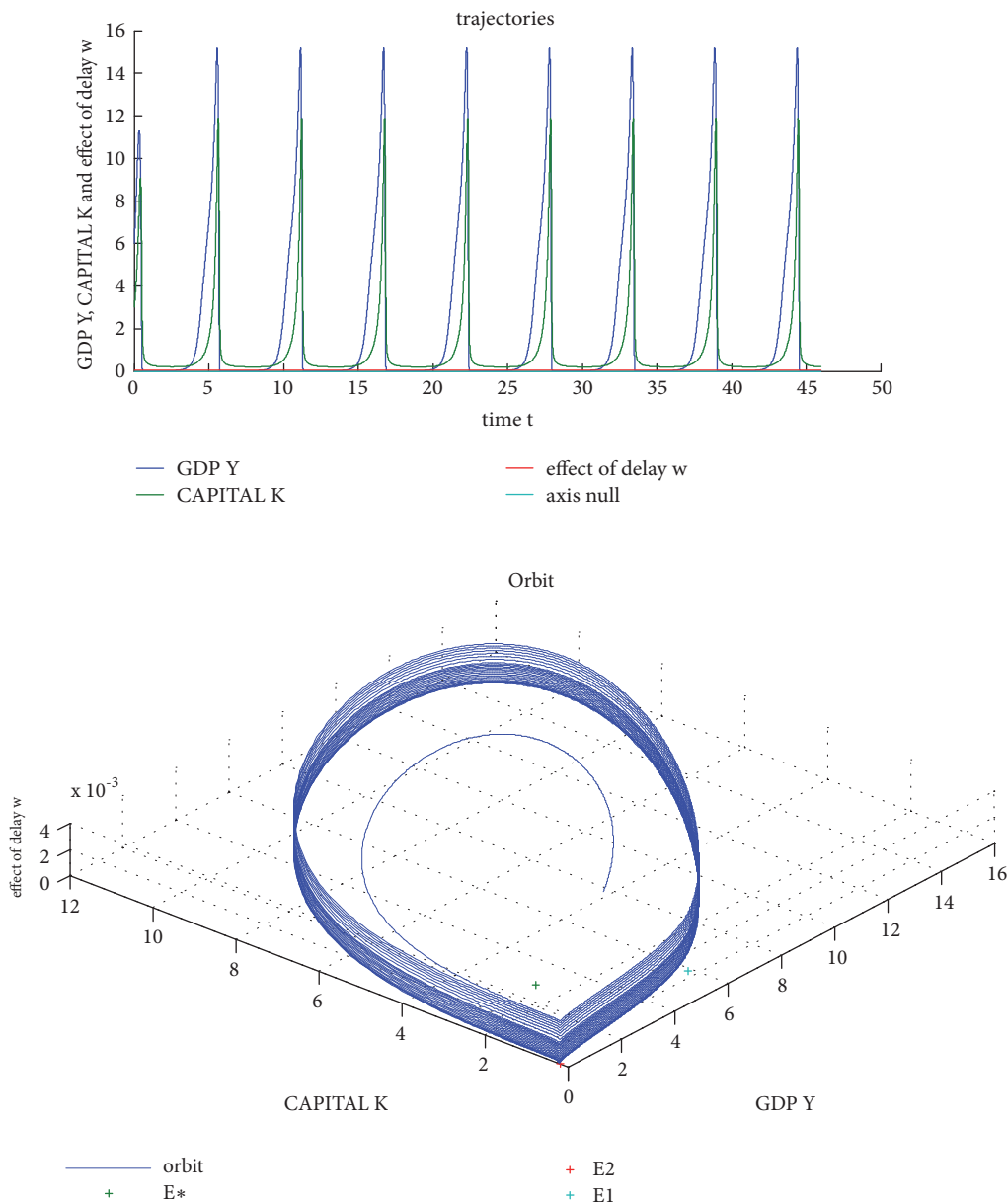
FIGURE 1: Trajectories and orbits with $u_0(\theta) = \varphi(\theta) = u(0)e^{A\theta}/(1 + u(0)(e^{A\theta} - 1))$, $\forall\theta[-T;0]$, where $A = (1/2T)\ln\{|u(0) - 1|/u(0)\}$ if $u(0) \neq 1$ and $A = -1/2T$ if $u(0) = 1$.

admits an attractor unit (set). Thus, under certain conditions, this model with delay is permanent. On the one hand, this permanence appears in the form of stationary growth of the capital stock and the product (stable interior equilibrium point) and, on the other hand, in the form of cyclic growth of the capital and the product (limiting cycle). In other words, this permanence avoids a shortage of the capital stock or the long-term production. In front of some other conditions, the dynamic stability with delay is also global; i.e., it does not depend on the level of the capital stock and the level of the production at the initial time. The consideration of the delay (due to the time for the economy to finance investments) can justify the bifurcation of an economic model of the stationary growth towards cyclic growth (see Theorem 12,

(3)(a)(i)). However, this delay in the model can stabilize an initially unstable equilibrium of the system (see Theorem 12, (3)(b)(ii)), which represents a major economic interest since it allows the saving to get rid of risks from the cyclic growth.

Data Availability

No data were used to support this study.

Conflicts of Interest

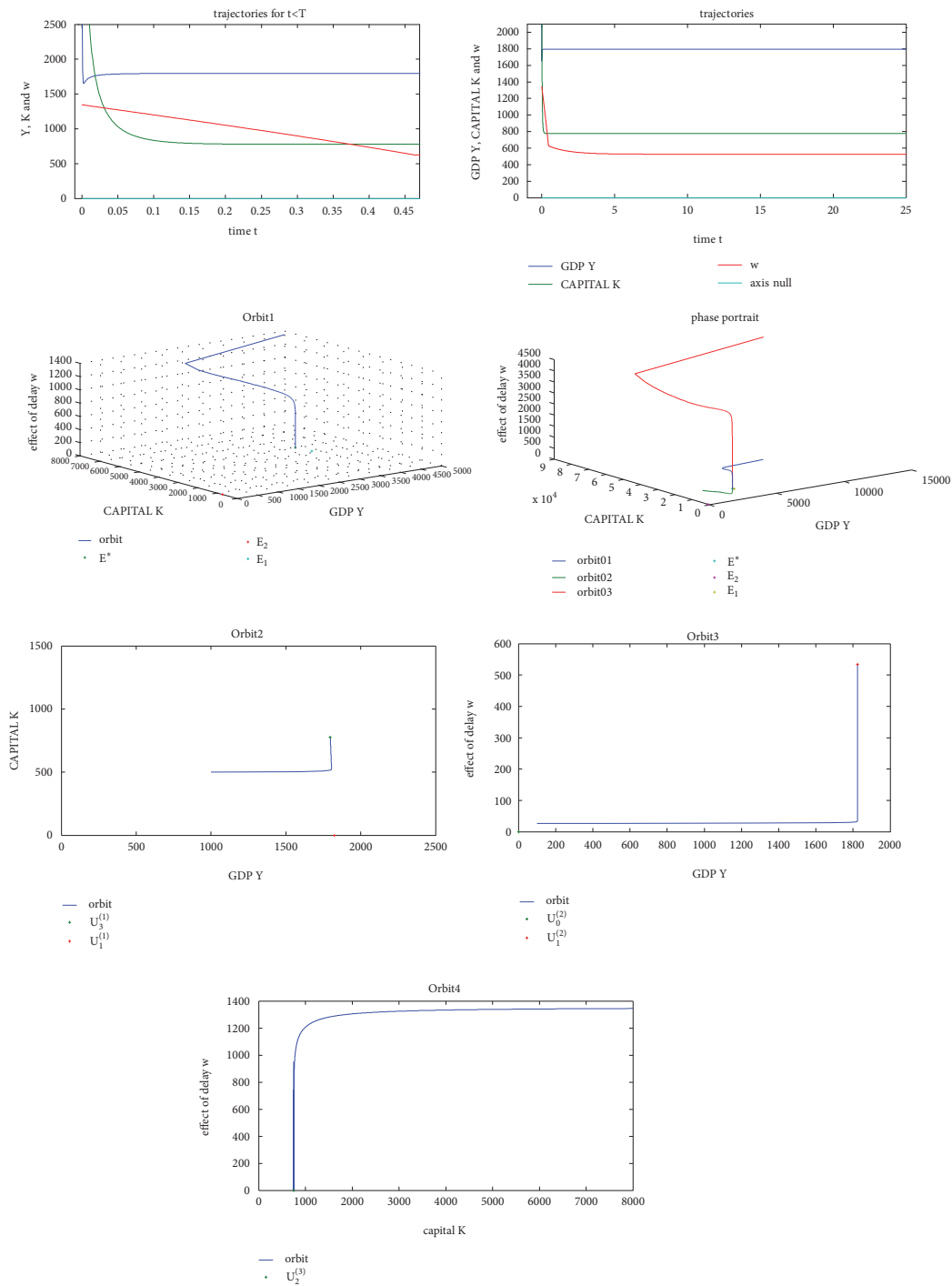The authors declare that there are no conflicts of interest for this paper.

FIGURE 2: Trajectories, orbits, and phase portrait with $u_0(\theta) = \varphi(\theta) = u(0)e^{A\theta}/(1 + u(0)(e^{A\theta} - 1)), \forall\theta[-T; 0]$, where $A = (1/2T)\ln\{|u(0) - 1|/u(0)\}$ if $u(0) \neq 1$ and $A = -1/2T$ if $u(0) = 1$.

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