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Research Article

Existence Theorems on Solvability of Constrained Inclusion Problems and Applications

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Let X be a real locally uniformly convex reflexive Banach space with locally uniformly convex dual space X^*. Let $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ be a maximal monotone operator and $C : X \supseteq D(C) \longrightarrow X^*$ be bounded and continuous with $D(T) \subseteq D(C)$. The paper provides new existence theorems concerning solvability of inclusion problems involving operators of the type $T + C$ provided that C is compact or T is of compact resolvents under weak boundary condition. The Nagumo degree mapping and homotopy invariance results are employed. The paper presents existence results under the weakest coercivity condition on $T+C$. The operator C is neither required to be defined everywhere nor required to be pseudomonotone type. The results are applied to prove existence of solution for nonlinear variational inequality problems.

1. Introduction: Preliminaries

In what follows, the norm of the spaces X and X^* will be denoted by $\|\cdot\|$. For $x \in X$ and $x^* \in X^*$, the pairing $\langle x^*, x\rangle$ denotes the value $x^*(x)$. Let X and Y be real Banach spaces. For an operator $T : X \longrightarrow 2^Y$, we define the domain $D(T)$ of T by $D(T) = \{x \in X : Tx \neq \emptyset\}$, and the range $R(T)$ of T by $R(T) = \bigcup_{x\in D(T)} Tx$. The symbol $G(T)$ denotes the graph of T given by $\{(x, x^*) : x \in D(T), x^* \in Tx\}$. An operator $T : X \supset D(T) \longrightarrow Y$ is "demicontinuous" if it is continuous from the strong topology of $D(T)$ to the weak topology of Y. It is "compact" if it is strongly continuous and maps bounded subsets of $D(T)$ to relatively compact subsets of Y. An operator $T : X \supset D(T) \longrightarrow 2^Y$ is "bounded" if it maps each bounded subset of $D(T)$ into a bounded subset of Y. The mapping $J : X \longrightarrow 2^{X^*}$ defined by

$$J(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \ \|x^*\| = \|x\|\right\} \qquad (1)$$

is called the "normalized duality mapping". It is known due to Hahn-Banach theorem that $J(x) \neq \emptyset$. In addition, the local uniform convexity of X and X^* implies that J is single valued, bounded, monotone, bicontinuous, and of type (S_+).

Definition 1. An operator $T : X \supset D(T) \longrightarrow 2^{X^*}$ is said to be

(i) "monotone" if $\langle u^* - v^*, x - y\rangle \geq 0$ for all (x, u^*) and (y, v^*) in $G(T)$;

(ii) "maximal monotone" if T is monotone and $\langle u^* - u_0^*, x - x_0\rangle \geq 0$ for every $(x, u^*) \in G(T)$ implies $x_0 \in D(T)$ and $u_0^* \in Tx_0$;

(iii) "coercive" if either $D(T)$ is bounded or there exists a function $\psi : [0, \infty) \longrightarrow (-\infty, \infty)$ such that $\psi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$ and $\langle y^*, x\rangle \geq \psi(\|x\|)\|x\|$ for all $x \in D(T)$ and $y^* \in Tx$;

(iv) "expansive" if there exists $\alpha > 0$ such that $\|u^* - v^*\| \geq \alpha\|x - y\|$ for all $x \in D(T)$, $y \in D(T)$, $u^* \in Tx$, and $v^* \in Ty$.

It is well-known that a monotone operator T is maximal monotone if and only if $R(T + \lambda J) = X^*$ for every $\lambda > 0$ (cf. Theorem 2.2 [1]) and $(T + \lambda J)^{-1} : X^* \longrightarrow D(T)$ is single valued, monotone, and demicontinuous. For each $\lambda > 0$, the operator $T_\lambda : X \longrightarrow X^*$, defined by $T_\lambda x = (T^{-1} + \lambda J^{-1})^{-1}x$, is the "Yosida approximant" of T. It is bounded, continuous, and maximal monotone such that $T_\lambda x \rightharpoonup T^{(0)}x$ as $\lambda \longrightarrow 0^+$, for every $x \in D(T)$, where $\|T^0 x\| = \inf\{\|y^*\| : y^* \in Tx\}$. The operator $J_\lambda : X \longrightarrow D(T)$ defined by $J_\lambda x = x - \lambda J^{-1}(T_\lambda x)$, is called the "Yosida resolvent" of T. It is continuous, $T_\lambda x \in$

$\overline{T(J_\lambda x)}$ for every $x \in X$ and $\lim_{\lambda \to 0} J_\lambda x = x$ for all $x \in \overline{coD(T)}$, where $coD(T)$ is the convex hull of $D(T)$. For each $x \in D(T)$, $\|T_\lambda x\| \leq |Tx|$ for all $\lambda > 0$, where $|Tx|$ denotes $\|T^0 x\|$. A maximal monotone operator T is called of compact resolvents if J_λ is compact for all $\lambda > 0$. For further references on monotonicity theory, the reader is referred to Pascali and Sburlan [2], Barbu [1], Zeidler [3], Kenmochi [4], and the references therein. The following Lemma is due to Brèzis, Crandal, and Pazy [5].

Lemma 2. *Let B be a maximal monotone set in $X \times X^*$. If $(u_n, u_n^*) \in B$ such that $u_n \rightharpoonup u$ in X, $u_n^* \rightharpoonup u^*$ in X^*, and*

$$\limsup_{n \to \infty} \langle u_n^* - u^*, u_n - u \rangle \leq 0, \tag{2}$$

then $(u, u^*) \in B$ and $\langle u_n^*, u_n \rangle \longrightarrow \langle u^*, u \rangle$ as $n \longrightarrow \infty$.

The main objective of this paper is to establish sufficient conditions which guarantee existence of solution for inclusions of the type $(T + C)(D(T) \cap B_R(0)) \ni f^*$ (where $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ is maximal monotone of compact resolvents and $C : X \supseteq D(C) \longrightarrow X^*$ is bounded and continuous with $D(T) \subseteq D(C)$) provided that there exists $R > 0$ such that $(T + C + \varepsilon J)(D(T) \cap \partial B_R(0)) \not\ni f^*$ for all $\varepsilon > 0$. Under this boundary condition, inclusion result of the type $f^* \in \overline{(T + C)(D(T) \cap B_R(0))}$ is included if C is compact with $D(T) \subseteq D(C)$ and T is arbitrary maximal monotone. The case where T is expansive maximal monotone and C is compact is also included. The operators of type C are available in the form of lower order term in many nonlinear differential equations in appropriate function spaces.

The paper is organized as follows. In Section 2, the main existence results (Theorems 3 and 6) are proved. The arguments of the proofs of Theorem 3 are based on Nagumo homotopy invariance result. Theorem 6 follows as a result of Theorem 3. Nagumo [6] developed a degree theory in a setting of linear convex topological space Y for operators of the type $I - B$, where $B : \overline{G} \longrightarrow Y$ is a compact operator, I is the identity mapping on Y, and G is a nonempty and open subset of X. The important contributions of Nagumo are (i) G is a nonempty and open subset of X (not necessarily bounded) and (ii) the degree is invariant under the homotopy $H(x,t) = x - W(t,x)$, $(t,x) \in [0,1] \times \overline{G}$ provided that $W : [0,1] \times \overline{G} \longrightarrow X$ is compact and $0 \notin H(t, \partial G)$ for all $t \in [0,1]$. For further references on Nagumo degree and related results, the reader is referred to the paper due to Nagumo [6, Theorem 5, 6, 7]. Throughout the paper d_{NA} stands for Nagumo degree. In Section 3, we demonstrated the applicability of the abstract results to prove existence of solution(s) for variational inequality problems.

Existence results concerning pseudomonotone perturbations of maximal monotone operators under coercivity condition can be found in the papers due to Kenmochi [4, 7, 8], Asfaw and Kartsatos [9], Le [10], Asfaw [11], and the references therein. For related results concerning existence of solution for inclusion problems of the type $Tu + Su \ni f^*$ in $D(T)$, where S is an everywhere defined bounded pseudomonotone operator under Leray-Schauder type boundary condition on $T + S$, we cite the results due to Asfaw and Kartsatos [9, Theorem 11, Theorem 13, pp. 127-133]. Analogous result for possibly unbounded single multivalued pseudomonotone operator S is due to Figueiredo [12]. However, the cases where S is not necessarily pseudomonotone type is not studied earlier. It is the purpose of the present paper to address analogous result for $T + C$, where C is possibly not everywhere defined compact or C is bounded and continuous, and T is of compact resolvents.

2. Main Results

In this section, we prove the main existence theorem.

Theorem 3. *Let $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ be maximal monotone and $C : X \supseteq D(C) \longrightarrow X^*$ with $D(T) \subseteq D(C)$. Let $f^* \in X^*$. Suppose there exists $R > 0$ such that $D(T) \cap B_R(0) \neq \emptyset$ and*

$$(T + C + \varepsilon J)(D(T) \cap \partial B_R(0)) \not\ni f^* \tag{3}$$

for all $\varepsilon > 0$. Then

(i) *$f^* \in \overline{(T + C)(D(T) \cap \overline{B}_R(0))}$ if C is compact;*

(ii) *$f^* \in (T + C)(D(T) \cap \overline{B}_R(0))$ if T is expansive and C is compact;*

(iii) *$f^* \in (T + C)(D(T) \cap \overline{B}_R(0))$ if $\overline{D(T)} \subseteq D(C)$ and $C : D(C) \longrightarrow X^*$ is completely continuous;*

(iv) *$f^* \in (T + C)(D(T) \cap \overline{B}_R(0))$ if T is of compact resolvent and, C is bounded and continuous.*

Proof. Suppose the hypotheses hold. We divide the proof into two steps.

Step 1. Let $\partial I_R : \overline{B}_R(0) \longrightarrow 2^{X^*}$ be the subdifferential of the indicator function I_R on $\overline{B}_R(0)$ and $A = T + \partial I_R$. The operator A is maximal monotone with bounded domain $D(A) = D(T) \cap \overline{B}_R(0)$ because $D(T) \cap D(\mathring{\partial} I_R) = D(T) \cap B_R(0) \neq \emptyset$. For each $\lambda > 0$, let $A_\lambda : X \longrightarrow X^*$ and $J_\lambda : X \longrightarrow D(A)$ be the Yosida approximant and resolvent of A, respectively. It is well-known that A_λ is bounded, continuous, and maximal monotone, and J_λ is bounded and continuous such that $A_\lambda x = (1/\lambda) J(x - J_\lambda x)$, $J_\lambda x \in D(A)$, and $A_\lambda x \in A(J_\lambda x)$ for all $x \in X$. Let

$$H_\varepsilon^\lambda(t, x) = x - W(t, x), \quad (t, x) \in [0,1] \times \overline{B}_R(0), \tag{4}$$

where $W : [0,1] \times \overline{B}_R(0) \longrightarrow X$ is given by

$$W(t, x) = (\varepsilon J + t A_\lambda)^{-1} (-t (C J_\lambda x - f^*)). \tag{5}$$

For each $\varepsilon > 0$ and $\lambda > 0$, we shall show that W is a compact operator. Let $(t_n, x_n) \longrightarrow (t_0, x_0)$, i.e., $t_n \longrightarrow t_0$ and $x_n \longrightarrow x_0$ as $n \longrightarrow \infty$. The continuity of C and J_λ implies $-t_n(C J_\lambda x_n - f^*) \longrightarrow -t_0(C J_\lambda x_0 - f^*)$ as $n \longrightarrow \infty$. Let

$$z_n = W(t_n, x_n) = (\varepsilon J + t_n A_\lambda)^{-1} (-t_n (C J_\lambda x_n - f^*)) \tag{6}$$

for all n, i.e., $\varepsilon J z_n + t_n A_\lambda z_n + t_n(CJ_\lambda x_n - f^*) = 0$ for all n. Fix $a_0 \in D(A)$. By applying the monotonicity of A_λ and boundedness of J_λ and C, we obtain that

$$\begin{aligned}
\varepsilon \|z_n\|^2 &= \varepsilon \langle Jz_n, z_n - a_0\rangle + \varepsilon \langle Jz_n, a_0\rangle \\
&= \varepsilon \langle Jz_n, a_0\rangle - t_n \langle A_\lambda z_n, z_n - a_0\rangle \\
&\quad - t_n \langle CJ_\lambda x_n - f^*, z_n - a_0\rangle \\
&= \varepsilon \langle Jz_n, a_0\rangle - t_n \langle A_\lambda z_n - A_\lambda a_0, z_n - a_0\rangle \\
&\quad + t_n \langle A_\lambda a_0, z_n - a_0\rangle \\
&\quad - t_n \langle CJ_\lambda x_n - f^*, z_n - a_0\rangle \\
&\le t_n \langle A_\lambda a_0, z_n - a_0\rangle \\
&\quad - t_n \langle CJ_\lambda x_n - f^*, z_n - a_0\rangle + \varepsilon \langle Jz_n, a_0\rangle \\
&\le (\|A_\lambda a_0\| + K)\|z_n - a_0\| + \varepsilon \|z_n\| \|a_0\| \\
&\le (|Aa_0| + K)\|z_n - a_0\| + \varepsilon \|z_n\| \|a_0\|
\end{aligned} \tag{7}$$

for all n; i.e., $\varepsilon\|z_n\|^2 \le (|Aa_0| + K)\|z_n - a_0\| + \varepsilon\|z_n\|\|a_0\|$ for all n, where K is an upper bound for $\{t_n C x_n - f^*\}$ and $|Aa_0| = \inf\{\|x^*\| : x^* \in Aa_0\}$. Thus, we get the boundedness of $\{z_n\}$. Assume without loss of generality that $z_n \rightharpoonup z_0$ as $n \longrightarrow \infty$. By the monotonicity of A_λ, we arrive at

$$\begin{aligned}
&\limsup_{n\to\infty} \langle Jz_n, z_n - z_0\rangle \\
&\quad \le \limsup_{n\to\infty} (-t_n \langle A_\lambda z_n, z_n - z_0\rangle) \\
&\quad\quad + \limsup_{n\to\infty} (-t_n \langle CJ_\lambda x_n - f^*, z_n - z_0\rangle) \\
&\quad \le -\liminf_{n\to\infty} t_n \langle A_\lambda z_n, z_n - z_0\rangle \\
&\quad\quad - \liminf_{n\to\infty} t_n \langle CJ_\lambda x_n - f^*, z_n - z_0\rangle \\
&\quad = -\liminf_{n\to\infty} t_n \langle A_\lambda z_n - A_\lambda z_0, z_n - z_0\rangle \\
&\quad\quad - \liminf_{n\to\infty} t_n \langle CJ_\lambda x_n - f^*, z_n - z_0\rangle \\
&\quad\quad + \lim_{n\to\infty} t_n \langle A_\lambda z_0, z_n - z_0\rangle \\
&\quad \le -\lim_{n\to\infty} t_n \langle CJ_\lambda x_n - f^*, z_n - z_0\rangle \\
&\quad\quad + \lim_{n\to\infty} t_n \langle A_\lambda z_0, z_n - z_0\rangle = 0.
\end{aligned} \tag{8}$$

Thus we conclude that $z_n \longrightarrow z_0$ as $n \longrightarrow \infty$ because J is of type (S_+). Since J and A_λ are continuous, it follows that $\varepsilon J z_0 + t_0(A_\lambda z_0 + (CJ_\lambda x_0 - f^*)) = 0$; i.e., we have

$$z_0 = (\varepsilon J + t_0 A_\lambda)^{-1} (-t_0 (CJ_\lambda x_0 - f^*)) = W(t_0, x_0). \tag{9}$$

We notice here that the above argument holds for any subsequence of $\{z_n\}$ (i.e., every subsequence of $\{z_n\}$ admits a convergent subsequence). Consequently, we conclude that $\{z_n\}$ converges z_0; i.e., the continuity of W is proved. Next we assume $\{(s_n, y_n)\}$ is bounded in $[0,1]\times\overline{G}$. The sequence $\{J_\lambda y_n\}$ is bounded because the sequences $\{s_n\}$ and $\{y_n\}$ are bounded. Since $J_\lambda y_n \in D(A) \subseteq D(C)$ for all n and C is compact, we can extract a subsequence, denoted again by $\{-s_n(CJ_\lambda y_n - f^*)\}$, such that $-s_n(CJ_\lambda y_n - f^*) \longrightarrow a_0^*$ as $n \longrightarrow \infty$. We notice here that

$$w_n = W(s_n, y_n) = (\varepsilon J + s_n A_\lambda)^{-1} (-s_n (CJ_\lambda y_n - f^*)) \tag{10}$$

for all n implies $\varepsilon J w_n + s_n(A_\lambda w_n + (CJ_\lambda y_n - f^*)) = 0$ for all n. By applying the argument used in the proof of continuity of W, it follows that $\{w_n\}$ is bounded and admits a convergence subsequence. Thus the compactness of W is proved. The same argument shows that W is compact if C is completely continuous with $\overline{D(T)} \subseteq D(C)$.

Next we fix $\varepsilon > 0$ temporarily and show that $0 \notin H_\varepsilon^\lambda(t, \partial B_R(0))$ for all $t \in [0,1]$ and sufficiently small $\lambda > 0$. Suppose not, i.e., there exists $\lambda_n \downarrow 0^+$, $\tau_n \in [0,1]$ and $u_n \in \partial G$ such that

$$\varepsilon J u_n + \tau_n \left(A_{\lambda_n} u_n + \left(CJ_{\lambda_n} u_n - f^*\right)\right) = 0 \tag{11}$$

for all n. If $\tau_n = 0$ for some n, then it follows that $\varepsilon J u_n = 0$; i.e., $u_n = 0$. But this is impossible because $u_n \in \partial B_R(0)$. Assume $\tau_n \in (0,1]$ for all n and $\tau_n \longrightarrow \tau_0 \in [0,1]$ as $n \longrightarrow \infty$. The boundedness of $\{CJ_{\lambda_n} u_n\}$ follows because of the boundedness of C and $D(A)$ is bounded and $J_{\lambda_n} u_n \in D(A)$ for all n; i.e., $\{A_{\lambda_n} u_n\}$ is bounded. Assume without loss of generality that $u_n \rightharpoonup u_0$, $A_{\lambda_n} u_n \rightharpoonup u_0^*$, and $CJ_{\lambda_n} u_n \longrightarrow b_0^*$ as $n \longrightarrow \infty$. By the monotonicity of J and (11), we claim that

$$\alpha = \liminf_{n\to\infty} \left\langle A_{\lambda_n} u_n, u_n - u_0\right\rangle \ge 0. \tag{12}$$

Suppose not, i.e., $\alpha < 0$. Then there exists a subsequence $\{\langle A_{\varepsilon_n} u_n, u_n - u_0\rangle\}$ that converges to α as $n \longrightarrow \infty$. The fact that $J_{\lambda_n} u_n \in D(A)$, $A_{\lambda_n} u_n \in A(J_{\lambda_n} u_n)$, and $J_{\lambda_n} u_n = u_n - \lambda_n J^{-1}(A_{\lambda_n} u_n)$ for all n implies

$$\begin{aligned}
&\limsup_{n\to\infty} \left\langle A_{\lambda_n} u_n, J_{\lambda_n} u_n - u_0\right\rangle \\
&\quad \le \limsup_{n\to\infty} \left\langle A_{\lambda_n} u_n, J_{\lambda_n} u_n - u_n\right\rangle \\
&\quad\quad + \limsup_{n\to\infty} \left\langle A_{\lambda_n} u_n, u_n - u_0\right\rangle \\
&\quad < -\liminf_{n\to\infty} \left(\lambda_n \left\|A_{\lambda_n} u_n\right\|^2\right) + \alpha \le \alpha < 0.
\end{aligned} \tag{13}$$

The maximality of A along with Lemma 2 implies $u_0 \in D(A)$, $u_0^* \in Au_0$, and $\langle A_{\lambda_n} u_n, J_{\lambda_n} u_n - u_0\rangle \longrightarrow 0$ as $n \longrightarrow \infty$; i.e., we get $\langle A_{\lambda_n} u_n, u_n - u_0\rangle \longrightarrow 0 = \alpha$. However, this is impossible; i.e., the claim holds. The case $\tau_0 = 0$ implies

$u_n \longrightarrow 0 \in \partial B_R(0)$ as $n \longrightarrow \infty$. However this is impossible. Assume that $\tau_0 \in (0,1]$. By using $\alpha \geq 0$ in (11), we obtain that

$$\begin{aligned}\limsup_{n\longrightarrow\infty} \langle Ju_n, u_n - u_0\rangle &= \limsup_{n\longrightarrow\infty} \left(-\tau_n \left\langle A_{\lambda_n} u_n + CJ_{\lambda_n} u_n - f^*, u_n - u_0\right\rangle\right) \\ &= -\tau_0 \liminf_{n\longrightarrow\infty} \left(\left\langle A_{\lambda_n} u_n + CJ_{\lambda_n} u_n - f^*, u_n - u_0\right\rangle\right) \\ &\leq -\tau_0 \liminf_{n\longrightarrow\infty} \left(\left\langle A_{\lambda_n} u_n, u_n - u_0\right\rangle\right) \\ &\quad + \liminf_{n\longrightarrow\infty} \left(\left\langle CJ_{\lambda_n} u_n - f^*, u_n - u_0\right\rangle\right) \leq 0.\end{aligned} \tag{14}$$

The (S_+) condition on J implies $u_n \longrightarrow u_0 \in D(T) \cap \partial B_R(0)$ and $J_{\lambda_n} u_n \longrightarrow u_0$ as $n \longrightarrow \infty$. The maximal monotonicity of A together with Lemma 2 implies that $u_0 \in D(A)$ and $u_0^* \in Au_0$. The continuity of C and J implies

$$\begin{aligned}&\varepsilon Ju_n + \tau_n \left(A_{\lambda_n} u_n + CJ_{\lambda_n} u_n - f^*\right) \longrightarrow \\ &\qquad \varepsilon Ju_0 + \tau_0 \left(u_0^* + Cu_0 - f^*\right)\end{aligned} \tag{15}$$

as $n \longrightarrow \infty$. Consequently, letting $n \longrightarrow \infty$ in (11) gives $\varepsilon\tau_0^{-1} Ju_0 + u_0^* + Cu_0 = f^*$; i.e., we obtain that

$$u_0^* + Cu_0 + \varepsilon\tau_0^{-1} Ju_0 \in Tu_0 + Cu_0 + \partial I_R(u_0). \tag{16}$$

This implies $u_0^* = a_0^* + b_0^*$ for some $a_0^* \in Tu_0$ and $b_0^* \in \partial I_R(u_0)$. However, it is well-known that

$$\partial I_R(x) = \begin{cases} \{0\} & \text{if } x \in B_R(0), \\ \{\lambda Jx : \lambda \geq 0\} & \text{if } x \in \partial B_R(0), \\ \emptyset & \text{if } x \in X \setminus \overline{B_R(0)}. \end{cases} \tag{17}$$

As a result, $b_0^* = \mu Ju_0$ for some $\mu \geq 0$; i.e., we get $(T + C + (\tau_0^{-1}\varepsilon + \mu)J)(D(T) \cap \partial B_R(0)) \ni f^*$. However, this is impossible because of the hypothesis of the theorem. Therefore, for fixed $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that $0 \notin H_\varepsilon^\lambda(t, \partial B_R(0))$ for all $t \in [0,1]$ and $\lambda \in (0, \lambda_0)$; i.e., $\{H_\lambda^\varepsilon(t,.)\}_{t\in[0,1]}$ is an acceptable Nagumo homotopy.

Step 2. Fix $\varepsilon > 0$. By using the admissible Nagumo homotopy $\{H_\varepsilon^\lambda(t,.)\}_{t\in[0,1]}$ obtained in Step 1 and applying the homotopy invariance properties of d_{NA}, we see that $d_{NA}(H_\lambda^\varepsilon(t,.), G, 0)$ is independent of $t \in [0,1]$; i.e.,

$$\begin{aligned}d_{NA}\left(H_\varepsilon^\lambda(1, B_R(0), 0\right) &= d_{NA}\left(H_\varepsilon^\lambda(0, B_R(0), 0\right) \\ &= d_{NA}\left(I, B_R(0), 0\right) = 1\end{aligned} \tag{18}$$

for all sufficiently small $\lambda > 0$. Thus for each $\lambda_n \downarrow 0^+$, there exists $v_n \in B_R(0)$ such that

$$\varepsilon Jv_n + A_{\lambda_n} v_n + CJ_{\lambda_n} v_n = f^* \tag{19}$$

for all n. The boundedness of $\{CJ_{\lambda_n} v_n\}$ and $\{A_{\lambda_n} v_n\}$ follows because of the boundedness of $\{v_n\}$, $D(A)$, and C. As a result we can easily follow the arguments used in the last part of Step 1 to conclude the existence of $z_\varepsilon \in D(T) \cap B_R(0)$ and $z_\varepsilon^* \in Az_\varepsilon$ such that $\varepsilon Jz_\varepsilon + z_\varepsilon^* + Cz_\varepsilon = f^*$; i.e., for each $\varepsilon_n \downarrow 0^+$, there exist $z_n \in D(T) \cap B_R(0)$ and $z_n^* \in Az_n$ such that

$$\varepsilon_n Jz_n + z_n^* + Cz_n = f^* \tag{20}$$

for all n; i.e., we get

$$\varepsilon_n Jz_n + a_n^* + b_n^* + Cz_n = f^* \tag{21}$$

for all n and for some $a_n^* \in \partial I_R(z_n)$ and $b_n^* \in Tz_n$. However, applying the boundary hypothesis on $T + C$, we conclude that $z_n \in B_R(0)$ for all n and $a_n^* = 0$ for all n. Consequently, we arrive at

$$\varepsilon_n Jz_n + b_n^* + Cz_n = f^* \tag{22}$$

for all n. The boundedness of $\{Jz_n\}$ implies that $b_n^* + Cz_n \longrightarrow f^*$; i.e., $f^* \in \overline{(T + C)(D(T) \cap B_R(0))}$. This completes the proof of (i). Next we prove (ii). Suppose T is expansive and C is compact. By the compactness of C we assume without loss of generality that $Cz_n \longrightarrow g_0^*$; i.e., $b_n^* \longrightarrow f^* - g_0^*$. The expansiveness of T implies $z_n \longrightarrow z_0 \in \overline{B}_R(0)$ as $n \longrightarrow \infty$. The maximality of T along with Lemma 2 yields $z_0 \in D(T) \subseteq D(C)$ and $f^* - g_0^* \in Tz_0$. As a result we conclude that $f^* - Cz_0 \in Tz_0$ (i.e., $f^* \in (T + C)(D(T) \cap \overline{B}_R(0))$) because of the continuity of C; i.e., (ii) holds.

(iii) Suppose C is completely continuous and $\overline{D(T)} \subseteq D(C)$. Assume by passing into a subsequence that $z_n \rightharpoonup d_0$ as $n \longrightarrow \infty$. The maximality of T implies that $\overline{D(T)} \cap \overline{B}_R(0)$ is closed and convex; i.e., it is weakly closed and $d_0 \in \overline{D(T)} \cap \overline{B}_R(0)$. The complete continuity of C implies $Cz_n \longrightarrow Cd_0$ and $b_n^* \longrightarrow f^* - Cd_0$; i.e., $d_0 \in D(T) \cap \overline{B}_R(0)$ and $f^* \in (T + C)(D(T) \cap \overline{B}_R(0))$; i.e., (iii) is proved.

(iv) Suppose T is of resolvent compact and C is bounded and continuous such that the boundary condition on $T + C$ holds. It is known due to Kartsatos [13, Lemma 3, pp. 1684] that J_λ is compact if and only if $(\lambda T + J)^{-1}$ is compact, and J_λ is compact for all $\lambda > 0$ if J_μ is compact for some $\mu > 0$. As a result, the compactness of $(T+J)^{-1}$ and J_1 is used equivalently. Since C is bounded and J_λ is compact, it follows that CJ_λ is a compact operator; i.e., we can follow the arguments used in the proof of the first part of Theorem 3 to conclude that $W : [0,1] \times \overline{B}_R(0) \longrightarrow X$ given by

$$W(t,x) = \left(\varepsilon J + tT_\lambda\right)^{-1} \left(-t\left(CJ_\lambda x - f^*\right)\right) \tag{23}$$

is a compact operator. By following exactly analogous arguments used in the proof of (i) through (iii) of this theorem, for each $\varepsilon_n \downarrow 0^+$ there exist $\tau_n \in D(T) \cap B_R(0)$ and $v_n^* \in T\tau_n$ such that

$$v_n^* + C\tau_n + \varepsilon_n J\tau_n = f^* \tag{24}$$

for all n; i.e., we get $v_n^* + J\tau_n = -C\tau_n - \varepsilon_n J\tau_n + J\tau_n$ for all n; i.e.,

$$\tau_n = (T + J)^{-1}\left(b_n\right) \tag{25}$$

for all n, where $b_n = -C\tau_n - \varepsilon_n J\tau_n + J\tau_n$ for all n. By the compactness of $(J+T)^{-1}$, there exists a subsequence of $\{b_n\}$, denoted again by $\{b_n\}$, such that $\tau_n = (T+J)^{-1}(b_n) \longrightarrow \tau_0$ as $n \longrightarrow \infty$. Assume without loss of generality (by passing into a subsequence) that $C\tau_n \rightharpoonup g^*$ and $v_n^* \rightharpoonup f^* - g^*$ as $n \longrightarrow \infty$. The maximal monotonicity of T along with Lemma 2 implies that $\tau_0 \in D(T)$ and $f^* - g^* \in T\tau_0$. On the other hand, the continuity of C implies $C\tau_n \longrightarrow C\tau_0 = g^*$; i.e., $v_n^* \longrightarrow f^* - C\tau_0$; i.e., $f^* \in (T+C)(D(T) \cap \overline{B}_R(0))$. The proof is complete. □

The following corollary holds.

Corollary 4. *Let T and C be as given in Theorem 3. The conclusions in Theorem 3 hold if*

(a) *the boundary condition*

$$(T + C + \varepsilon J)(D(T) \cap \partial B_R(0)) \not\ni f^* \tag{26}$$

for all $\varepsilon > 0$ is replaced by

$$\left(T + C + \varepsilon\widetilde{J}\right)(D(T) \cap \partial B_R(0)) \not\ni f^* \tag{27}$$

for all $\varepsilon > 0$ and some $u_0 \in D(T) \cap B_R(0)$, where $\widetilde{J}(x) = J(x - u_0)$ for $x \in X$;

(b) *there exist $R > 0$ and $v_0 \in \overline{B}_R(0)$ such that*

$$\langle u^* + Cx - f^*, x - v_0 \rangle > 0 \tag{28}$$

for all $x \in D(T) \cap \partial B_R(0)$ and $u^* \in Tx$;

(c) *$0 \in T(0)$ and $\overline{B}_R(0)$ is replaced by a nonempty, closed, bounded, and convex subset K of X such that $0 \in \mathring{K}$ and εJ is replaced by $\varepsilon\partial I_K$, and*

$$(T + C + \varepsilon\partial I_K)(D(T) \cap \partial K) \not\ni f^* \tag{29}$$

for all $\varepsilon > 0$, where ∂I_K is the subdifferential of the indicator function on K.

Proof. (a) The proof for the analogous result under the boundary condition involving $\widetilde{J}$ follows because the operator $\widetilde{J} : X \longrightarrow X^*$ inherits all the properties of J; i.e., $\widetilde{J}$ is bounded, continuous, monotone, and of type (S_+).

(b) The side condition in (b) is equivalent to $(T + C + \varepsilon\partial I_R)(D(T) \cap \partial B_R(0)) \not\ni f^*$ for all $\varepsilon > 0$ because of the definition of $f^* - u^* - Cx \notin \partial I_R(x)$ for all $x \in D(T) \cap \partial B_R(0)$, $u^* \in Tx$, and the fact that $\partial I_R(x) = \{\varepsilon Jx : \varepsilon \geq 0\}$ for all $x \in \partial B_R(0)$. This implies that the condition in (b) is equivalent to the boundary condition in Theorem 3.

(c) Suppose $(T + C + \varepsilon\partial I_K)(D(T) \cap \partial K) \not\ni f^*$ for all $\varepsilon > 0$. The maximal monotonicity of $A = T + \partial I_K$ follows because $D(T) \cap \mathring{K} \neq \emptyset$. It is not difficult to see that boundary condition on $T + C + \varepsilon\partial K$ implies that there exists $y_0 \in K$ such that

$$\langle u^* + Cx - f^*, x - y_0 \rangle > 0 \tag{30}$$

for all $x \in D(T) \cap \partial\mathring{K}$ and $u^* \in Tx$. The proof of (c) follows based on the arguments of the proofs of (i) through (iii) of Theorem 3 by using $G = \mathring{K}$ instead of $B_R(0)$. The details are omitted here. To prove (iv) of Theorem 3 under (c), we shall show that A is resolvent compact. Let $\{f_n^*\}$ be a sequence in X^* such that $f_n^* \longrightarrow f_0^*$ as $n \longrightarrow \infty$ and $x_n = (J + A)^{-1}(f_n^*)$ for all n; i.e., $x_n \in D(A) \cap K$ for all n. Let $u_n^* \in Tx_n$ and $v_n^* \in \partial I_K(x_n)$ such that

$$u_n^* + v_n^* + Jx_n = f_n^* \tag{31}$$

for all n. The boundedness of $\{x_n\}$ follows because K is bounded. Assume without loss of generality that $x_n \rightharpoonup x_0$. Since K is closed and convex (i.e., it is weakly closed), we have $x_0 \in K$. In addition, the condition $0 \in T(0)$ yields

$$\begin{aligned} \langle v_n^*, x_n \rangle &= -\langle u_n^*, x_n \rangle - \|x_n\|^2 + \|f_n^*\| \|x_n\| \\ &\leq \|f_n^*\| \|x_n\| \leq M \end{aligned} \tag{32}$$

for all n, where M is an upper bound for $\{\|f_n^*\| \|x_n\|\}$. Since $D(\mathring{\partial I_K}) = \mathring{K} \neq \emptyset$, it follows that ∂I_K is strongly quasibounded, which yields the boundedness of $\{v_n^*\}$, i.e., $\{u_n^*\}$ is bounded. Assume without loss of generality that $v_n^* \rightharpoonup v_0^*$ and $u_n^* \rightharpoonup u_0^*$ as $n \longrightarrow \infty$. By following the arguments used in the proof of Theorem 3 and using the fact that J is bounded of type (S_+), we get $x_n \longrightarrow x_0$ as $n \longrightarrow \infty$ and $u_0^* + v_0^* + Jx_0 = f_0^*$; i.e., $x_0 = (J + A)^{-1}(f_0^*)$. Since this convergence holds for each subsequence of $\{x_n\}$, we conclude that $(J + A)^{-1}$ is continuous. Next we assume $\{f_n^*\}$ is bounded. Then the compactness of $(J + T)^{-1}$ implies the existence of a subsequence, denoted by $\{f_n^* - v_n^*\}$, such that

$$x_n = (J + T)^{-1}\left(f_n^* - v_n^*\right) = (J + A)^{-1}\left(f_n^*\right) \longrightarrow y_0 \tag{33}$$

as $n \longrightarrow \infty$. Thus the compactness of $(J + A)^{-1}$ follows; i.e., J_λ is compact for all $\lambda > 0$. The remaining proof follows as in the arguments of the proof of Theorem 3. The detail is omitted here. □

Next we give the following surjectivity result.

Corollary 5. *Let T and C be as given in Theorem 3. Assume that $T + C$ is coercive; i.e., there exists $v_0 \in X$ such that*

$$\inf_{x \in D(T), u^* \in Tx} \frac{\langle u^* + Cx, x - v_0 \rangle}{\|x\|} \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty. \tag{34}$$

Then the following conclusions hold.

(i) *$\overline{R(T + C)} = X^*$ if C is compact.*

(ii) *$R(T + C) = X^*$ if T is expansive and C is compact.*

(iii) *$R(T + C) = X^*$ if C is completely continuous.*

(iv) *$R(T + C) = X^*$ if T is of resolvent compact and C is bounded and continuous.*

Proof. Let $\widetilde{J}x = J(x - v_0)$ for $x \in X$ and some $u_0 \in D(T)$. It is known that $\widetilde{J}$ is bounded, monotone, continuous, and of type (S_+). Fix $f^* \in X^*$. It is enough to show that the boundary condition in Theorem 3 holds. The coercivity of $T + C$ implies

$$\begin{aligned} &\frac{\langle u^* + Cx - f^*, x - v_0 \rangle}{\|x\|} \\ &\quad \geq \frac{\langle u^* + Cx, x - v_0 \rangle}{\|x\|} - \frac{\|f^*\| \|x - v_0\|}{\|x\|} \\ &\quad \geq \frac{\langle u^* + Cx, x - v_0 \rangle}{\|x\|} - \|f^*\| - \frac{\|f^*\| \|v_0\|}{\|x\|} \end{aligned} \tag{35}$$

for all $x \in D(T) \setminus \{0\}$ and $u^* \in Tx$. The coercivity of $T + C$ implies that the right side in (35) approaches to ∞ as $\|x\| \longrightarrow \infty$; i.e., there exists $R = R(f^*) > 0$ such that

$$\langle u^* + Cx - f^*, x - v_0 \rangle > 0 \tag{36}$$

for all $x \in D(T) \cap \partial B_R(0)$ and $u^* \in Tx$. Thus we have

$$\begin{aligned} &\left\langle u^* + Cx + \varepsilon \widetilde{J}x - f^*, x - v_0 \right\rangle \\ &\quad = \langle u^* + Cx - f^*, x - v_0 \rangle + \varepsilon \langle J(x - v_0), x - v_0 \rangle \\ &\quad > \varepsilon \|x - v_0\|^2 \end{aligned} \tag{37}$$

for all $x \in D(T) \cap \partial B_R(0)$ and $u^* \in Tx$; i.e., we get

$$\left(T + C + \varepsilon \widetilde{J}\right)(D(T) \cap \partial B_R(0)) \not\ni f^* \tag{38}$$

for all $\varepsilon > 0$. Consequently, the conclusions (i) through (iv) follows based on the conclusion of Theorem 3. □

It is worth noticing that Theorem 3 is new in the sense that the conclusion required only the boundary condition $(T + C + \varepsilon J)(D(T) \cap \partial B_R(0)) \not\ni f^*$ for all $\varepsilon > 0$. For analogous results under such boundary condition, the reader is referred to Figueiredo [12] (for single multivalued pseudomonotone operator S) and Asfaw and Kartsatos [9, Theorem 11 and Theorem 13] (for pseudomonotone perturbations of maximal monotone operator).

Next we prove the following result.

Theorem 6. *Let $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ be maximal monotone and $C : D(C) \longrightarrow X^*$ be such that $D(T) \subseteq D(C)$. Assume that $d \geq 0$, $\mu \geq 0$, and $Q > 0$ such that*

$$\langle u^* + Cx, x \rangle \geq -d \|x\|^2 - \mu \tag{39}$$

for all $x \in D(T)$ with $\|x\| \geq Q$, and

$$\lim_{x \in D(T), \|x\| \longrightarrow \infty} \frac{|Tx + Cx|}{\|x\|^2} > 0. \tag{40}$$

Then the following hold.

(i) *$\overline{R(T + C)} = X^*$ if C is compact.*

(ii) *$T + C$ is surjective if C is completely continuous and $\overline{D(T)} \subseteq D(C)$*

(iii) *$T + C$ is surjective if T is expansive and C is compact;*

(iv) *$T + C$ is surjective if T is of compact resolvent and C is bounded and continuous.*

Proof. Fix $\varepsilon > 0$ and $f^* \in X*$. Let $\widetilde{J} : X \longrightarrow X^*$ be defined by $\widetilde{J}x = \|x\| Jx$, $x \in X$. It is well-known that $\widetilde{J}$ is bounded, continuous, maximal monotone, and of type (S_+). Then applying (39) gives

$$\left\langle u^* + Cx + \varepsilon \widetilde{J}x, x \right\rangle \geq \|x\|^3 \psi_\varepsilon(\|x\|) \tag{41}$$

for all $x \in D(T) \setminus \{0\}$ and $u^* \in Tx$, where

$$\psi_\varepsilon(t) = \varepsilon - dt^{-1} - \mu t^{-3} - \|f^*\| t^{-2}, \quad t > 0. \tag{42}$$

As a result, we get

$$\frac{\left\langle u^* + Cx + \varepsilon \widetilde{J}x, x \right\rangle}{\|x\|} \longrightarrow \infty \tag{43}$$

as $\|x\| \longrightarrow \infty$. The operator $A_\varepsilon = T + \varepsilon \widetilde{J}$ is maximal monotone. By using the operators A_ε and C in Corollary 5, we conclude that $\overline{(T + C + \varepsilon \widetilde{J})(D(T) \cap \overline{B}_{r_\varepsilon}(0))} \ni f^*$. Then there exist $x_n \in D(T) \cap \overline{B}_{r_\varepsilon}(0)$ and $u_n^* \in Tx_n$ such that

$$u_n^* + Cx_n + \varepsilon \widetilde{J}x_n \longrightarrow f^* \tag{44}$$

as $n \longrightarrow \infty$. The compactness of C implies that $Cx_n \longrightarrow a_0^*$ (for some subsequence $\{x_n\}$). By applying the maximality of T together with Lemma 2 and following the arguments used in the proof of Theorem 3 we arrive at $f^* \in (T + C + \varepsilon \widetilde{J})(D(T) \cap \overline{B}_{r_\varepsilon}(0))$. The surjectivity of $T + C + \varepsilon \widetilde{J}$ follows because $f^* \in X^*$ is arbitrary. Next we give the proof of (i). Since $R(T + C + \varepsilon \widetilde{J}) = X^*$ for all $\varepsilon > 0$, for each $\varepsilon_n \downarrow 0^+$, there exist $x_n \in D(T)$ and $u_n^* \in Tx_n$ such that

$$u_n^* + Cx_n + \varepsilon_n \|x_n\| Jx_n = f^* \tag{45}$$

for all n; i.e., (39) gives

$$\begin{aligned} \varepsilon_n \|x_n\|^3 &= \varepsilon_n \langle \|x_n\| Jx_n, x_n \rangle = -\langle u_n^* + Cx_n - f^*, x_n \rangle \\ &\leq d \|x_n\|^2 + \mu + \|f^*\| \|x_n\| \end{aligned} \tag{46}$$

for all n. We shall show that $\{x_n\}$ is bounded. Suppose not, i.e., there exists a subsequence, denoted again by $\{x_n\}$, such that $\|x_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Dividing (46) by $\|x_n\|$ for all large n implies

$$\varepsilon_n \|x_n\|^2 \leq d \|x_n\| + \frac{\mu}{\|x_n\|} + \|f^*\| \tag{47}$$

for all large n; i.e., (45) implies

$$\begin{aligned} \|u_n^* + Cx_n\| &\leq \varepsilon_n \|x_n\|^2 + \|f^*\| \\ &\leq d \|x_n\| + \frac{\mu}{\|x_n\|} + 2 \|f^*\| \end{aligned} \tag{48}$$

for all large n; i.e., we get

$$\frac{\|u_n^* + Cx_n\|}{\|x_n\|^2} \longrightarrow 0 \quad as\ n \longrightarrow \infty. \tag{49}$$

However, this is a contradiction to (40). As a result the sequence $\{x_n\}$ is bounded. Letting $n \longrightarrow \infty$ in (45) gives $f^* \in \overline{R(T+C)}$. The proofs of (ii), (iii), and (iv) follow based on the arguments used in the proof of (i) and Theorem 3. The details are omitted here. □

In [13], Kartsatos proved that $\overline{R(T+C)} = X^*$ if C is compact and T is maximal monotone with $D(C) = D(T)$ provided that there exists $\beta : [0,\infty) \longrightarrow [0,\infty)$ such that $\beta(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $Q > 0$ such that

$$\langle u^* + Cx, x\rangle \geq -\beta(\|x\|)\|x\|^2 \tag{50}$$

for all $x \in D(T)$, $u^* \in Tx$, and $\|x\| \geq Q$ and

$$\liminf_{x\in D(T),\|x\|\longrightarrow\infty} \frac{|Tx + Cx|}{\|x\|} > 0. \tag{51}$$

It is worth mentioning here that (50) implies that for each $d > 0$ there exists $Q > 0$ such that $\beta(\|x\|) < d$ for all $\|x\| \geq Q$; i.e., (50) implies (39) and (40) implies (51). On the other hand, Theorem 6 due to Kartsatos [13] holds if $D(T) \subseteq D(C)$ is used instead of $D(C) = D(T)$. The proof follows based on the proof of (i) of Theorem 6 by using J instead of $\widetilde{J}$. However, it is worth noticing that condition (39) is natural in applications than that of (50).

3. Applications

In this section, we demonstrate the applicability of the result to prove existence of solution for variational inequality problems.

Example 7. Let Ω be a nonempty, bounded, and open subset of $\mathbb{R}^n$ with $n \geq 1$ and $\Gamma = \partial\Omega$ is of smooth boundary. Let $X = L^2(\Omega)$ and K be a nonempty, closed, and convex subset of X with nonempty interior. Let $j : \mathbb{R} \longrightarrow (-\infty,\infty]$ be proper, convex, and lower semicontinuous function, $p \in [2,\infty)$ and $\lambda > 0$. Let $\Phi_p^\lambda : X \longrightarrow (-\infty,\infty]$ be defined by

$$\Phi_p^\lambda(u) = \frac{1}{p}\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx + \frac{\lambda}{p}\int_\Omega |u|^p\, dx + \int_\Gamma j(u)\, d\sigma \tag{52}$$

if $u \in W^{1,p}(\Omega)$ and $j(u) \in L^1(\Gamma)$, and $\Phi_p^\lambda(u) = \infty$ otherwise. It is known that $\partial\Phi_p^\lambda : X \supseteq D(\partial\Phi_p^\lambda) \longrightarrow 2^X$ is given by

$$\partial\Phi_p^\lambda(u) = -\Delta_p^\lambda u \tag{53}$$

for each $u \in D(\partial\Phi_p^\lambda)$, where

$$\Delta_p^\lambda u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) - \lambda u|u|^{p-2}, \tag{54}$$

domain of $\partial\Phi_p^\lambda$ given by

$$D\left(\partial\Phi_p^\lambda\right) = \left\{u \in W^{1,p}(\Omega) : \Delta_\lambda^p u \in L^2(\Omega), -\frac{\partial u}{\partial \nu_p}(x) \in \partial j(u(x))\ \ a.e.\ x \in \Gamma\right\} \tag{55}$$

and

$$\frac{\partial u}{\partial \nu_p} = \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\cos(n, e_i), \tag{56}$$

where n is the outward normal on Γ and $\{e_1, e_2, \ldots, e_n\}$ is the canonical base in $\mathbb{R}^n$. Assume, further, that

(C_1) $g : \Omega\times\mathbb{R} \longmapsto \mathbb{R}$ be Carathéodory function; i.e., $x \longmapsto g(x,s)$ is measurable for almost all $s \in \mathbb{R}$ and $s \longmapsto g(x,s)$ is continuous for almost all $x \in \Omega$;

(C_2) there exist $d \geq 0$ and $h \in L^2(Q)$ such that

$$|g(x,s)| \leq h(x) + d|s| \tag{57}$$

for all $(x,s) \in \Omega\times\mathbb{R}$.

Let $C : X \longrightarrow X$ be defined by $Cu(.) = f(., u(.))$ for $u \in X$. In the following theorem, we prove the solvability of variational inequality problem VIP$(C, \Phi_p^\lambda, K, f^*)$; i.e., we find $u \in K \cap D(\Phi_p^\lambda)$ such that

$$\langle Cu - f^*, v - u\rangle \geq \Phi_p^\lambda(u) - \Phi_p^\lambda(v) \tag{58}$$

for all $x \in$ K.

Theorem 8. *Let $f \in X$ and Φ_p^λ be as in (52). Suppose (C_1) and (C_2) are satisfied. If $d < 2^{-1}\lambda$ for $p = 2$ or $\lambda > 0$ for $p \in (2,\infty)$, then the problem* VIP$(C, \Phi_p^\lambda, K, f^*)$ *is solvable.*

Proof. Let $A = \partial\Phi_p^\lambda$ be the subdifferential of Φ_p^λ in the sense of convex analysis. It is well-known that A is an m-accretive operator (equivalently, A is maximal monotone) with compact resolvents (cf. Proposition 2.2.2 and Proposition 2.2.3 [14]). By applying conditions (C_1) and (C_2), it follows that C is bounded and continuous. Let $\partial I_K : K \longrightarrow 2^X$ be the subdifferential of the indicator function on K. The m-accretivity (maximal monotonicity) of $T = A + \partial I_K$ follows because of the condition $D(A) \cap \mathring{K} \neq \emptyset$. In addition, it is not difficult to see that T is of resolvent compact m-accretive operator. Next we show that $T + C$ satisfies the boundary condition in Theorem 3. Let $f^* \in X$ be the functional generated by f. Choose $\mu_0 \in D(\Phi_p^\lambda)$. By applying the definition of A and conditions (C_1) and (C_2), we arrive at

$$\begin{aligned}\langle g^* + Cu, u - \mu_0\rangle &\geq \Phi_p^\lambda(u) - \Phi_p^\lambda(\mu_0) + \langle Cu, u - \mu_0\rangle \\ &\geq \frac{\lambda}{p}\|u\|^p - \Phi_p^\lambda(\mu_0) - \|Cu\|\|u\| - \|Cu\|\|\mu_0\| \\ &\geq \frac{\lambda}{p}\|u\|^p - \Phi_p^\lambda(\mu_0) - (\|h\| + d\|u\|)\|u\| - (\|h\| + d\|u\|)\|\mu_0\|\end{aligned} \tag{59}$$

for all $u \in D(A)$ and $g^* \in Au$. As a result, we obtain the estimate

$$\frac{\langle g^* + Cu, u - \mu_0\rangle}{\|u\|} \geq \frac{\lambda}{p}\|u\|^{p-1} - \|u\|^{-1}\Phi_p^\lambda(\mu_0) - (\|h\| + d\|u\|) - \frac{(\|h\| + d\|u\|)\|\mu_0\|}{\|u\|} \tag{60}$$

for $u \in D(A) \setminus \{0\}$ and $g^* \in Au$. Since $\lambda < 2^{-1}d$ (if $p = 2$) or $\lambda > 0$ (if $p \in (2, \infty)$), the right side of (60) approaches to ∞ as $\|u\| \longrightarrow \infty$ and

$$\inf_{u\in D(A),\, g^*\in Au} \frac{\langle g^* + Cu, u - \mu_0\rangle}{\|u\|} \longrightarrow \infty \tag{61}$$

as $\|u\| \longrightarrow \infty$; i.e., the coercivity condition on $A + C$ is satisfied. Thus for each $f^* \in X$ we conclude that the inclusion problem $Au + Cu \ni f^*$ is solvable; i.e., the problem $\mathrm{VIP}(C, \Phi_p^\lambda, K, f^*)$ is solvable. This completes the proof. □

Next we present the following example.

Example 9. Let Ω be a nonempty, bounded, and open subset of $\mathbb{R}^N$ with smooth boundary and $N \geq 1$, $T > 0$, $Q = (0, T) \times \Omega$, $H = L^2(0, T; H_0^1(\Omega))$, $X = L^2(Q)$, and $V = H_0^1(\Omega)$. Let $\lambda > 0$ and $A : X \supseteq D(A) \longrightarrow X$ be defined by

$$Au = \frac{\partial u}{\partial t} - \lambda\Delta u \tag{62}$$

for $u \in D(A)$, where

$$D(A) = \left\{u \in L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) : \frac{\partial u}{\partial t} \in L^2(Q), u(x, 0) = u(x, T)\right\}. \tag{63}$$

Let $\langle , \rangle$ denote the inner product in X; i.e., for $g^* \in X$ and $u \in H$; we have $\langle g^*, u\rangle = \langle g^*, u\rangle_{(X,X)}$ with X^* identified with X. The norm of $u \in H$ is the normed induced from X. It is well-known that A is maximal monotone, surjective, $N(A) = \{0\}$, and

$$\langle Au, u\rangle \geq c_0\|u\|_H^2 \tag{64}$$

for all $u \in D(A)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(Q)$, it follows that $A^{-1} : H \longrightarrow H$ is a compact operator; i.e., for each $\varepsilon > 0$ the resolvent $(A + \varepsilon I)^{-1} : H \longrightarrow H$ is compact. Assume, further, that

(C_3) $g : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is Carathéodory function; i.e., $(x, t) \longmapsto g(x, t, s)$ is measurable for almost all $s \in \mathbb{R}$, and $s \longmapsto g(x, t, s)$ is continuous for almost all $(x, t) \in Q$;

(C_4) there exist $c \geq 0$ and $k \in L^2(Q)$ such that

$$|g(x, t, s)| \leq c|s| + k(x, t) \tag{65}$$

for all $(x, t, s) \in Q \times \mathbb{R}$;

(C_5) $\Phi : X \longrightarrow (-\infty, \infty]$ is a proper, convex, and lower semicontinuous function;

(C_6) there exists a nonnegative constant α such that $D(\Phi) \subseteq M$, where

$$M = \left\{u \in X : \int_Q g(x, t, u)\, u\, d\mu \geq -\alpha\|u\|\right\}. \tag{66}$$

We notice here that the set M defined in (C_6) is nonempty if $s \longrightarrow g(x, t, s)$ is nondecreasing for almost all $(x, t) \in Q$ such that $g(x, t, 0) = 0$ for all $(x, t) \in Q$. Actually, condition (C_4) implies that the unit ball in X is contained in M with $\alpha = c + \|k\|_{L^2(Q)}$. In Theorem 10 below, we prove existence of solution(s) for variational inequality problem denoted by $\mathrm{VIP}(A + C, \Phi, K, f^*)$, i.e., finding $u \in D(A) \cap D(\phi) \cap K$ such that

$$\langle Au - f^*, v - u\rangle + \int_Q g(x, t, u)(v - u)\, d\mu \geq \Phi(u) - \Phi(v) \tag{67}$$

for all $v \in K$, where K is a nonempty, closed, and convex subset of X with $\mathring{K} \neq \emptyset$.

Theorem 10. *Let $A : X \supseteq D(A) \longrightarrow X$ be as given in (62) and $f \in L^2(Q)$. Assume, further, that $\mathring{K} \cap D(\mathring{\Phi}) \neq \emptyset$ and conditions (C_3) through (C_6) are satisfied. Then the problem* $\mathrm{VIP}(A + C, \Phi, K, f^*)$ *is solvable.*

Proof. Let $f \in L^2(Q)$, A be as given in (62), and $C : X \longrightarrow X$ be defined by

$$\langle Cu, v\rangle = \int_Q g(x, t, u)\, u\, d\mu, \quad u \in X. \tag{68}$$

The operator A is linear maximal monotone, surjective, one to one, and densely defined. The density of $D(A)$ in X implies $D(A) \cap D(\mathring{\partial} I_K) \cap D(\mathring{\Phi}) \neq \emptyset$. As a result, we conclude that

$T = \partial I_K + \partial\phi + A$ is maximal monotone (cf. Rockafellar [15]). The arguments used in the proof of (c) of Corollary 4 imply that T is resolvent compact. In addition, the continuity of C follows based on the conditions (C_3) and (C_4). The proof follows if we prove the coercivity of $T+C$. Let us denote again the restriction of C on M by C. The definition on $M = D(C)$ as given in (C_6) implies that $\langle Cu, u\rangle \geq -\alpha\|u\|$ for all $u \in M$. The conditions on Φ imply that there exist $h^* \in X^*$ and $\beta \in \mathbb{R}$ such that $\Phi(u) \geq \langle h^*, u\rangle - \beta \geq -\|h^*\|\|u\| - \beta$ for all $u \in X$. Choose $w_0 \in D(A) \cap D(\Phi)$. The monotonicity of A and definition of $\partial\Phi(w_0)$ imply

$$\begin{aligned}\langle Tu + Cu, u - w_0\rangle &= \langle Au - Aw_0, u - w_0\rangle \\ &\quad + \langle Aw_0, u - w_0\rangle + \Phi(u) \\ &\quad - \Phi(w_0) + \langle Cu, u - w_0\rangle \\ &\geq c_0 \|u - w_0\|_H^2 - \|Aw_0\| \|u - w_0\| \\ &\quad - \|h^*\| \|u\| - \beta - \Phi(w_0) \\ &\quad + \langle Cu, u\rangle - \langle Cu, w_0\rangle \\ &\geq c_0 \|u - w_0\|_H^2 - \|Aw_0\| \|u - w_0\| \\ &\quad - \|h^*\| \|u\| - \beta - \Phi(w_0) \\ &\quad - \alpha\|u\| - \|Cu\| \|w_0\| \\ &\geq c_0 \|u - w_0\|^2 - \|Aw_0\| \|u - w_0\| \\ &\quad - \|h^*\| \|u\| - \beta - \Phi(w_0) \\ &\quad - \alpha\|u\| - (d\|u\| + \|k\|) \|w_0\|\end{aligned} \tag{69}$$

for all $u \in D(T) \cap D(\Phi)$. We notice that $D(T) \subseteq M$ because $D(\Phi) \subseteq M$. In addition, the right side of the above inequality approaches ∞ as $\|u\| \longrightarrow \infty$; i.e., we get

$$\frac{\langle Tu + Cu, u - w_0\rangle}{\|u\|} \longrightarrow \infty \quad as \ \|u\| \longrightarrow \infty; \tag{70}$$

that is, $T+C$ is coercive. Consequently, we conclude that $T+C$ is surjective. Thus for each $f \in L^2(Q)$, the problem in (67) admits at least one weak solution. The proof is completed. □

The reader can find plenty of resolvent compact maximal monotone operators in the paper due to Brèzis and Nirenberg [16] and in the books due to Vrabie [14], Barbu [1, 17], Showalter [18], and the references therein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of the paper.

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