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*Research Article*

# On the Convex and Convex-Concave Solutions of Opposing Mixed Convection Boundary Layer Flow in a Porous Medium

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In this paper, we are concerned with the solution of the third-order nonlinear differential equation $f''' + ff'' + \beta f'(f' - 1) = 0$, satisfying the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = b < 0$, and $f'(t) \longrightarrow \lambda$, as $t \longrightarrow +\infty$, where $\lambda \in \{0, 1\}$ and $0 < \beta < 1$. The problem arises in the study of the opposing mixed convection approximation in a porous medium. We prove the existence, nonexistence, and the sign of convex and convex-concave solutions of the problem above according to the mixed convection parameter $b < 0$ and the temperature parameter $0 < \beta < 1$.

## 1. Introduction

Owing to their numerous applications in industrial manufacturing processes, the convection phenomena about heated or cooled surfaces embedded in fluid-saturated porous media have attracted considerable attention during the last few decades. In this paper, our interest focuses on the analysis of the boundary value problems $\mathscr{P}_{\lambda(a,b)}$

$$\mathscr{P}_{\lambda(a,b)} \quad \begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0 \\ f(0) = a, \quad a \in \mathbb{R} \\ f'(0) = b < 0 \\ f'(t) \longrightarrow \lambda \quad \text{as } t \longrightarrow +\infty \end{cases}$$

where $\lambda \in \{0, 1\}$. This problem derives from the study of mixed convection boundary layer near a semi-infinite vertical plate embedded in a saturated porous medium, with a prescribed power law of the distance from the leading edge for the temperature. The parameter $\beta$ is a temperature power-law profile and $b$ is the mixed convection parameter, namely, $b = R_a/Pe - 1$, with $R_a$ the Rayleigh number and $P_e$ the Péclet number. The interested reader can consult references [1, 2] for more details on the physical derivation and the numerical treatments.

Mathematical results about the problem $\mathscr{P}_{\lambda(a,b)}$ with $\lambda = 1$ can be found in [3–7]. The case where $a \geq 0$, $b \geq 0$, $\beta > 0$ and $\lambda \in \{0, 1\}$ was treated by Aïboudi and al. in [3], and the results obtained generalize the ones of [6]. In [4], Brighi and Hoernel established some results about the existence and uniqueness of convex and concave solution of $\mathscr{P}_{1(a,b)}$ where $-2 < \beta < 0$ and $b > 0$. These results can be recovered from [8], where the general equation $f''' + ff'' + \mathbf{g}(f') = 0$ is studied.

In [5], some theoretical results can be found about the problem $\mathscr{P}_{1(0,b)}$ with $-2 < \beta < 0$, $b = 1 + \varepsilon$, and $\varepsilon < -1$. In particular, the authors prove that there exist $\varepsilon_* \in (-1.807, -1.806)$ and $\varepsilon^* \in (-1.193, -1.192)$, such that

(i) $\mathscr{P}_{1(0,b)}$ has no convex solution for any $\beta < 0$ and each $\varepsilon \leq \varepsilon_*$.

(ii) $\mathscr{P}_{1(0,b)}$ has a convex solution for each $\beta < 0$ and each $\varepsilon \in [\varepsilon^*, -1)$.

In [7] one can find an interesting new result about the existence of convex solutions of $\mathscr{P}_{1(0,b)}$ where $0 < \beta <$

1 under some conditions. In [5, 7], the method used by the authors allows them to prove the existence of a convex solution for the case $a = 0$ and seems difficult to generalize for $a \neq 0$.

The problem $\mathscr{P}_{\lambda(a,b)}$ with $\beta = 0$ is the well known Blasius problem. For a broad view, see [9]. See also [10].

Great interest is given to analytical studies of similarity solutions because of their applications in different fields, for example, in magnetohydrodynamic (see [11–13]) or in boundary layer flows (see [8, 14]).

The main goal of this paper is to study the question of existence and nonexistence of the solutions of $\mathscr{P}_{\lambda(a,b)}$ with $0 < \beta < 1$ and $\lambda \in \{0, 1\}$. We will focus our attention on convex and convex-concave solutions of the equation

$$f''' + ff'' + \beta f' \left(f' - 1\right) = 0. \tag{1}$$

As usual, to get a convex or convex-concave solution of $\mathscr{P}_{\lambda(a,b)}$, we use the shooting technique which consists of finding the values of a parameter $c \geq 0$ for which the solution of (1) satisfying the initial conditions $f(0) = a$, $f'(0) = b$, and $f''(0) = c$ exists on $[0, +\infty)$ and is such that $f'(t) \longrightarrow \lambda$ as $t \longrightarrow +\infty$. We denote by $f_c$ the solution of the following initial value problem and by $[0, T_c)$ the right maximal interval of existence:

$$\begin{aligned} &f''' + ff'' + \beta f' \left(f' - 1\right) = 0 \\ &f(0) = a \\ &f'(0) = b < 0 \\ &f''(0) = c \geq 0 \end{aligned} \qquad \mathscr{P}_{(a,b,c)}$$

## 2. On Blasius Equation

In this section, we recall some basic properties of the subsolutions and $\varepsilon$-subsolutions of the Blasius equation. Let $I \subset \mathbb{R}$ be an interval and $f : I \longrightarrow \mathbb{R}$ be a function.

*Definition 1.* We say that $f$ is a subsolution of the Blasius equation $f''' + ff'' = 0$ if $f$ is of class $C^3$ and if $f''' + ff'' \leq 0$ on $I$.

*Definition 2* (let $\varepsilon > 0$). We say that $f$ is an $\varepsilon$-subsolution of the Blasius equation $f''' + ff'' = 0$ if $f$ is of class $C^3$ and if $f''' + ff'' \leq -\varepsilon$ on $I$.

**Proposition 3** (let $t_0 \in \mathbb{R}$). *There does not exist nonpositive concave subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

*Proof.* See [8], Proposition 2.11. □

**Proposition 4** (let $\varepsilon > 0$ and $t_0 \in \mathbb{R}$). *There does not exist any $\varepsilon$-subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

*Proof.* See [8], Proposition 2.18. □

## 3. Preliminary Results

**Proposition 5.** *Let $f$ be a solution of (1) on some maximal interval $I = (T_-, T_+)$.*

(1) *If $F$ is any antiderivative of $f$ on $I$, then $(f''e^F)' = -\beta f'(f' - 1)e^F$.*

(2) *Assume that $T_+ = +\infty$ and that $f'(t) \longrightarrow \lambda \in \mathbb{R}$ as $t \longrightarrow +\infty$. If moreover $f$ is of constant sign at infinity, then $f''(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.*

(3) *If $T_+ = +\infty$ and if $f'(t) \longrightarrow \lambda \in \mathbb{R}$ as $t \longrightarrow +\infty$, then $\lambda = 0$ or $\lambda = 1$.*

(4) *If $T_+ < +\infty$, then $f''$ and $f'$ are unbounded near $T_+$.*

(5) *If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t_0) = \mu$, where $\mu = 0$ or 1, then, for all $t \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$.*

*Proof.* The first item follows immediately from (1). For the proof of items (2)-(5), see [8], Proposition 3.1 with $g(x) = \beta x(x - 1)$. □

**Lemma 6.** *Let $\beta \in (0, 1]$ and $f$ be a solution of (1) on some maximal interval $I = (T_-, T_+)$. If there exists $t_0 \in I$ such that*

$$\begin{aligned} &f'(t_0) > 1 \text{ and} \\ &f(t_0)\left(1 - f'(t_0)\right) \leq f''(t_0) \leq 0, \end{aligned} \tag{2}$$

*then $T_+ = +\infty$ and $f'(t) \longrightarrow 1$ as $t \longrightarrow +\infty$. Moreover, $f'' < 0$ on $[t_0, +\infty)$.*

*Proof.* See [3], Lemma 9. □

## 4. The Boundary Value Problem in the Convex and Convex-Concave Case with $0<\beta<1$

In the following, we take $a, b \in \mathbb{R}$ and $\lambda \in \{0, 1\}$ with $b < 0$ and $0 < \beta < 1$. We are interested here in convex and convex-concave solutions of the boundary value problem $\mathscr{P}_{\lambda(a,b)}$. As mentioned in the introduction, we will use the shooting method to find these solutions. Define the following sets:

$$\begin{aligned} C_1 &= \left\{c \geq 0 : f_c' \leq 0 \text{ and } f_c'' \geq 0 \text{ on } [0, T_c)\right\}, \\ C_2 &= \left\{c \geq 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c' \right. \\ &\quad < 0 \text{ on } (0, t_c), \; f_c' > 0 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f_c'' \\ &\quad \left. > 0 \text{ on } (0, t_c + \varepsilon_c)\right\}, \\ C_3 &= \left\{c \geq 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c'' \right. \\ &\quad > 0 \text{ on } (0, s_c), \; f_c'' < 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f_c' \\ &\quad \left. < 0 \text{ on } (0, s_c + \varepsilon_c)\right\}. \end{aligned} \tag{3}$$

*Remark 7.* It is easy to prove that $C_2$ and $C_3$ are disjoint nonempty open subsets of $[0, +\infty)$ and that there exist $c_0 >$

$c_* > 0$ such that $C_2 = (c_0, +\infty)$, $C_3 = [0, c_*)$, and $C_1 \cup C_2 \cup C_3 = [0, +\infty)$ (see Appendix A of [8] with $g(x) = \beta x(x-1)$ and $\beta > 0$).

**Lemma 8** (let $\beta > 0$). *Then, $f_c$ is a convex solution of the boundary value problem $\mathcal{P}_{0(a,b)}$ if and only if $c \in C_1$.*

*Proof.* See Appendix A of [8] with $g(x) = \beta x(x-1)$ and $\beta > 0$. □

**Lemma 9** (let $\beta > 0$). *If $c \in C_3$, then $T_c < +\infty$. Moreover, $f_c$ is convex-concave, decreasing and $f_c'(t) \longrightarrow -\infty$ as $t \longrightarrow T_c$.*

*Proof.* If $c \in C_3$ then there exists $s_c \in [0, T_c)$ such that $f_c'(s_c) < 0$ and $f_c''(s_c) = 0$. From Proposition 5, items (1) and (3), we have $f_c''(t) < 0$ and $f_c'(t) < 0$ for all $t \in (s_c, T_c)$, and $f_c'(T) \longrightarrow -\infty$ as $t \longrightarrow T_c$. Thus, $f_c$ is convex-concave solution on $[0, T_c)$ and $f_c'(t) \longrightarrow -\infty$ as $t \longrightarrow T_c$.

Let us assume that $T_c = +\infty$; then there exists $t_0 \in (s_c, +\infty)$ such that $f_c'$ and $f_c$ are negative on $t \in (t_0, +\infty)$ and we obtain $f_c''' + f_c f_c'' = -\beta f_c'(f_c' - 1) < 0$ on $(t_0, +\infty)$. Hence, $f_c$ is a nonpositive concave subsolution of the Blasius equation on $(t_0, +\infty)$. This contradicts the Proposition 3 and thus $T_c < +\infty$. □

*Remark 10.* From Proposition 5, items (1), (3), and (5), if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $\mathcal{P}_{(a,b,c)}$:

(1) $f_c$ is convex and $f_c'(t) \longrightarrow +\infty$ as $t \longrightarrow T_c$ (with $T_c \leq +\infty$).

(2) There exists a point $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$.

(3) $f_c$ is a convex solution of $\mathcal{P}_{1(a,b)}$.

The next proposition shows that case (1) cannot hold.

**Proposition 11** (let $\beta > 0$). *There does not exist $c \geq 0$, such that $f_c$ is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \longrightarrow +\infty$ as $t \longrightarrow T_c$.*

*Proof.* Assume that $f_c$ is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \longrightarrow +\infty$ as $t \longrightarrow T_c$. Then there exists $t_0 \in [0, T_c)$ such that, for all $t \in [t_0, T_c)$, $f_c'(t) > 1$ and

$$\begin{aligned} f_c'''(t) + f_c(t) f_c''(t) &= -\beta f_c'(t)\left(f_c'(t) - 1\right) \\ &< -\beta f_c'(t_0)\left(f_c'(t_0) - 1\right) = -\varepsilon. \end{aligned} \tag{4}$$

Consequently, $f_c$ is a $\varepsilon$-subsolution of the Blasius equation on $[t_0, T_c)$. Therefore from Proposition 4 we have $T_c < +\infty$.

Furthermore, there exists $t_1 \in [t_0, T_c)$ such that $f_c(t_1) = \alpha > 0$ and $f_c'(t_1) > 1$, and then $f_c'''(t) + f_c(t) f_c''(t) < 0$ and $f_c(t) > f_c(t_1) = \alpha$ for all $t \in [t_1, T_c)$. Thus,

$$f_c'''(t) < -\alpha f_c''(t) \tag{5}$$

for all $t \in [t_1, T_c)$. Next, integrating (5) on $[t_1, t]$ for $t_1 < t < T_c$, we obtain $f_c''(t) - f_c''(t_1) < -\alpha(f_c'(t) - f_c'(t_1))$ and using Proposition 5, item (4), yields a contradiction as $t \longrightarrow T_c$. □

## 5. The $a \leq 0$ Case

**Lemma 12** (let $0 < \beta < 1$ and $a \leq 0$). *If $c \geq 0$ and if there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$, then $f_c(t_0) > 0$.*

*Proof.* Let $c \geq 0$ and assume that there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$.

Let us consider the function $H_c = f_c'' + f_c(f_c' - \beta)$. Since $H_c' = (1-\beta) f_c'^2 \geq 0$ on $[0, T_c)$, then $H_c$ is nondecreasing on $[0, T_c)$ and hence

$$\begin{aligned} 0 \leq H_c(0) = c + a(b - \beta) &< H_c(t_0) \\ &= f_c(t_0)\left(f_c'(t_0) - \beta\right). \end{aligned} \tag{6}$$

Thus, $f_c(t_0) > 0$. □

For the rest of this section we will set $a^* = -\sqrt{(1-b^2)/(\beta - 2b)}$.

**Proposition 13** (let $0 < \beta < 1$). *If either $b \leq -1$ or $b \in (-1, 0]$ and $a \leq a^*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution.*

*Proof.* Suppose that $b \leq -1$ and that $f_c$ is a convex solution of the boundary value problem $\mathcal{P}_{1(a,b)}$. Then, there exists $t_* > 0$ such that $f_c(t_*) = 0$.

Let $K_c = 2 f_c f_c'' - f_c'^2 + f_c^2(2f_c' - \beta)$. From (1), we obtain $K_c' = 2(2-\beta) f_c f_c'^2 < 0$ on $(0, t_*)$. Therefore, $K_c$ is decreasing on $(0, t_*)$ and hence $K_c(0) > K_c(t_*)$. It follows that

$$f_c'^2(t_*) > -2ac + b^2 + a^2(\beta - 2b) \geq b^2, \tag{7}$$

which implies that $f_c'(t_*) > 1$. This is a contradiction. The same contradiction is obtained where $b \in (-1, 0]$ and $a \leq a^*$. □

**Theorem 14.** *Let $0 < \beta < 1$ and $a, b \in \mathbb{R}$ with $b < 0$ and $a \leq 0$ and $0 < \beta < 1$.*

(1) *The boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one convex solution.*

(2) *If either $b \leq -1$ or $b \in (-1, 0]$ and $a \leq a^*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution and has infinitely many convex-concave solutions.*

*Proof.* The first result follows from Remark 7 and Lemma 8. The second result follows from Remark 7, Remark 10, Proposition 11, Proposition 13, and Lemma 6. □

## 6. The $a>0$ Case

Let $a, b \in \mathbb{R}$ with $b < 0$ and $a > 0$. We assume $0 < \beta < 1$ and consider the solution $f_c$ of the initial value problem $P_{(a,b,c)}$ on the right maximal interval of existence $[0, T_c)$.

Let us set $b^* = \max\{-(1/2)a^2, -\beta/(1-\beta)\}$.

**Lemma 15** (let $0 < \beta < 1$. let $c \geq 0$). *If $b \in (b^*, 0)$ and if there exists $t_* \in (0, T_c)$ such that $t_*$ is the first point where $f_c(t_*) = 0$, then $f_c'(t_*) < 0$ and $f_c''(t_*) < 0$.*

*Proof.* Let $t_* \in (0, T_c)$ be such that $f_c > 0$ on $[0, t_*)$ and $f_c(t_*) = 0$. Suppose that $f_c'' > 0$ on $[0, t_*)$. Then, necessarily, we have $f_c' < 0$ on $[0, t_*)$. Moreover, since $f_c'$ is increasing and $b > b^*$, we also have $f_c' > -\beta/(1-\beta)$ on $[0, t_*)$.

Let $E_c = f_c'' + f_c f_c'$. From (1), we have $E_c' = (1-\beta) f_c'^2 + \beta f_c'$. Consequently, $E_c' < 0$ on $[0, t_*)$ and since $E_c(t_*) = f_c''(t_*) \geq 0$, it follows that $E_c > 0$ on $[0, t_*)$. Integrating from 0 to $t_*$ gives

$$0 < \int_0^{t_*} E_c(t)\,\mathrm{d}t = f_c'(t_*) - b - \frac{1}{2}a^2. \tag{8}$$

Thus $f_c'(t_*) > b + (1/2)a^2 \geq 0$ which is a contradiction.

Therefore, there exists $t_0 \in [0, t_*)$ such that $f_c'' > 0$ on $(0, t_0)$ and $f_c''(t_0) = 0$. From Proposition 5, items (1) and (5), we have either $f_c'(t_0) < 0$ or $f_c'(t_0) > 1$. The second case cannot happen. Assume, for the sake of contradiction, that $f_c'(t_0) > 1$. Then $f_c'' \geq 0$ and $f_c > 0$ on $[0, t_0]$, so that we have $f_c(t_0)(1 - f_c'(t_0)) \leq f_c''(t_0) \leq 0$. From Lemma 6, we obtain that $T_c = +\infty$, $f_c'(t) \longrightarrow 1$ as $t \longrightarrow +\infty$, and $f_c'' < 0$ on $[t_0, +\infty)$. It follows that $f_c$ is a positive convex-concave solution of the boundary value problem $\mathcal{P}_{1(a,b)}$ on $[0, +\infty)$, which contradicts the existence of $t_*$. Consequently, we have $f_c'(t_0) < 0$. This implies that $f_c' < 0$ on $[0, t_0]$ and that $c \in C_3$. By virtue of Lemma 9, we see that $f_c''$ remains negative after $t_0$. The proof is complete. □

**Lemma 16** (let $0 < \beta < 1$). *If there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$, then $f_c(t_0) > 0$.*

*Proof.* Assume that there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$, $f_c'(t_0) > 1$, and $f_c(t_0) < 0$. Then, there would exist $t_1 < t_0$ such that $f_c(t_1) = 0$.

Let $H_c = f_c'' + f_c(f_c' - \beta)$. From (1), we have $H_c' = (1-\beta) f_c'^2 \geq 0$ on $[0, T_c)$. Therefore, $H_c$ is nondecreasing on $[0, T_c)$. Since $H_c(t_0) = f_c(t_0)(f_c'(t_0) - \beta) < 0$, we get $f_c''(t_1) = H_c(t_1) < 0$. But, this and Proposition 5, item (1), imply that $f_c''$ remains negative on $(t_1, T_c)$, a contradiction. Hence $f_c(t_0) > 0$. □

**Lemma 17.** *If $0 < \beta < 1$ and $b \in (b^*, 0)$, then there exists $c_0 \in C_2$ such that if $c \geq c_0$ then $f_c$ is a convex-concave solution of $\mathcal{P}_{1(a,b)}$.*

*Proof [let $c \in C_2$].* From Remark 10 and Proposition 11, we see that either $f_c$ is a convex solution of $\mathcal{P}_{1(a,b)}$ or there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$. Now, as we have seen in the proof of Lemma 15, in the second case, $f_c$ is a convex-concave solution of $\mathcal{P}_{1(a,b)}$.

Let $c \in C_2$ be such that $f_c$ is a convex solution of $\mathcal{P}_{1(a,b)}$. Therefore, we have $b < f_c' < 1$ on $[0, +\infty)$ and, from Lemma 15, we have $f_c > 0$. It follows that

$$\begin{aligned}\left(f_c'' + f_c\left(f_c' - 1\right)\right)' &= (1-\beta) f_c'\left(f_c' - 1\right)\\ &\geq -\frac{1}{4}(1-\beta)\end{aligned} \tag{9}$$

on $[0, +\infty)$. Integrating between 0 and $t \geq 0$, and using the fact that $f_c > 0$, we obtain

$$\begin{aligned} f_c''(t) &\geq -\frac{1}{4}(1-\beta)t + a(b-1) + c\\ &\quad - f_c(t)\left(f_c'(t) - 1\right)\\ &\geq -\frac{1}{4}(1-\beta)t + a(b-1) + c.\end{aligned} \tag{10}$$

Integrating once again we get

$$\forall t \geq 0, \tag{11}$$

$$1 > f_c'(t) \geq -\frac{1}{8}(1-\beta)t^2 + (a(b-1) + c)t + b. \tag{12}$$

Let us set $P_c(t) = -(1/8)(1-\beta)t^2 + (a(b-1)+c)t + b - 1$. We have $P_c(t) < 0$ for all $t \geq 0$. It means that $P_c$ has no positive roots. Thus $c$ cannot be too large, because, on the contrary, its discriminant $\Delta = (a(b-1)+c)^2 + (1/2)(1-\beta)(b-1)$ and $a(b-1)+c$ would be positive, and hence the polynomial $P_c$ would have two positive roots, a contradiction.

Therefore, there exists $c_0 > 0$ such that $f_c$ is convex-concave solution of the problem $\mathcal{P}_{1(a,b)}$ for $c \geq c_0$. This completes the proof. □

**Theorem 18.** *Let $a, b \in \mathbb{R}$, with $b < 0$, $a > 0$, and $0 < \beta < 1$.*

(1) *The boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one convex solution. If in addition $b \in (b^*, 0)$, then any convex solution of $\mathcal{P}_{0(a,b)}$ is positive.*

(2) *If $b \in (b^*, 0)$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has infinitely many positive convex-concave solutions.*

*Proof.* The first part of (1) follows from Remark 7 and Lemma 8. The second part follows from Lemma 15, because if there was a point $t_*$ such that $f_c > 0$ on $[0, t_*)$ and $f_c(t_*) = 0$ then $f_c''(t_*) < 0$, a contradiction. The second result follows from Remark 7, Remark 10, Proposition 11, Lemma 16, and Lemma 6. □

## 7. Conclusion

In this work, in particular in Theorems 14 and 18, we have presented some new and important results about the boundary value problems $\mathcal{P}_{0(a,b)}$ and $\mathcal{P}_{1(a,b)}$, which we summarize below. The parameters $\beta$ and $b$ satisfy $0 < \beta < 1$ and $b < 0$. The constants $a_*$ and $b_*$ are defined in Sections 5 and 6.

(1) For $a \leq 0$ :

  (a) The boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one convex solution.

  (b) If either $b \leq -1$ or $b \in (-1, 0]$ and $a \leq a^*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution and has infinitely many convex-concave solutions.

(2) For $a > 0$ :

(a) If $b \in (b^*, 0)$, then the boundary value problem $\mathscr{P}_{0(a,b)}$ has at least one positive convex solution.
(b) If $b \in (b^*, 0)$, then the boundary value problem $\mathscr{P}_{1(a,b)}$ has infinitely many positive convex-concave solutions.

Numerical simulations prompt us to formulate the following conjecture.

**Conjecture 19.** *Let $a, b \in \mathbb{R}$, with $b \leq -1$, $a > 0$, and $0 < \beta < 1$. The boundary value problem $\mathscr{P}_{1(a,b)}$ has no convex solution.*

To finish, we give the following proposition concerning the case $a = 0$.

**Proposition 20** (let $\beta < 2$). *If $b \leq -1$, then the boundary value problem $\mathscr{P}_{1(0,b)}$ has no convex solution.*

*Proof.* Assume that $f_c$ is a convex solution of the boundary value problem $\mathscr{P}_{1(0,b)}$. Then, there exists $t_* \geq 0$, such that $f_c < 0$ on $(0, t_*)$, $f_c(t_*) = 0$, and $f_c'(t_*) > 0$. Consider again the function

$$K_c = 2 f_c f_c'' - f_c'^2 + f_c^2 \left(2 f_c' - \beta\right). \quad (13)$$

We have $K_c' = 2(2-\beta) f_c f_c'^2 < 0$ on $(0, t_*)$. Thus, $K_c$ is a decreasing function and hence $K_c(0) > K_c(t_*)$. It follows that $f_c'^2(t_*) > b^2$ which implies that $f_c'(t_*) > 1$, which is a contradiction. □

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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