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Research Article

Time Scale Inequalities of the Ostrowski Type for Functions Differentiable on the Coordinates

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In 2016, some inequalities of the Ostrowski type for functions (of two variables) differentiable on the coordinates were established. In this paper, we extend these results to an arbitrary time scale by means of a parameter $\lambda \in [0,1]$. The aforementioned results are regained for the case when the time scale $\mathbb{T} = \mathbb{R}$. Besides extension, our results are employed to the continuous and discrete calculus to get some new inequalities in this direction.

1. Introduction

To find a bound for the difference of a function and its integral mean, the Ukraine-born mathematician Ostrowski [1], in 1938, established the subsequent result which is nowadays celebrated as the Ostrowski inequality.

Theorem 1. *Let $G : [\alpha, \beta] \to \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable in (α, β) and its derivative $G' : (\alpha, \beta) \to \mathbb{R}$ is bounded in (α, β). If $|G'(s)| \le \mathscr{M}$ for all $s \in [\alpha, \beta]$, then we have*

$$\left|G(x) - \frac{1}{\beta-\alpha}\int_\alpha^\beta G(s)\,ds\right| \le \left(\frac{1}{4} + \frac{(x-(\alpha+\beta)/2)^2}{(\beta-\alpha)^2}\right)(\beta-\alpha)\mathscr{M}, \tag{1}$$

for all $x \in [\alpha, \beta]$. The inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller one.

In 2001, Cheng [2] gave the following improvement of the above inequality.

Theorem 2. *Let $G : [\alpha, \beta] \to \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable in (α, β) such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \le G'(s) \le \Gamma$ for all $s \in [\alpha, \beta]$. Then for all $x \in [\alpha, \beta]$, one gets*

$$\left|\frac{1}{2}G(x) - \frac{(x-\beta)G(\beta) - (x-\alpha)G(\alpha)}{2(\beta-\alpha)} - \frac{1}{\beta-\alpha}\int_\alpha^\beta G(s)\,ds\right| \le \frac{(x-\alpha)^2 + (\beta-x)^2}{8(\beta-\alpha)}(\Gamma-\gamma). \tag{2}$$

Recently, Farid [3] extended Theorems 1 and 2 to functions of two variables that are differentiable on their coordinates. Specifically, he proved the following two theorems.

Theorem 3. *Let $G : \mathscr{I} \times \mathscr{J} \to \mathbb{R}$, where $\mathscr{I}, \mathscr{J}$ are open intervals in $\mathbb{R}$, be a mapping such that for $\alpha_1, \beta_1 \in \mathscr{I}$, $\alpha_2, \beta_2 \in \mathscr{J}$, $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, the partial mappings*

$$\begin{aligned} G_y &: [\alpha_1, \beta_1] \longrightarrow \mathbb{R}, \\ &G_y(\xi) := G(\xi, y), \\ G_x &: [\alpha_2, \beta_2] \longrightarrow \mathbb{R}, \\ &G_x(\zeta) := G(x, \zeta), \end{aligned} \tag{3}$$

defined for all $y \in [\alpha_2, \beta_2]$ *and* $x \in [\alpha_1, \beta_1]$, *are differentiable, and* $|G_y'(s)| \le \mathscr{M}$, $s \in [\alpha_1, \beta_1]$, $|G_x'(s)| \le \mathscr{N}$, $s \in [\alpha_2, \beta_2]$. *Then we have*

$$\begin{aligned}
&\left|\int_{\alpha_1}^{\beta_1} \frac{G(x,\alpha_2)+G(x,\beta_2)}{2} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1,y)+G(\beta_1,y)}{2} dy - \left(\frac{1}{\beta_1-\alpha_1}+\frac{1}{\beta_2-\alpha_2}\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\right| \\
&\quad \le \frac{\mathscr{M}+\mathscr{N}}{2}(\beta_1-\alpha_1)(\beta_2-\alpha_2),
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\left|\int_{\alpha_1}^{\beta_1} \frac{G(x,\alpha_2)+G(x,\beta_2)}{4(\beta_1-\alpha_1)} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1,y)+G(\beta_1,y)}{4(\beta_2-\alpha_2)} dy - \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\right| \\
&\quad \le \frac{\mathscr{M}(\beta_1-\alpha_1)+\mathscr{N}(\beta_2-\alpha_2)}{4},
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\left|\int_{\alpha_1}^{\beta_1} G\left(x,\frac{\alpha_2+\beta_2}{2}\right) dx + \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1+\beta_1}{2},y\right) dy - \left(\frac{1}{\beta_1-\alpha_1}+\frac{1}{\beta_2-\alpha_2}\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\right| \\
&\quad \le \frac{\mathscr{M}+\mathscr{N}}{4}(\beta_1-\alpha_1)(\beta_2-\alpha_2),
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\left|\frac{1}{2(\beta_1-\alpha_1)}\int_{\alpha_1}^{\beta_1} G\left(x,\frac{\alpha_2+\beta_2}{2}\right) dx + \frac{1}{2(\beta_2-\alpha_2)}\int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1+\beta_1}{2},y\right) dy - \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\right| \\
&\quad \le \frac{\mathscr{M}(\beta_1-\alpha_1)+\mathscr{N}(\beta_2-\alpha_2)}{8}.
\end{aligned} \tag{7}$$

Remark 4. Inequality (7) is the correct version of inequality (2.18) as presented in [[3], Theorem 2.6].

Theorem 5. *Let* $G : \mathscr{I} \times \mathscr{J} \to \mathbb{R}$, *where* $\mathscr{I}, \mathscr{J}$ *are open intervals in* $\mathbb{R}$, *be a mapping such that for* $\alpha_1, \beta_1 \in \mathscr{I}$, $\alpha_2, \beta_2 \in \mathscr{J}$, $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, *the partial mappings*

$$\begin{aligned}
&G_y : [\alpha_1, \beta_1] \longrightarrow \mathbb{R}, \\
&G_y(\xi) := G(\xi, y), \\
&G_x : [\alpha_2, \beta_2] \longrightarrow \mathbb{R}, \\
&G_x(\zeta) := G(x, \zeta),
\end{aligned} \tag{8}$$

defined for all $y \in [\alpha_2, \beta_2]$ *and* $x \in [\alpha_1, \beta_1]$, *are differentiable with* $\gamma_y \le G_y'(s) \le \Gamma_y$, $s \in [\alpha_1, \beta_1]$, $\gamma_x \le G_x'(s) \le \Gamma_x$, $s \in [\alpha_2, \beta_2]$. *Then we have*

$$\begin{aligned}
&\left|\int_{\alpha_1}^{\beta_1} \frac{G(x,\alpha_2)+G(x,\beta_2)}{2} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1,y)+G(\beta_1,y)}{2} dy - \left(\frac{1}{\beta_1-\alpha_1}+\frac{1}{\beta_2-\alpha_2}\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\right| \\
&\quad \le \frac{\Gamma_x+\Gamma_y-(\gamma_x+\gamma_y)}{8}(\beta_1-\alpha_1)(\beta_2-\alpha_2), \\
&\left|\int_{\alpha_1}^{\beta_1} \frac{G(x,\alpha_2)+G(x,\beta_2)}{4(\beta_1-\alpha_1)} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1,y)+G(\beta_1,y)}{4(\beta_2-\alpha_2)} dy - \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\right| \\
&\quad \le \frac{(\beta_1-\alpha_1)(\Gamma_y-\gamma_y)+(\beta_2-\alpha_2)(\Gamma_x-\gamma_x)}{16}.
\end{aligned} \tag{9}$$

In 1988, the idea of time scales [4] was initiated so as to bring together the continuous and discrete analysis into a unified fold. Since the introduction of this subject, many classical integral results have been extended to time scales. "A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$." We shall presume, all over this work, that the reader is familiar with the theory of time scale (see [5, 6] for more on this subject). We present here a result of Bohner and Matthews [7] which is embedded in Theorem 6 below. This result extends Theorem 1 to time scales. For more improvements and generalizations around this result, we refer the interested reader to see the papers [8–12] and the references therein.

Theorem 6. *Let* $\alpha, \beta, s, t \in \mathbb{T}$, $\alpha < \beta$ *and* $G : [\alpha, \beta] \to \mathbb{R}$ *be differentiable. Then for all* $t \in [\alpha, \beta]$, *we have*

$$\left|G(t) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} G(\sigma(s))\,\Delta s\right| \le \frac{\mathscr{M}}{\beta-\alpha}\left(h_2(t,\alpha)+h_2(t,\beta)\right), \tag{10}$$

where $h_2(t,s) = \int_s^t (\tau - s)\Delta\tau$ *for all* $s, t \in \mathbb{T}$ *and* $\mathscr{M} = \sup_{\alpha<t<\beta} |G^{\Delta}(t)| < \infty$. *This inequality is sharp in the sense that the right-hand side of (10) cannot be replaced by a smaller one.*

It is our purpose in this paper to extend inequalities (4), (5), (6), (7), and (9) to time scales by means of a parameter $\lambda \in [0,1]$. In Section 2, we frame and prove the main results followed by applications to the continuous and discrete calculus.

2. Main Results

In this section, we will present our results involving double integrals. For some recent results in this regard, see [13–18]. The proofs of our findings shall be anchored on the subsequent lemmas.

Lemma 7 (see [9]). *Let $\alpha, \beta, t, x \in \mathbb{T}, \alpha < \beta$ and $G : [\alpha, \beta] \to \mathbb{R}$ be differentiable. Then*

$$\begin{aligned} &\left|(1-\lambda) G(x) + \lambda \frac{G(\alpha) + G(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta G(\sigma(t))\Delta t\right| \\ &\le \frac{\mathcal{M}}{\beta-\alpha}\left[h_2\left(\alpha, \alpha+\lambda\frac{\beta-\alpha}{2}\right) + h_2\left(x, \alpha+\lambda\frac{\beta-\alpha}{2}\right) + h_2\left(x, \beta-\lambda\frac{\beta-\alpha}{2}\right) + h_2\left(\beta, \beta-\lambda\frac{\beta-\alpha}{2}\right)\right], \end{aligned} \tag{11}$$

for all $\lambda \in [0,1]$ such that $\alpha + \lambda((\beta-\alpha)/2)$ and $\beta - \lambda((\beta-\alpha)/2)$ are in $\mathbb{T}$ and $x \in [\alpha+\lambda((\beta-\alpha)/2), \beta-\lambda((\beta-\alpha)/2)] \cap \mathbb{T}$, where $\mathcal{M} := \sup_{\alpha<x<\beta} |G^\Delta(x)| < \infty$. This is sharp provided that

$$\frac{\lambda}{2}\alpha(\beta-\alpha) + \frac{\lambda^2}{4}(\beta-\alpha)^2 \le \int_\alpha^{\alpha+\lambda((\beta-\alpha)/2)} t\Delta t. \tag{12}$$

Lemma 8 (see [8]). *Let $\alpha, \beta, t, x \in \mathbb{T}, \alpha < \beta$ and $G : [\alpha, \beta] \to \mathbb{R}$ be differentiable. If $G^\Delta \in C_{rd}(\mathbb{T}, \mathbb{R})$ and there exist $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le G^\Delta(t) \le \Gamma$ for all $t \in [\alpha, \beta]$, then for all $x \in [\alpha, \beta]$, we have*

$$\begin{aligned} &\left|\left(1-\frac{\lambda}{2}\right)G(x) + \lambda\frac{(x-\alpha)G(\alpha) + (\beta-x)G(\beta)}{2(\beta-\alpha)} - \frac{1}{\beta-\alpha}\int_\alpha^\beta G(\sigma(t))\Delta t - \frac{\Gamma+\gamma}{2}\frac{1}{\beta-\alpha}\left[h_2(x,\alpha) - h_2(x,\beta) - \lambda\left(\frac{(x-\alpha)^2-(\beta-x)^2}{2}\right)\right]\right| \\ &\le \frac{\Gamma-\gamma}{2(\beta-\alpha)}\left[h_2\left(\alpha, \alpha+\lambda\frac{x-\alpha}{2}\right) + h_2\left(x, \alpha + \lambda\frac{x-\alpha}{2}\right) + h_2\left(x, \beta-\lambda\frac{\beta-x}{2}\right) + h_2\left(\beta, \beta - \lambda\frac{\beta-x}{2}\right)\right], \end{aligned} \tag{13}$$

for all $\lambda \in [0,1]$ such that $\alpha + \lambda((x-\alpha)/2)$ and $\beta - \lambda((\beta-x)/2)$ are in $\mathbb{T}$.

We now formulate and prove our first result.

Theorem 9. *Let $\alpha_1, \beta_1, x \in \mathbb{T}_1$, $\alpha_2, \beta_2, y \in \mathbb{T}_2$, with $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ and $G : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \to \mathbb{R}$ be such that the partial mappings*

$$\begin{aligned} &G_y : [\alpha_1, \beta_1] \longrightarrow \mathbb{R}, \\ &\quad G_y(\xi) := G(\xi, y), \\ &G_x : [\alpha_2, \beta_2] \longrightarrow \mathbb{R}, \\ &\quad G_x(\zeta) := G(x, \zeta), \end{aligned} \tag{14}$$

defined for all $y \in [\alpha_2, \beta_2]$ and $x \in [\alpha_1, \beta_1]$, are differentiable. If $\mathcal{M} = \sup_{\alpha_1<t<\beta_1} |G_y^\Delta(t)|$ and $\mathcal{N} = \sup_{\alpha_2<t<\beta_2} |G_x^\Delta(t)|$, then the succeeding inequalities

$$\begin{aligned} &\left|\int_{\alpha_1}^{\beta_1} \frac{G(x,\alpha_2)+G(x,\beta_2)}{2}\Delta x + \int_{\alpha_2}^{\beta_2}\frac{G(\alpha_1,y)+G(\beta_1,y)}{2}\Delta y - \frac{1}{\beta_1-\alpha_1}\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x),y)\Delta y\,\Delta x - \frac{1}{\beta_2-\alpha_2}\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,\sigma(y))\Delta y\,\Delta x\right| \\ &\le \frac{\mathcal{M}(\beta_2-\alpha_2)}{2(\beta_1-\alpha_1)}\left[3h_2\left(\alpha_1, \alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) + h_2\left(\beta_1, \alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) + 3h_2\left(\beta_1, \beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right) + h_2\left(\alpha_1, \beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right] \\ &+ \frac{\mathcal{N}(\beta_1-\alpha_1)}{2(\beta_2-\alpha_2)}\left[3h_2\left(\alpha_2, \alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right) + h_2\left(\beta_2, \alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right) + 3h_2\left(\beta_2, \beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right) + h_2\left(\alpha_2, \beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right], \end{aligned} \tag{15}$$

$$\left|\int_{\alpha_1}^{\beta_1}\frac{G(x,\alpha_2)+G(x,\beta_2)}{4(\beta_1-\alpha_1)}\Delta x + \int_{\alpha_2}^{\beta_2}\frac{G(\alpha_1,y)+G(\beta_1,y)}{4(\beta_2-\alpha_2)}\Delta y\right.$$

$$
\begin{aligned}
&- \frac{1}{2(\beta_1-\alpha_1)(\beta_2-\alpha_2)} \\
&\cdot \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} [G(x,\sigma(y)) + G(\sigma(x),y)]\,\Delta y\,\Delta x \Bigg| \\
&\le \frac{\mathscr{M}}{4(\beta_1-\alpha_1)}\left[3h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\
&+ h_2\left(\beta_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&+ 3h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&\left.+ h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right] \\
&+ \frac{\mathscr{N}}{4(\beta_2-\alpha_2)}\left[3h_2\left(\alpha_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)\right. \\
&+ h_2\left(\beta_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right) \\
&+ 3h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right) \\
&\left.+ h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right],
\end{aligned}
\tag{16}
$$

hold for all $\lambda \in [0,1]$ *such that* $(\alpha_1+\lambda((\beta_1-\alpha_1)/2), \alpha_2+\lambda((\beta_2-\alpha_2)/2)) \in \mathbb{T}_1\times\mathbb{T}_2$ *and* $(\beta_1-\lambda((\beta_1-\alpha_1)/2), \beta_2-\lambda((\beta_2-\alpha_2)/2)) \in \mathbb{T}_1\times\mathbb{T}_2$.

Proof. Applying Lemma 7 to G_y at $x=\beta_1$, we get

$$
\begin{aligned}
&\left|\left(1-\frac{\lambda}{2}\right)G(\beta_1,y) + \frac{\lambda}{2}G(\alpha_1,y)\right. \\
&\left.- \frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1} G(\sigma(x),y)\,\Delta x\right| \\
&\le \frac{\mathscr{M}}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\
&+ h_2\left(\beta_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&\left.+ 2h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right].
\end{aligned}
\tag{17}
$$

Integrating (17) over $[\alpha_2,\beta_2]$ gives

$$
\begin{aligned}
&\left|\left(1-\frac{\lambda}{2}\right)\int_{\alpha_2}^{\beta_2} G(\beta_1,y)\,\Delta y + \frac{\lambda}{2}\int_{\alpha_2}^{\beta_2} G(\alpha_1,y)\,\Delta y\right. \\
&\left.- \frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x),y)\,\Delta y\,\Delta x\right| \\
&\le \frac{\mathscr{M}(\beta_2-\alpha_2)}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\
&+ h_2\left(\beta_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&\left.+ 2h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right].
\end{aligned}
\tag{18}
$$

Applying, again, Lemma 7 to G_y at $x=\alpha_1$, and integrating over $[\alpha_2,\beta_2]$ give

$$
\begin{aligned}
&\left|\left(1-\frac{\lambda}{2}\right)\int_{\alpha_2}^{\beta_2} G(\alpha_1,y)\,\Delta y + \frac{\lambda}{2}\int_{\alpha_2}^{\beta_2} G(\beta_1,y)\,\Delta y\right. \\
&\left.- \frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x),y)\,\Delta y\,\Delta x\right| \\
&\le \frac{\mathscr{M}(\beta_2-\alpha_2)}{\beta_1-\alpha_1}\left[2h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\
&+ h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&\left.+ h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right].
\end{aligned}
\tag{19}
$$

Using (18) and (19), we have

$$
\begin{aligned}
&\left|\int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1,y)+G(\beta_1,y)}{2}\Delta y\right. \\
&\left.- \frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x),y)\,\Delta y\,\Delta x\right| \\
&\le \frac{\mathscr{M}(\beta_2-\alpha_2)}{2(\beta_1-\alpha_1)}\left[3h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\
&+ h_2\left(\beta_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&+ 3h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right) \\
&\left.+ h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right].
\end{aligned}
\tag{20}
$$

Similarly, doing the same thing for G_x at $y=\alpha_2$ and $y=\beta_2$, and then integrating the resultant inequality over $[\alpha_1,\beta_1]$, we get

$$
\begin{aligned}
&\left|\int_{\alpha_1}^{\beta_1} \frac{G(x,\alpha_2)+G(x,\beta_2)}{2}\Delta x\right. \\
&\left.- \frac{1}{\beta_2-\alpha_2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,\sigma(y))\,\Delta y\,\Delta x\right| \\
&\le \frac{\mathscr{N}(\beta_1-\alpha_1)}{2(\beta_2-\alpha_2)}\left[3h_2\left(\alpha_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)\right. \\
&+ h_2\left(\beta_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)
\end{aligned}
$$

$$+3h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)$$
$$+h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\Bigg]. \quad (21)$$

Using (20) and (21) amounts to (15). Also, from (20) and (21), we get

$$\left|\int_{\alpha_2}^{\beta_2}\frac{G(\alpha_1,y)+G(\beta_1,y)}{2(\beta_2-\alpha_2)}\Delta y-\frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}G(\sigma(x),y)\,\Delta y\,\Delta x\right|$$
$$\le\frac{\mathscr{M}}{2(\beta_1-\alpha_1)}\Bigg[3h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\beta_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)+3h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\Bigg], \quad (22)$$

$$\left|\int_{\alpha_1}^{\beta_1}\frac{G(x,\alpha_2)+G(x,\beta_2)}{2(\beta_1-\alpha_1)}\Delta x-\frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}G(x,\sigma(y))\,\Delta y\,\Delta x\right|$$
$$\le\frac{\mathscr{N}}{2(\beta_2-\alpha_2)}\Bigg[3h_2\left(\alpha_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\beta_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)+3h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\Bigg]. \quad (23)$$

Combining (22) and (23), one gets (16). □

Theorem 10. *Under the assumptions of Theorem 9 and suppose also the intervals contain the mid points, then we have*

$$\Bigg|(1-\lambda)\left[\int_{\alpha_1}^{\beta_1}G\left(x,\frac{\alpha_2+\beta_2}{2}\right)\Delta x+\int_{\alpha_2}^{\beta_2}G\left(\frac{\alpha_1+\beta_1}{2},y\right)\Delta y\right]+\frac{\lambda}{2}\cdot\int_{\alpha_1}^{\beta_1}[G(x,\alpha_2)+G(x,\beta_2)]\,\Delta x+\frac{\lambda}{2}\cdot\int_{\alpha_2}^{\beta_2}[G(\alpha_1,y)+G(\beta_1,y)]\,\Delta y-\frac{1}{\beta_1-\alpha_1}\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}G(\sigma(x),y)\,\Delta y\,\Delta x-\frac{1}{\beta_2-\alpha_2}\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}G(x,\sigma(y))\,\Delta y\,\Delta x\Bigg|$$
$$\le\frac{\mathscr{M}(\beta_2-\alpha_2)}{\beta_1-\alpha_1}\Bigg[h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\frac{\alpha_1+\beta_1}{2},\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\frac{\alpha_1+\beta_1}{2},\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\Bigg]$$
$$+\frac{\mathscr{N}(\beta_1-\alpha_1)}{\beta_2-\alpha_2}\Bigg[h_2\left(\alpha_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\frac{\alpha_2+\beta_2}{2},\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\frac{\alpha_2+\beta_2}{2},\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\Bigg], \quad (24)$$

$$\Bigg|\frac{1-\lambda}{2(\beta_1-\alpha_1)}\int_{\alpha_2}^{\beta_2}G\left(x,\frac{\alpha_2+\beta_2}{2}\right)\Delta x+\frac{1-\lambda}{2(\beta_2-\alpha_2)}\cdot\int_{\alpha_2}^{\beta_2}G\left(\frac{\alpha_1+\beta_1}{2},y\right)\Delta y+\frac{\lambda}{4(\beta_1-\alpha_1)}\cdot\int_{\alpha_1}^{\beta_1}[G(x,\alpha_2)+G(x,\beta_2)]\,\Delta x+\frac{\lambda}{4(\beta_2-\alpha_2)}\cdot\int_{\alpha_2}^{\beta_2}[G(\alpha_1,y)+G(\beta_1,y)]\,\Delta y-\frac{1}{2(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}[G(\sigma(x),y)+G(x,\sigma(y))]\,\Delta y\,\Delta x\Bigg|$$
$$\le\frac{\mathscr{M}}{2(\beta_1-\alpha_1)}\Bigg[h_2\left(\alpha_1,\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\frac{\alpha_1+\beta_1}{2},\alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\frac{\alpha_1+\beta_1}{2},\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\Bigg]$$
$$+\frac{\mathscr{N}}{2(\beta_2-\alpha_2)}\Bigg[h_2\left(\alpha_2,\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\frac{\alpha_2+\beta_2}{2},\alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\frac{\alpha_2+\beta_2}{2},\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)+h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\Bigg]. \quad (25)$$

Proof. Next, we now apply Lemma 7 to G_y at $x = (\alpha_1 + \beta_1)/2$ and thereafter integrate the resulting inequality over $[\alpha_2, \beta_2]$ to get

$$\begin{aligned}&\Bigg|(1-\lambda)\int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1+\beta_1}{2}, y\right)\Delta y \\ &\quad + \lambda\frac{\int_{\alpha_2}^{\beta_2} G(\alpha_1, y)\,\Delta y + \int_{\alpha_2}^{\beta_2} G(\beta_1, y)\,\Delta y}{2} \\ &\quad - \frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x), y)\,\Delta y\,\Delta x\Bigg| \\ &\leq \frac{\mathcal{M}(\beta_2-\alpha_2)}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1, \alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\ &\quad + h_2\left(\frac{\alpha_1+\beta_1}{2}, \alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) \\ &\quad + h_2\left(\frac{\alpha_1+\beta_1}{2}, \beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right) \\ &\quad \left.+ h_2\left(\beta_1, \beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right].\end{aligned} \tag{26}$$

Similarly, if one applies Lemma 7 to G_x at $y = (\alpha_2+\beta_2)/2$ and thereafter integrate the resulting inequality over $[\alpha_1, \beta_1]$, one gets

$$\begin{aligned}&\Bigg|(1-\lambda)\int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2+\beta_2}{2}\right)\Delta x \\ &\quad + \lambda\frac{\int_{\alpha_1}^{\beta_1} G(x, \alpha_2)\,\Delta x + \int_{\alpha_1}^{\beta_1} G(x, \beta_2)\,\Delta x}{2} \\ &\quad - \frac{1}{\beta_2-\alpha_2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x, \sigma(y))\,\Delta y\,\Delta x\Bigg| \\ &\leq \frac{\mathcal{N}(\beta_1-\alpha_1)}{\beta_2-\alpha_2}\left[h_2\left(\alpha_2, \alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)\right. \\ &\quad + h_2\left(\frac{\alpha_2+\beta_2}{2}, \alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right) \\ &\quad + h_2\left(\frac{\alpha_2+\beta_2}{2}, \beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right) \\ &\quad \left.+ h_2\left(\beta_2, \beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right].\end{aligned} \tag{27}$$

Combining (26) and (27), we get (24). Finally, from (26) and (27), we get

$$\begin{aligned}&\Bigg|\frac{1-\lambda}{\beta_2-\alpha_2}\int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1+\beta_1}{2}, y\right)\Delta y \\ &\quad + \frac{\lambda}{2(\beta_2-\alpha_2)}\int_{\alpha_2}^{\beta_2}\left[G(\alpha_1, y) + G(\beta_1, y)\right]\Delta y \\ &\quad - \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x), y)\,\Delta y\,\Delta x\Bigg| \\ &\leq \frac{\mathcal{M}}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1, \alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\ &\quad + h_2\left(\frac{\alpha_1+\beta_1}{2}, \alpha_1+\lambda\frac{\beta_1-\alpha_1}{2}\right) \\ &\quad + h_2\left(\frac{\alpha_1+\beta_1}{2}, \beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right) \\ &\quad \left.+ h_2\left(\beta_1, \beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right], \\ &\Bigg|\frac{1-\lambda}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2+\beta_2}{2}\right)\Delta x \\ &\quad + \frac{\lambda}{2(\beta_1-\alpha_1)}\int_{\alpha_1}^{\beta_1}\left[G(x, \alpha_2) + G(x, \beta_2)\right]\Delta x \\ &\quad - \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x, \sigma(y))\,\Delta y\,\Delta x\Bigg| \\ &\leq \frac{\mathcal{N}}{\beta_2-\alpha_2}\left[h_2\left(\alpha_2, \alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right)\right. \\ &\quad + h_2\left(\frac{\alpha_2+\beta_2}{2}, \alpha_2+\lambda\frac{\beta_2-\alpha_2}{2}\right) \\ &\quad + h_2\left(\frac{\alpha_2+\beta_2}{2}, \beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right) \\ &\quad \left.+ h_2\left(\beta_2, \beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right].\end{aligned} \tag{28}$$

Using (28) amounts to (25). Thus, the proof of Theorem 9 is complete. □

Corollary 11. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorems 9 and 10, then we obtain the inequality*

$$\begin{aligned}&\Bigg|\int_{\alpha_1}^{\beta_1}\frac{G(x, \alpha_2) + G(x, \beta_2)}{2}dx \\ &\quad + \int_{\alpha_2}^{\beta_2}\frac{G(\alpha_1, y) + G(\beta_1, y)}{2}dy - \left(\frac{1}{\beta_1-\alpha_1}\right. \\ &\quad \left.+ \frac{1}{\beta_2-\alpha_2}\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x, y)\,dy\,dx\Bigg| \\ &\leq \frac{\lambda^2-\lambda+1}{2}(\beta_1-\alpha_1)(\beta_2-\alpha_2)(\mathcal{M}+\mathcal{N}), \\ &\Bigg|\int_{\alpha_1}^{\beta_1}\frac{G(x, \alpha_2) + G(x, \beta_2)}{4(\beta_1-\alpha_1)}dx \\ &\quad + \int_{\alpha_2}^{\beta_2}\frac{G(\alpha_1, y) + G(\beta_1, y)}{4(\beta_2-\alpha_2)}dy\end{aligned}$$

$$- \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)} \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx \Bigg| \le \frac{\lambda^2-\lambda+1}{4}\left(\mathscr{M}(\beta_1-\alpha_1)+\mathscr{N}(\beta_2-\alpha_2)\right),$$

$$\Bigg|(1-\lambda)\left[\int_{\alpha_1}^{\beta_1} G\left(x,\frac{\alpha_2+\beta_2}{2}\right)dx + \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1+\beta_1}{2},y\right)dy\right] + \frac{\lambda}{2} \cdot \int_{\alpha_1}^{\beta_1}\left[G(x,\alpha_2)+G(x,\beta_2)\right]dx + \frac{\lambda}{2} \cdot \int_{\alpha_2}^{\beta_2}\left[G(\alpha_1,y)+G(\beta_1,y)\right]dy - \left(\frac{1}{\beta_1-\alpha_1} + \frac{1}{\beta_2-\alpha_2}\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\Bigg| \le \frac{2\lambda^2-2\lambda+1}{4}(\beta_1-\alpha_1)(\beta_2-\alpha_2)(\mathscr{M}+\mathscr{N}),$$

$$\Bigg|\frac{1-\lambda}{2(\beta_1-\alpha_1)}\int_{\alpha_2}^{\beta_2} G\left(x,\frac{\alpha_2+\beta_2}{2}\right)dx + \frac{1-\lambda}{2(\beta_2-\alpha_2)} \cdot \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1+\beta_1}{2},y\right)dy + \frac{\lambda}{4(\beta_1-\alpha_1)} \cdot \int_{\alpha_1}^{\beta_1}\left[G(x,\alpha_2)+G(x,\beta_2)\right]dx + \frac{\lambda}{4(\beta_2-\alpha_2)} \cdot \int_{\alpha_2}^{\beta_2}\left[G(\alpha_1,y)+G(\beta_1,y)\right]dy - \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\Bigg| \le \frac{2\lambda^2-2\lambda+1}{8}\left(\mathscr{M}(\beta_1-\alpha_1)+\mathscr{N}(\beta_2-\alpha_2)\right). \tag{29}$$

Remark 12. Corollary 11 becomes Theorem 3 if $\lambda = 0$.

Corollary 13. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorems 9 and 10, then we get the succeeding inequalities*

$$\Bigg|\sum_{x=\alpha_1}^{\beta_1-1}\frac{G(x,\alpha_2)+G(x,\beta_2)}{2} + \sum_{y=\alpha_2}^{\beta_2-1}\frac{G(\alpha_1,y)+G(\beta_1,y)}{2} - \frac{1}{\beta_1-\alpha_1} \cdot \sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G(x+1,y) - \frac{1}{\beta_2-\alpha_2} \cdot \sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G(x,y+1)\Bigg| \le \frac{\lambda^2-\lambda+1}{2}(\beta_1-\alpha_1)(\beta_2-\alpha_2)(\mathscr{M}+\mathscr{N}),$$

$$\Bigg|\sum_{x=\alpha_1}^{\beta_1-1}\frac{G(x,\alpha_2)+G(x,\beta_2)}{4(\beta_1-\alpha_1)} + \sum_{y=\alpha_2}^{\beta_2-1}\frac{G(\alpha_1,y)+G(\beta_1,y)}{4(\beta_2-\alpha_2)} - \frac{1}{2(\beta_1-\alpha_1)(\beta_2-\alpha_2)} \cdot \sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G\left[(x,y+1)+G(x+1,y)\right]\Bigg| \le \frac{\lambda^2-\lambda+1}{4}\left[\mathscr{M}(\beta_1-\alpha_1)+\mathscr{N}(\beta_2-\alpha_2)\right],$$

$$\Bigg|(1-\lambda)\left[\sum_{x=\alpha_1}^{\beta_1-1} G\left(x,\frac{\alpha_2+\beta_2}{2}\right) + \sum_{y=\alpha_2}^{\beta_2-1} G\left(\frac{\alpha_1+\beta_1}{2},y\right)\right] + \frac{\lambda}{2}\sum_{x=\alpha_1}^{\beta_1-1}\left[G(x,\alpha_2)+G(x,\beta_2)\right] + \frac{\lambda}{2} \cdot \sum_{y=\alpha_2}^{\beta_2-1}\left[G(\alpha_1,y)+G(\beta_1,y)\right] - \frac{1}{\beta_1-\alpha_1} \cdot \sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G(x+1,y) - \frac{1}{\beta_2-\alpha_2} \cdot \sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G(x,y+1)\Bigg| \le \frac{2\lambda^2-2\lambda+1}{4}(\beta_1-\alpha_1) \cdot (\beta_2-\alpha_2)(\mathscr{M}+\mathscr{N}),$$

$$\Bigg|\frac{1-\lambda}{2(\beta_1-\alpha_1)}\sum_{y=\alpha_2}^{\beta_2-1} G\left(x,\frac{\alpha_2+\beta_2}{2}\right) + \frac{1-\lambda}{2(\beta_2-\alpha_2)} \cdot \sum_{y=\alpha_2}^{\beta_2-1} G\left(\frac{\alpha_1+\beta_1}{2},y\right) + \frac{\lambda}{4(\beta_1-\alpha_1)} \cdot \sum_{x=\alpha_1}^{\beta_1-1}\left[G(x,\alpha_2)+G(x,\beta_2)\right] + \frac{\lambda}{4(\beta_2-\alpha_2)} \cdot \sum_{y=\alpha_2}^{\beta_2-1}\left[G(\alpha_1,y)+G(\beta_1,y)\right] - \frac{1}{2(\beta_1-\alpha_1)(\beta_2-\alpha_2)}$$

$$\cdot \sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1}\left[G(x+1,y)+G(x,y+1)\right]\Bigg|$$
$$\leq \frac{2\lambda^2-2\lambda+1}{8}\left[\mathscr{M}(\beta_1-\alpha_1)+\mathscr{N}(\beta_2-\alpha_2)\right]. \tag{30}$$

Theorem 14. *Let $\alpha_1,\beta_1,x \in \mathbb{T}_1$, $\alpha_2,\beta_2,y \in \mathbb{T}_2$, with $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ and $G : [\alpha_1,\beta_1]\times[\alpha_2,\beta_2] \to \mathbb{R}$ be such that the partial mappings*

$$\begin{aligned} G_y &: [\alpha_1,\beta_1] \longrightarrow \mathbb{R}, \\ G_y(\xi) &:= G(\xi,y), \\ G_x &: [\alpha_2,\beta_2] \longrightarrow \mathbb{R}, \\ G_x(\zeta) &:= G(x,\zeta), \end{aligned} \tag{31}$$

defined for all $y \in [\alpha_2,\beta_2]$ and $x \in [\alpha_1,\beta_1]$, are differentiable. If $G_x^{\Delta}, G_y^{\Delta} \in C_{rd}(\mathbb{T},\mathbb{R})$ and there exist $\gamma_x,\gamma_y,\Gamma_x,\Gamma_y \in \mathbb{R}$ such that $\gamma_y \leq G_y^{\Delta}(t) \leq \Gamma_y$, $t \in [\alpha_1,\beta_1]$, $\gamma_x \leq G_x^{\Delta}(t) \leq \Gamma_x$, $t \in [\alpha_2,\beta_2]$, then the succeeding inequalities

$$\begin{aligned} &\Bigg|\left(1-\frac{\lambda}{2}\right)\left[\int_{\alpha_1}^{\beta_1} G(x,\alpha_2)\,\Delta x+\int_{\alpha_2}^{\beta_2} G(\alpha_1,y)\,\Delta y\right] \\ &\quad+\frac{\lambda}{2}\left[\int_{\alpha_1}^{\beta_1} G(x,\beta_2)\,\Delta x+\int_{\alpha_2}^{\beta_2} G(\beta_1,y)\,\Delta y\right]-\frac{1}{\beta_1-\alpha_1} \\ &\quad\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(\sigma(x),y)\,\Delta y\,\Delta x-\frac{1}{\beta_2-\alpha_2} \\ &\quad\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,\sigma(y))\,\Delta y\,\Delta x+\frac{\Gamma_x+\gamma_x}{2} \\ &\quad\cdot\frac{\beta_1-\alpha_1}{\beta_2-\alpha_2}\left[h_2(\alpha_2,\beta_2)-\frac{\lambda}{2}(\beta_2-\alpha_2)^2\right]+\frac{\Gamma_y+\gamma_y}{2} \\ &\quad\cdot\frac{\beta_2-\alpha_2}{\beta_1-\alpha_1}\left[h_2(\alpha_1,\beta_1)-\frac{\lambda}{2}(\beta_1-\alpha_1)^2\right]\Bigg| \leq \frac{\Gamma_x-\gamma_x}{2} \\ &\quad\cdot\frac{\beta_1-\alpha_1}{\beta_2-\alpha_2}\left[h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right. \\ &\quad\left.+h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right]+\frac{\Gamma_y-\gamma_y}{2} \\ &\quad\cdot\frac{\beta_2-\alpha_2}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\ &\quad\left.+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right], \end{aligned} \tag{32}$$

$$\begin{aligned} &\Bigg|\frac{1}{4(\beta_1-\alpha_1)}\left[\int_{\alpha_1}^{\beta_1}((2-\lambda)G(x,\alpha_2)+\lambda G(x,\beta_2))\,\Delta x\right] \\ &\quad+\frac{1}{4(\beta_2-\alpha_2)}\left[\int_{\alpha_2}^{\beta_2}((2-\lambda)G(\alpha_1,y)+\lambda G(\beta_1,y))\,\Delta y\right] \\ &\quad-\frac{1}{2(\beta_1-\alpha_1)(\beta_2-\alpha_2)} \\ &\quad\cdot\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}(G(x,\sigma(y))+G(\sigma(x),y))\,\Delta y\,\Delta x+\frac{\Gamma_y+\gamma_y}{4} \\ &\quad\cdot\frac{1}{\beta_1-\alpha_1}\left[h_2(\alpha_1,\beta_1)-\frac{\lambda}{2}(\beta_1-\alpha_1)^2\right]+\frac{\Gamma_x+\gamma_x}{4} \\ &\quad\cdot\frac{1}{\beta_2-\alpha_2}\left[h_2(\alpha_2,\beta_2)-\frac{\lambda}{2}(\beta_2-\alpha_2)^2\right]\Bigg| \leq \frac{\Gamma_y-\gamma_y}{4} \\ &\quad\cdot\frac{1}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\ &\quad\left.+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right]+\frac{\Gamma_x-\gamma_x}{4} \\ &\quad\cdot\frac{1}{\beta_2-\alpha_2}\left[h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right. \\ &\quad\left.+h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right], \end{aligned} \tag{33}$$

hold for all $\lambda \in [0,1]$ such that $\beta_1-\lambda((\beta_1-\alpha_1)/2) \in \mathbb{T}_1$ and $\beta_2-\lambda((\beta_2-\alpha_2)/2) \in \mathbb{T}_2$.

Proof. Applying Lemma 8 to the mapping G_y at $x=\alpha_1$ gives

$$\begin{aligned} &\Bigg|\left(1-\frac{\lambda}{2}\right)G(\alpha_1,y)+\frac{\lambda}{2}G(\beta_1,y) \\ &\quad-\frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{b} f(\sigma(t),y)\,\Delta y \\ &\quad+\frac{\Gamma_y+\gamma_y}{2}\frac{1}{\beta_1-\alpha_1}\left[h_2(\alpha_1,\beta_1)-\frac{\lambda}{2}(\beta_1-\alpha_1)^2\right]\Bigg| \\ &\quad\leq\frac{\Gamma_y-\gamma_y}{2}\frac{1}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\ &\quad\left.+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right]. \end{aligned} \tag{34}$$

Integrating (34) over $[\alpha_2,\beta_2]$ yields

$$\begin{aligned} &\Bigg|\left(1-\frac{\lambda}{2}\right)\int_{\alpha_2}^{d} f(\alpha_1,y)\,\Delta y+\frac{\lambda}{2}\int_{\alpha_2}^{d} f(\beta_1,y)\,\Delta y \\ &\quad-\frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{d} f(\sigma(x),y)\,\Delta y\,\Delta x \\ &\quad+\frac{\Gamma_y+\gamma_y}{2}\frac{\beta_2-\alpha_2}{\beta_1-\alpha_1}\left[h_2(\alpha_1,\beta_1)-\frac{\lambda}{2}(\beta_1-\alpha_1)^2\right]\Bigg| \\ &\quad\leq\frac{\Gamma_y-\gamma_y}{2}\frac{\beta_2-\alpha_2}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right. \\ &\quad\left.+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right]. \end{aligned} \tag{35}$$

Similarly, applying Lemma 8 to the mapping G_x at $y=\alpha_2$ and then integrating the resulting inequality over $[\alpha_1,\beta_1]$ give

$$\begin{aligned}&\left|\left(1-\frac{\lambda}{2}\right)\int_{\alpha_1}^{b} f(x,\alpha_2)\,\Delta x+\frac{\lambda}{2}\int_{\alpha_1}^{b} f(x,\beta_2)\,\Delta x\right.\\&-\frac{1}{\beta_2-\alpha_2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{d} f(x,\sigma(y))\,\Delta y\,\Delta x\\&\left.+\frac{\Gamma_x+\gamma_x}{2}\frac{\beta_1-\alpha_1}{\beta_2-\alpha_2}\left[h_2(\alpha_2,\beta_2)-\frac{\lambda}{2}(\beta_2-\alpha_2)^2\right]\right|\\&\leq\frac{\Gamma_x-\gamma_x}{2}\frac{\beta_1-\alpha_1}{\beta_2-\alpha_2}\left[h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right.\\&\left.+h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right].\end{aligned}\tag{36}$$

Using (35) and (36), we get (32). Also, we obtain from (35) and (36) the following inequalities:

$$\begin{aligned}&\left|\frac{(1-\lambda/2)}{\beta_2-\alpha_2}\int_{\alpha_2}^{d} f(\alpha_1,y)\,\Delta y\right.\\&+\frac{\lambda}{2(\beta_2-\alpha_2)}\int_{\alpha_2}^{d} f(\beta_1,y)\,\Delta y\\&-\frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{d} f(\sigma(x),y)\,\Delta y\,\Delta x\\&\left.+\frac{\Gamma_y+\gamma_y}{2}\frac{1}{\beta_1-\alpha_1}\left[h_2(\alpha_1,\beta_1)-\frac{\lambda}{2}(\beta_1-\alpha_1)^2\right]\right|\\&\leq\frac{\Gamma_y-\gamma_y}{2}\frac{1}{\beta_1-\alpha_1}\left[h_2\left(\alpha_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right.\\&\left.+h_2\left(\beta_1,\beta_1-\lambda\frac{\beta_1-\alpha_1}{2}\right)\right],\\&\left|\frac{(1-\lambda/2)}{\beta_1-\alpha_1}\int_{\alpha_1}^{b} f(x,\alpha_2)\,\Delta x\right.\\&+\frac{\lambda}{2(\beta_1-\alpha_1)}\int_{\alpha_1}^{b} f(x,\beta_2)\,\Delta x\\&-\frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{d} f(x,\sigma(y))\,\Delta y\,\Delta x\\&\left.+\frac{\Gamma_x+\gamma_x}{2}\frac{1}{\beta_2-\alpha_2}\left[h_2(\alpha_2,\beta_2)-\frac{\lambda}{2}(\beta_2-\alpha_2)^2\right]\right|\\&\leq\frac{\Gamma_x-\gamma_x}{2}\frac{1}{\beta_2-\alpha_2}\left[h_2\left(\alpha_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right.\\&\left.+h_2\left(\beta_2,\beta_2-\lambda\frac{\beta_2-\alpha_2}{2}\right)\right].\end{aligned}\tag{37}$$

Using (37) amounts to (33). That completes the proof of Theorem 14. □

Corollary 15. *If we let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$ in Theorem 10, then the succeeding inequalities hold:*

$$\begin{aligned}&\left|\left(1-\frac{\lambda}{2}\right)\left[\int_{\alpha_1}^{\beta_1} G(x,\alpha_2)\,dx+\int_{\alpha_2}^{\beta_2} G(\alpha_1,y)\,dy\right]\right.\\&+\frac{\lambda}{2}\left[\int_{\alpha_1}^{\beta_1} G(x,\beta_2)\,dx+\int_{\alpha_2}^{\beta_2} G(\beta_1,y)\,dy\right]\\&-\left(\frac{1}{\beta_1-\alpha_1}+\frac{1}{\beta_2-\alpha_2}\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\\&\left.+\frac{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}{4}(1-\lambda)\left(\Gamma_x+\gamma_x+\Gamma_y+\gamma_y\right)\right|\\&\leq\frac{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}{8}\left((\lambda-1)^2+1\right)\left(\Gamma_x-\gamma_x+\Gamma_y\right.\\&\left.-\gamma_y\right),\\&\left|\frac{1}{4(\beta_1-\alpha_1)}\left[\int_{\alpha_1}^{\beta_1}((2-\lambda)G(x,\alpha_2)+\lambda G(x,\beta_2))\,dx\right]\right.\\&+\frac{1}{4(\beta_2-\alpha_2)}\left[\int_{\alpha_2}^{\beta_2}((2-\lambda)G(\alpha_1,y)+\lambda G(\beta_1,y))\,dy\right]\\&-\frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} G(x,y)\,dy\,dx\\&\left.+\frac{1-\lambda}{8}\left((\Gamma_y+\gamma_y)(\beta_1-\alpha_1)+(\Gamma_x+\gamma_x)(\beta_2-\alpha_2)\right)\right|\\&\leq\frac{(\lambda-1)^2+1}{16}\left((\Gamma_y-\gamma_y)(\beta_1-\alpha_1)\right.\\&\left.+(\Gamma_x-\gamma_x)(\beta_2-\alpha_2)\right).\end{aligned}\tag{38}$$

Remark 16. Corollary 15 becomes Theorem 5 if $\lambda=1$.

Corollary 17. *If we let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$ in Theorem 10, then the succeeding inequalities hold:*

$$\begin{aligned}&\left|\left(1-\frac{\lambda}{2}\right)\left[\sum_{x=\alpha_1}^{\beta_1-1} G(x,\alpha_2)+\sum_{y=\alpha_2}^{\beta_2-1} G(\alpha_1,y)\right]\right.\\&+\frac{\lambda}{2}\left[\sum_{x=\alpha_1}^{\beta_1-1} G(x,\beta_2)+\sum_{y=\alpha_2}^{\beta_2-1} G(\beta_1,y)\right]-\frac{1}{\beta_1-\alpha_1}\\&\cdot\sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G(x+1,y)-\frac{1}{\beta_2-\alpha_2}\sum_{x=\alpha_1}^{\beta_1-1}\sum_{y=\alpha_2}^{\beta_2-1} G(x,y+1)\\&+\frac{\Gamma_x+\gamma_x}{4}(\beta_1-\alpha_1)\left[(\beta_2-\alpha_2+1)-\lambda(\beta_2-\alpha_2)\right]\\&\left.+\frac{\Gamma_y+\gamma_y}{4}(\beta_2-\alpha_2)\left[(\beta_1-\alpha_1+1)-\lambda(\beta_1-\alpha_1)\right]\right|\\&\leq\frac{\Gamma_x-\gamma_x}{8}(\beta_1-\alpha_1)\left((\beta_2-\alpha_2)\left(\lambda^2-2\lambda+2\right)-2\lambda\right.\\&\left.+2\right)+\frac{\Gamma_y-\gamma_y}{8}(\beta_2-\alpha_2)\left((\beta_1-\alpha_1)\left(\lambda^2-2\lambda+2\right)\right.\\&\left.-2\lambda+2\right),\end{aligned}$$

$$
\begin{aligned}
&\left| \frac{1}{4(\beta_1-\alpha_1)} \left[\sum_{x=\alpha_1}^{\beta_1-1} ((2-\lambda) G(x,\alpha_2) + \lambda G(x,\beta_2)) \right] \right. \\
&\quad + \frac{1}{4(\beta_2-\alpha_2)} \left[\sum_{y=\alpha_2}^{\beta_2-1} ((2-\lambda) G(\alpha_1,y) + \lambda G(\beta_1,y)) \right] \\
&\quad - \frac{1}{2(\beta_1-\alpha_1)(\beta_2-\alpha_2)} \\
&\quad \cdot \sum_{x=\alpha_1}^{\beta_1-1} \sum_{y=\alpha_2}^{\beta_2-1} (G(x,y+1) + G(x+1,y)) \\
&\quad + \frac{\Gamma_y+\gamma_y}{8} [(\beta_1-\alpha_1+1) - \lambda(\beta_1-\alpha_1)] \\
&\quad \left. + \frac{\Gamma_x+\gamma_x}{8} [(\beta_2-\alpha_2+1) - \lambda(\beta_2-\alpha_2)] \right| \\
&\quad \le \frac{\Gamma_y-\gamma_y}{16} \left((\beta_1-\alpha_1)\left(\lambda^2-2\lambda+2\right) - 2\lambda+2 \right) \\
&\quad + \frac{\Gamma_x-\gamma_x}{16} \left((\beta_2-\alpha_2)\left(\lambda^2-2\lambda+2\right) - 2\lambda+2 \right).
\end{aligned}
\tag{39}
$$

3. Conclusion

Three main theorems are hereby established. The results of Farid [3] are obtained as special cases of our results. Loads of interesting new inequalities can be obtained by choosing different values of $\lambda \in [0,1]$, and considering a different time scale different from $\mathbb{R}$ and $\mathbb{Z}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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