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Research Article

Existence and Uniqueness of Solution of Stochastic Dynamic Systems with Markov Switching and Concentration Points

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In this article the problem of existence and uniqueness of solutions of stochastic differential equations with jumps and concentration points are solved. The theoretical results are illustrated by one example.

1. Introduction

First, consider the works that are relevant to this subject. Note that number of these works are very small, since the existence of points of condensation is not very often encountered in real processes. However, the relevant equations can significantly enhance the understanding of the dynamics of real processes. In addition, this mathematical model can be a very good comparison for the classical model, the ordinary differential equations, stochastic differential equations, functional differential equations, and impulse equations. On the other hand, equations with concentration points cannot be considered as equations with Poison integral, because for these equations points of condensation do not exist with probability 1.

In paper [1] differential equations with delay in simplest form

$$\dot{x}(t) + ax(t-\tau) = \sum_{j=1}^{\infty} b_j x(t_j -) \delta(t - t_j) \quad (1)$$

with infinity impulses are considered. However, in Theorems 2.1 and 3.1 [1] it is supposed that impulses satisfy the inequality $t_j - t_{j-1} > T = \text{const}$; that is, $t_j \to \infty$, if $j \to \infty$. According to Theorem 2.1 [1], one of the exponential stability conditions is

$$1 + |b_j| \le M \quad \text{for } j = 1, 2, 3, \ldots \quad (2)$$

whence influence of impulses in determining case is obvious. In Section 3 of this paper authors consider determining differential-difference system with one delay and perturbation $x(t_i+) - x(t_i-) = b_i x(t_i-)$. Then one of the conditions of oscillation is

$$\lim_{i\to\infty} \sup (1 + b_i)^{-1} \int_{t_i}^{t_i = \min\{\tau, T\}} p(s)\, ds > 1. \quad (3)$$

On the other hand, Theorem 3.3 [1] consists of sufficient conditions about condition on the value of jumps $b_i > 0$, $i = 1, 2, 3, \ldots$, and $\sum_{i=1}^{\infty} d_i < \infty$.

In [2] delay depends on the time for differential equations with delay, and there is a condition on impulses $\lim_{k\to\infty} t_k = \infty$. Under the given conditions the boundedness of the solutions by exponential functions $k \cdot e^{\gamma t}$ is proved (Theorems 3.1, 4.1 [2]). Differential equations with impulses are used in a lot of application problems (however, besides delay effect, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics, and telecommunications; artificial electronic systems, neural networks such as Hopfield neural networks, bidirectional neural networks, and recurrent neural networks often are subject to impulsive perturbations which can affect dynamical behaviors of the systems just as time delays). Authors note influence of the jumps on the capability analysis; namely, the condition on the jump's moments is $(\ln(\gamma_k)/(t_k - t_{k-1})) \leq \gamma$, where γ and γ_k are described in Section 3. Thus, authors use weaker condition in comparison with $t_k - t_{k-1} > \text{const} > 0$, but existence of concentration points is not considered.

A significant contribution to the study of impulse systems of differential equations was made by Ukrainian academician Samoilenko. The monograph [3] studies impulse systems of ordinary differential equations; in the monograph [4] authors consider counted systems of ordinary differential equation in the next form

$$\frac{dx}{dt} = A(t)\, x \tag{4}$$

with impulse perturbations

$$\Delta x(t_j) = B_j x(t_j - 0). \tag{5}$$

For these systems problems of existence and uniqueness of the solution, limited periodicity is considered. In paper [5] the main problems of ordinary differential equations with stochastic parameters and perturbations are considered. As in papers [3, 4], authors suppose that $t_i - t_{i-1} > T$. In papers [3–5] the focus is on the existence of periodic solutions of the system and authors prove that existence of impulses changes qualitative characteristic of solutions (stability and periodicity). Beside this, in these papers there is a supposition about continuity on (t_i, t_{i+1}), as opposed to continuity on $[t_i, t_{i+1})$, as in this paper and in [6–8].

In [9] authors consider the second-order system, which describes behavior of $[y, \dot{y}]$ for the solution of 2nd order differential equation with impulse perturbations. By contrast, in this paper the delay process is found such that solution is non-Markov process in classic perception, but the condition on impulses is the same: $|t_i - t_{i-1}| > T$. The feature of this paper is transition of non-Markov process of perturbation to Markov process using additional variables. Based on this approach we can build finite-dimensional distributions.

Problem of existence and uniqueness of solution of impulse systems without the concentration points is considered in [10]. Also the problem of stability of solution using discontinuous impulses is considered.

All above-listed papers do not contain concentration points and cannot be used for describing systems with increase on the short time interval resonance.

The problem of existence and uniqueness of solution of dynamic systems with concentration points for determinate dynamic differential equations is solved in [11]. This paper is one of the first papers where the concentration points are considered. The examples of real processes, which are described by impulse differential equations, for which the condition $t_k - t_{k-1} > \text{const} > 0$ does not hold are considered in the paper [12]. Existence and uniqueness of random stochastic dynamic systems with permanent delay in the absence of the concentration points are considered in [8].

The sufficient conditions of existence and uniqueness of the solution of the systems of stochastic differential-difference equations with Markov switching with concentration points are shown in this paper. Thus the paper is actual and timely.

2. Problem Definition

Consider stochastic differential-difference equation

$$\begin{aligned} dx(t) &= a(t, \xi(t), x(t), x(t-r))\, dt \\ &\quad + b(t, \xi(t), x(t), x(t-r))\, dw(t), \end{aligned} \tag{6}$$

for $t \in \mathbb{R}_+ \setminus \mathbf{T}$ with Markov's switching

$$\begin{aligned} \Delta x(t_k) &= x(t_k) - x(t_k-) \\ &= g(t_k-, \xi(t_k-), \eta_k, x(t_k-)), \\ &\qquad t_k \in \mathbf{T} := \{t_k \uparrow,\ k = 1, 2, \ldots\}, \end{aligned} \tag{7}$$

and the initial conditions

$$\begin{gathered} x(t) = \varphi(t), \\ -r \leq t \leq 0, \\ r > 0, \\ \varphi \in \mathbf{D}, \\ \xi(0) = y \in \mathbf{Y}, \\ \eta_0 = h \in \mathbf{H}. \end{gathered} \tag{8}$$

Here $\xi(t)$ is the Markov process with values in the measured space $(\mathbf{Y}, \mathscr{Y})$ with generator Q; $\eta_k,\ k \geq 0$ is the Markov chain with values in the measured space $(\mathbf{H}, \mathscr{H})$, which is described by the matrix of transition probabilities $P(y, A) = P\{\eta_k \in A \mid \eta_{k-1} = y\}$, $y \in \mathbf{H}$, $A \in \mathscr{H}$; $w(t)$ is one-dimensional Wiener process. It should be noted that $w(t), \xi(t),\ t \geq 0$, and $\eta_k,\ k \geq 0$, are independent [13]; $\mathbf{D} \equiv \mathbf{D}([-r, 0], \mathbb{R}^m)$ is the Skorokhod space of the right continuous functions with left-hand limits [14] with the norm

$$\|\varphi\| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|. \tag{9}$$

Let measured mapping $a : \mathbb{R}_+ \times \mathbf{Y} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$; $b : \mathbb{R}_+ \times \mathbf{Y} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$; $g : \mathbb{R}_+ \times \mathbf{Y} \times \mathbf{H} \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies the bounded condition and Lipschitz condition $\forall t \in [0, T],\ y \in \mathbf{Y},\ h \in \mathbf{H}$:

$$\begin{aligned} &|a(t, y, \phi_1, \phi_2)|^2 + |b(t, y, \phi_1, \phi_2)|^2 + |g(t, y, h, \phi_3)|^2 \\ &\quad \leq C\left(1 + |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right), \\ &\qquad \forall t \geq 0,\ y \in \mathbf{Y},\ h \in \mathbf{H},\ \varphi_i \in \mathbb{R}^m,\ i = 1, 2, 3; \\ &|a(t, y, \phi_1, \phi_2) - a(t, y, \psi_1, \psi_2)|^2 \\ &\quad + |b(t, y, \phi_1, \phi_2) - b(t, y, \psi_1, \psi_2)|^2 \\ &\quad \leq L\left(|\phi_1 - \psi_1|^2 + |\phi_2 - \psi_2|^2\right) \\ &\qquad \forall t \geq 0,\ y \in \mathbf{Y},\ \phi_i, \psi_i \in \mathbb{R}^m,\ i = 1, 2; \\ &|g(t_k, y, h, \phi_3) - g(t_k, y, h, \psi_3)|^2 \leq l_k |\phi_3 - \psi_3|^2, \\ &\qquad \phi_3, \psi_3 \in \mathbb{R}^m,\ \sum_{k=1}^{\infty} l_k < \infty. \end{aligned} \tag{10}$$

Consider the case when the concentration point

$$\lim_{k\to\infty} t_k = t^* \in [0,T], \quad (11)$$

presents on $[0,T]$, where we learn existence and uniqueness of Cauchy problem (6)–(8) solution.

2.1. The Main Result. In this part of the work consider the problem of existence and uniqueness of solutions of stochastic differential-difference equations with impulse perturbations. Note that the conditions considered in this theorem are not elementary, since they have a quite complicated form.

Theorem 1. *Let*

(1) *the condition (10) hold;*

(2) $\sum_{k=1}^{\infty} \gamma_k < \infty$, $\gamma_k = \sup_{x\in\mathbb{R}^m, y\in\mathbf{Y}, h\in\mathbf{H}} |g(t_k, y, h, x)|$;

(3) *the condition*

$$\lim_{\varepsilon\downarrow 0} \left(\ln\varepsilon + N_\varepsilon \sum_{k=1}^{N_\varepsilon} l_k \right) = -\infty, \quad (12)$$

where

$$N_\varepsilon := \inf\left\{ k \ge 1 : \sum_{m=k}^{\infty} \gamma_m < \varepsilon \right\}, \quad (13)$$

hold.

Then exists a unique solution of the Cauchy's problems (6)–(8).

Proof.

(I) Existing. Let us determine the following process:

$$\begin{aligned} \tilde{x}(t) &= \tilde{x}(t_k) + \int_{t_k}^{t} a(s, \xi(s), \tilde{x}(s), \tilde{x}(s-r))\, ds \\ &\quad + \int_{t_k}^{t} b(s, \xi(s), \tilde{x}(s), \tilde{x}(s-r))\, dw(s), \\ &\qquad t \in (t_k, t_{k+1}), \\ \tilde{x}(t_k) &= \tilde{x}(t_k-) + g(t_k-, \xi(t_k-), \eta_k, \tilde{x}(t_k-)), \\ &\qquad k \ge 0, \end{aligned} \quad (14)$$

which determines the initial condition

$$\begin{aligned} x(\tau) &= \varphi(\tau), \\ -r &\le \tau \le 0, \\ \xi(0) &= y \in \mathbf{Y}, \\ \eta_0 &= h \in \mathbf{H}. \end{aligned} \quad (15)$$

Consider $T' < t^*$. Then, for interval $[0,T']$, we can use classical theorem of existence and uniqueness [8]. Besides this, in the paper [15] it is proved that $x \in L_2[0,T']$. The next inequality

$$\begin{aligned} \mathbf{E}|x(t)| &\le \mathbf{E}|x(0)| \\ &\quad + \int_0^{T'} \mathbf{E}|a(s, \xi(s), x(s), x(s-r))|\, ds \\ &\quad + \sqrt{\int_0^{T'} \mathbf{E}|b(s, \xi(s), x(s), x(s-r))|^2\, ds} \\ &\quad + \sum_{k=1}^{k'} \mathbf{E}|g(t_k-, \xi(t_k-), \eta_k, x(t_k-))| \end{aligned} \quad (16)$$

also holds. Using condition (7), we get

$$\lim_{T'\to t^*} \mathbf{E}|x(t)| < \infty. \quad (17)$$

This inequality proves the existence of the first moment for x; it means $x \in L_1[0,t^*]$ or $x \in L_1[0,T]$. Existence is proved.

(II) Uniqueness. Let two solutions $x^{(1)}(t)$, $x^{(2)}(t)$, $t \ge 0$ exist. Consider estimation $\mathbf{E}|x^{(1)}(t) - x^{(2)}(t)|^2$, $t \ge 0$, using its integral form and Lipschitz condition

$$\begin{aligned} I(t) &= \mathbf{E}\left|x^{(1)}(t) - x^{(2)}(t)\right|^2 \le K \int_0^t \mathbf{E}\left|x^{(1)}(s) \right. \\ &\quad \left. - x^{(2)}(s)\right|^2 ds \\ &\quad + \mathbf{E}\left(\sum_{k=1}^{N(t)} \left| g\left(t_k-, \xi(t_k-), \eta_k, x^{(1)}(t_k-)\right) \right. \right. \\ &\quad \left. \left. - g\left(t_k-, \xi(t_k-), \eta_k, x^{(2)}(t_k-)\right) \right| \right)^2 \\ &\le K \int_0^t \mathbf{E}\left|x^{(1)}(s) - x^{(2)}(s)\right|^2 ds \\ &\quad + 2\mathbf{E}\left(\sum_{k=1}^{N_\varepsilon(t)} \left| g\left(t_k-, \xi(t_k-), \eta_k, x^{(1)}(t_k-)\right) \right. \right. \\ &\quad \left. \left. - g\left(t_k-, \xi(t_k-), \eta_k, x^{(2)}(t_k-)\right) \right| \right)^2 \\ &\quad + 2\mathbf{E}\left(\sum_{k=N_\varepsilon(t)+1}^{\infty} \left| g\left(t_k-, \xi(t_k-), \eta_k, x^{(1)}(t_k-)\right) \right. \right. \\ &\quad \left. \left. - g\left(t_k-, \xi(t_k-), \eta_k, x^{(2)}(t_k-)\right) \right| \right)^2, \end{aligned} \quad (18)$$

where $N(t) = \sup\{k \in \mathbb{N} \mid t_k < t\} + 1$, $N_\varepsilon(t) = N(t) \wedge N_\varepsilon$, and $K = K(T, L)$. Let us use the Lipschitz condition for the second term of the last part of inequality and limited condition for the third one. Then we get

$$\begin{aligned} I(t) &\le K\int_0^t \mathbf{E}\left|x^{(1)}(s)-x^{(2)}(s)\right|^2 ds \\ &\quad + 2N_\varepsilon \sum_{k=1}^{N_\varepsilon(t)} l_k \mathbf{E}\left|x^{(1)}(t_k)-x^{(2)}(t_k)\right|^2 + 2\varepsilon. \end{aligned} \tag{19}$$

According to [11]

$$\begin{aligned} I(t) &\le 2\varepsilon \prod_{k=1}^{N_\varepsilon(t)} \left(1+2N_\varepsilon l_k\right) e^{KT} \\ &\le e^{KT+2N_\varepsilon \sum_{k=1}^{N_\varepsilon(t)} l_k + \ln\varepsilon + \ln 2}. \end{aligned} \tag{20}$$

Then, according to the theorem's condition (3)

$$\lim_{\varepsilon\downarrow 0}\left(KT + 2N_\varepsilon \sum_{k=1}^{N_\varepsilon(t)} l_k + \ln\varepsilon + \ln 2\right) = -\infty; \tag{21}$$

that is, $I(t) = 0$. This completes the proof of uniqueness. Theorem is proved. □

2.2. Model Example. Consider linear stochastic differential-difference equation

$$\begin{aligned} dx(t) &= -a(\xi(t))x(t)\,dt - b(\xi(t))x(t-r)\,dt \\ &\quad + \sigma(\xi(t))\,dw(t), \quad t\ge 0,\ r>0, \end{aligned} \tag{22}$$

with impulse contagion

$$\begin{aligned} \Delta x\left(2-\frac{1}{k}\right) &= x\left(2-\frac{1}{k}-\right) \\ &\quad + e^{-\alpha k\eta_k}\left(\left|x\left(2-\frac{1}{k}-\right)\right| \wedge 1\right), \\ &\qquad k \longrightarrow \infty, \end{aligned} \tag{23}$$

and initial condition

$$\begin{aligned} x(\theta) &= 1, \\ -r &\le \theta \le 0, \\ \xi(0) &= y_0 \in \mathbf{Y}, \\ \eta_0 &= 1. \end{aligned} \tag{24}$$

Here a, b, σ are constants that depend on Markov process ξ with generator $Q = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, where η_k, $k \ge 0$ is Markov chain with two nonabsorbing states $h_1 = 1$ i $h_2 = 2$ and transition matrix $P = \left(\begin{smallmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix}\right)$.

Let us define the values of the parameter α that the solution of the systems (22)–(24) exists.

Define the value of N_ε using equality

$$\sum_{m=k}^{\infty} \gamma_m = \sum_{m=k}^{\infty} e^{-\alpha m} = \frac{e^{-\alpha k}}{1-e^{-\alpha}}. \tag{25}$$

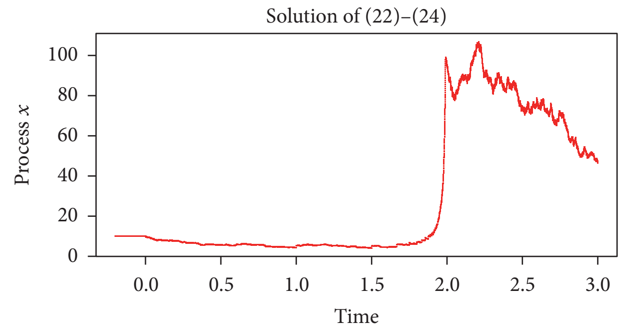

Figure 1

So,

$$N_\varepsilon = \left[-\frac{\ln\varepsilon\left(1-e^{-\alpha}\right)}{\alpha} + 1\right], \tag{26}$$

and the theorem's condition (3) as $\alpha > 1$ is

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\left(\ln\varepsilon + \left[-\frac{\ln\varepsilon\left(1-e^{-\alpha}\right)}{\alpha}+1\right]\sum_{k=1}^{N_\varepsilon} l_k\right) \\ &= \lim_{\varepsilon\downarrow 0}\left(\ln\varepsilon + \left[-\frac{\ln\varepsilon\left(1-e^{-\alpha}\right)}{\alpha}+1\right]\frac{1}{1-e^{-\alpha}}\right) \\ &= -\infty, \end{aligned} \tag{27}$$

as $\alpha(1-e^{-\alpha}) > 1$.

Let us give the R-realization of the problems (22)–(24) solution, using the following values:

If $\xi = 1$: $a = -1$, $b = -0.3$, $\sigma = 0.3$;

If $\xi = 2$: $a = 0.5$, $b = 0.04$, $\sigma = 2.1$;

$\eta_k \in \{1, 2\}$;

$\alpha = 1.673$, $h = 0.0001$, $r = 0.2$, $\varphi \equiv 10$.

As shown in Figure 1, concentration is in the point $t = 2$, and then process's behavior is continuous and stable for $t > 2$.

3. Conclusion

Usually, in mathematical describing of real processes with short-term perturbations evolution one supposes that perturbations are momentary and mathematical model is dynamic system with discontinuous trajectories. In this case the important class of the systems with impulse impact frequency increasing is lost. This paper is one of the important steps in learning of such systems and development of the quality persistence theory and learning the stabilization problem.

On the other hand, this paper significantly expands the class of equalities, for which we can consider the conditions of resistance, existence of periodic and quasi-periodic solutions, and tasks of the optimal control.

Additional Points

Annotation. The sufficient conditions of existence and uniqueness of the strong solution of stochastic dynamic systems with random structure with Markov switching and concentration points are proved in the paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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