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Research Article

An Asymptotic-Numerical Hybrid Method for Solving Singularly Perturbed Linear Delay Differential Equations

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In this work, approximations to the solutions of singularly perturbed second-order linear delay differential equations are studied. We firstly use two-term Taylor series expansion for the delayed convection term and obtain a singularly perturbed ordinary differential equation (ODE). Later, an efficient and simple asymptotic method so called Successive Complementary Expansion Method (SCEM) is employed to obtain a uniformly valid approximation to this corresponding singularly perturbed ODE. As the final step, we employ a numerical procedure to solve the resulting equations that come from SCEM procedure. In order to show efficiency of this numerical-asymptotic hybrid method, we compare the results with exact solutions if possible; if not we compare with the results that are obtained by other reported methods.

1. Introduction

Almost all physical phenomena in nature are modelled using differential equations, and singularly perturbed problems are vital class of these kinds of problems. In general, a singular perturbation problem is defined as a differential equation that is controlled by a positive small parameter $0 < \varepsilon \ll 1$ that exists as multiplier to the highest derivative term in the differential equation. As ε tends to zero, the solution of problem exhibits interesting behaviors (rapid changes) since the order of the equation reduces. The region where these rapid changes occur is called *inner region* and the region in which the solution changes mildly is called *outer region*. As mentioned in [1, 2], these kinds of problems arise in almost all applied natural sciences. Some of these can be given as mechanical and electrical systems, celestial mechanics, fluid and solid mechanics, electromagnetics, particle and quantum physics, chemical and biochemical reactions, and economics and financial mathematics. Various methods are employed to solve singular perturbation problems analytically, numerically, or asymptotically such as the method of matched asymptotic expansions (MMAE), the method of multiple scales, the method of WKB approximation, Poincaré-Lindstedt method and periodic averaging method. Rigorous analysis and applications of these methods can be found in [3–8].

Modelling automatic systems often involve the idea of control because feedback is necessary in order to maintain a stable state. But much of this feedback require a finite time to sense information and react to it. A general way for describing this process is to formulate a delay differential equation (difference-differential equation). Delay differential equations (DDE) are widely used for modelling problems in population dynamics, nonlinear optics, fluid mechanics, mechanical engineering, evolutionary biology, and even modelling of HIV infection and human pupil-light reflex. One can refer to [10–14] for general theory and applications of DDEs.

In this paper, we study an important class of delay differential equations: singularly perturbed linear delay differential equations. A singularly perturbed delay differential equation is a differential equation in which the highest-order derivative is multiplied by a positive small ε parameter and involving at least one delay term. We restrict our attention to singularly perturbed second-order ordinary delay differential equations that contains the delay in convection term. Various methods have been used to solve singularly perturbed DDEs such as finite difference methods [9, 15, 16], finite element methods

[17, 18], homotopy perturbation method [19, 20], reproducing kernel method [21, 22], spline collocation methods [23, 24], and asymptotic approaches [25, 26]. We use an asymptotic-numerical hybrid method in order to find uniformly valid approximations to singularly perturbed ODEs. At the first step, two-term Taylor series expansion is used to vanish delayed term. Secondly, to obtain a uniformly valid approximation an efficient and easily applicable asymptotic method so called Successive Complementary Expansion Method (SCEM) that was introduced in [27] is employed. Finally, a numerical approach is used to solve resulting equations that come from SCEM process.

2. Description of the Method

In this section, we first give a short overview of asymptotic approximations and then explain Successive Complementary Expansion Method by which we obtain highly accurate approximations to solutions of singularly perturbed linear DDEs.

Consider two continuous functions of real numbers $g(\varepsilon)$ and $h(\varepsilon)$ that depend on a positive small parameter ε; then $g(\varepsilon) = O(h(\varepsilon))$ for $\varepsilon \to 0$ if there exist positive constants K and ε_0 such that $K \in (0, \varepsilon_0]$ with $|g(\varepsilon)| \leqslant K|h(\varepsilon)|$ for $\varepsilon \to 0$, and $g(\varepsilon) = o(h(\varepsilon))$ for $\varepsilon \to 0$ if $\lim_{\varepsilon\to 0}(g(\varepsilon)/h(\varepsilon)) = 0$. Let E be a set of real functions that depend on ε, strictly positive and continuous in $(0, \varepsilon_0]$, such that $\lim_{\varepsilon\to 0}\delta(\varepsilon)$ exists and $\delta_1(\varepsilon)\delta_2(\varepsilon) \in E$ for each $\delta_1(\varepsilon), \delta_2(\varepsilon) \in E$. A function $\delta_i(\varepsilon)$ that satisfies these conditions is called *order function*. Given two functions $\phi(x, \varepsilon)$ and $\phi_a(x, \varepsilon)$ defined in a domain Ω are asymptotically identical to order $\delta(\varepsilon)$ if their difference is asymptotically smaller than $\delta(\varepsilon)$, where $\delta(\varepsilon)$ is an order function; that is,

$$\phi(x, \varepsilon) - \phi_a(x, \varepsilon) = o(\delta(\varepsilon)), \tag{1}$$

where ε is a positive small parameter arising from the physical problem under consideration. The function $\phi_a(x, \varepsilon)$ is named as *asymptotic approximation* of the function $\phi(x, \varepsilon)$. Asymptotic approximations in general form are defined by

$$\phi_a(x, \varepsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\,\varphi_i(x, \varepsilon), \tag{2}$$

where $\delta_{i+1}(\varepsilon) = o(\delta_i(\varepsilon))$, as $\varepsilon \to 0$. Under these conditions, the approximation (2) is named as *generalized asymptotic expansion*. If the expansion (2) is written in the form of

$$\phi_a(x, \varepsilon) = E_0\phi = \sum_{i=1}^{n} \delta_i^{(0)}(\varepsilon)\,\varphi_i^{(0)}(x), \tag{3}$$

then it is called *regular asymptotic expansion* where the special operator E_0 is *outer expansion operator* of a given order $\delta(\varepsilon)$. Thus, $\phi - E_0\phi = o(\delta(\varepsilon))$. For more detailed information about the asymptotic approximations, we refer the interested reader to [3–8, 28, 29].

Interesting behaviors occur when the function $\phi(x, \varepsilon)$ is not regular in Ω so (2) or (3) is valid only in a restricted region $\Omega_0 \in \Omega$, called the outer region. We introduce an inner domain which can be formally denoted as $\Omega_1 = \Omega - \Omega_0$ and corresponding inner layer variable, located near the point $x = x_0$, as $\overline{x} = (x - x_0)/\eta(\varepsilon)$, with $\eta(\varepsilon)$ being the order of thickness of the boundary layer (the region in which the rapid changes-behaviors occur). If a regular expansion can be constructed in Ω_1, one can write down the approximation as

$$\phi_a(x, \varepsilon) = E_1\phi = \sum_{i=1}^{n} \delta_i^{(1)}(\varepsilon)\,\varphi_i^{(1)}(\overline{x}), \tag{4}$$

where the *inner expansion operator* E_1, defined in Ω_1, is of the same order $\delta(\varepsilon)$ as the outer expansion operator E_0; that is, $\phi - E_1\phi = o(\delta(\varepsilon))$. Thus,

$$\phi_a = E_0\phi + E_1\phi - E_1E_0\phi \tag{5}$$

is clearly uniformly valid approximation (UVA) [28–30].

Now let us consider second-order singularly perturbed DDE in its general form (delay in the convection term):

$$\varepsilon y''(x) + p(x)\,y'(x - \delta) + q(x)\,y(x) = r(x), \tag{6}$$

where $0 < \varepsilon \ll 1$ small parameter and $0 < x < 1$. Boundary and interval conditions are given as

$$y(x) = \phi(x), \quad -\delta \le x \le 0, \quad y(1) = \gamma, \tag{7}$$

where $p(x), q(x), r(x)$, and $\phi(x)$ are smooth functions, $\gamma \in \mathbb{R}$, and δ is delay term.

As ε tends to zero, the order of the differential equation reduces and so a layer occurs in the solution. The sign of $p(x)$ on the interval $[0, 1]$ determines the type of the layer. If the sign changes on the interval, interior layer behavior occurs in the solution. If the sign of $p(x)$ does not change, there are two possibilities: if $p(x) < 0$ on $[0, 1]$, then a boundary layer occurs at the right end (near the point $x = 1$) and if $p(x) > 0$ on $[0, 1]$, then a boundary layer occurs at the left end (near the point $x = 0$).

Using Taylor series expansion we linearize the convection term; that is, $y'(x - \delta) = y'(x) - \delta y''(x)$ and substituting it into (6) one can reach

$$(\varepsilon - \delta p(x))\,y''(x) + p(x)\,y'(x) + q(x)\,y(x) = r(x). \tag{8}$$

Letting $\delta = \kappa\varepsilon$, where $\kappa \in \mathbb{R}$

$$\varepsilon(1 - \kappa p(x))\,y''(x) + p(x)\,y'(x) + q(x)\,y(x) = r(x) \tag{9}$$

is found and it is clear that (9) is a singularly perturbed ordinary differential equation for $\varepsilon \to 0$ with the same boundary and interval conditions as given by (7). SCEM procedure is applicable at this stage.

The uniformly valid SCEM approximation is in the regular form given by

$$y_n^{\text{scem}}(x, \overline{x}, \varepsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\left[y_i(x) + \Psi_i(\overline{x})\right], \tag{10}$$

where $\{\delta_i(\varepsilon)\}$ is an asymptotic sequence and functions $\Psi_i(\overline{x})$ are the complementary functions that depend on $\overline{x}$.

If the functions $y_i(x)$ and $\Psi_i(\bar{x})$ depend also on ε, the uniformly valid SCEM approximation is called *generalized SCEM approximation* and given by

$$y_{ng}^{\text{scem}}(x,\bar{x},\varepsilon)=\sum_{i=1}^{n}\delta_i(\varepsilon)\left[y_i(x,\varepsilon)+\Psi_i(\bar{x},\varepsilon)\right]. \tag{11}$$

If only one-term SCEM approximation is desired, then one seeks a uniformly valid SCEM approximation in the form of

$$y_1^{\text{scem}}(x,\bar{x},\varepsilon)=y_1(x,\varepsilon)+\Psi_1(\bar{x},\varepsilon). \tag{12}$$

To improve the accuracy of approximation, (12) can be iterated using (11). It means that successive complementary terms will be added to the approximation. To this end, second SCEM approximation will be in the form of

$$\begin{aligned} y_2^{\text{scem}}(x,\bar{x},\varepsilon)&=y_1(x,\varepsilon)+\Psi_1(\bar{x},\varepsilon)\\ &\quad+\varepsilon\left(y_2(x,\varepsilon)+\Psi_2(\bar{x},\varepsilon)\right). \end{aligned} \tag{13}$$

In this work, we seek an approximation in our calculations in the form of (13).

Now, let us assume that problem (9) has a left boundary layer (near the point $x=0$) and let $y_{\text{out}}(x)$ be asymptotic approximation to the outer solution and let $\Psi(\bar{x})$ be the complementary solution, where $\bar{x}=x/\varepsilon$ is boundary layer (stretching) variable. If approximations

$$\begin{aligned} y_{\text{out}}(x,\varepsilon)&=y_1(x)+\varepsilon y_2(x)+\varepsilon^2y_3(x)+\cdots,\\ \Psi(\bar{x},\varepsilon)&=\Psi_1(\bar{x},\varepsilon)+\varepsilon\Psi_2(\bar{x},\varepsilon)+\varepsilon^2\Psi_3(\bar{x},\varepsilon)+\cdots \end{aligned} \tag{14}$$

are substituted into (9) and if each term is balanced with respect to the powers of ε (we balance just the terms $O(1)$ and $O(\varepsilon)$),

$$\begin{aligned} p(x)y_1'(x,\varepsilon)+q(x)y_1(x,\varepsilon)&=r(x),\\ y_1(1,\varepsilon)&=\gamma,\\ y_1''(x,\varepsilon)+p(x)y_2'(x,\varepsilon)+q(x)y_2(x,\varepsilon)&=0,\\ y_2(1,\varepsilon)&=0 \end{aligned} \tag{15}$$

are found. If the same procedure is applied for equations that involve complementary functions

$$\Psi_1''(\bar{x},\varepsilon)+p(x)\Psi_1'(\bar{x},\varepsilon)=r(x) \tag{16}$$

with the boundary conditions

$$\begin{aligned} &\Psi_1(0,\varepsilon)=\phi(0)-\gamma,\\ &\Psi_1\left(\frac{1}{\varepsilon},\varepsilon\right)=0,\\ &\Psi_2''(\bar{x},\varepsilon)-\frac{\kappa}{\varepsilon}p(x)\Psi_1''(\bar{x},\varepsilon)+p(x)\Psi_2'(\bar{x},\varepsilon)\\ &\quad+q(x)\Psi_1(\bar{x},\varepsilon)=0 \end{aligned} \tag{17}$$

with the boundary conditions

$$\begin{aligned} \Psi_1(0,\varepsilon)&=-y_2(0,\varepsilon),\\ \Psi_1\left(\frac{1}{\varepsilon},\varepsilon\right)&=0 \end{aligned} \tag{18}$$

being obtained and so (13) gives uniformly valid second SCEM approximation.

3. Illustrative Examples

3.1. Left Boundary Layer Problem. Consider singularly perturbed DDE that exhibits a boundary layer at the left end of the interval:

$$\varepsilon y''(x)+y'(x-\delta)-y(x)=0,\quad 0\le x\le 1, \tag{19}$$

with the boundary conditions $y(0)=1$ and $y(1)=1$. The exact solution of this problem is given by $y(x)=((1-e^{m_2})e^{m_1x}+(e^{m_1}-1)e^{m_2x})/(e^{m_1}-e^{m_2})$, where $m_{1,2}=(-1\pm\sqrt{1+4(\varepsilon-\delta)})/2(\varepsilon-\delta)$. As the first step, we use two-term Taylor expansion for $y'(x-\delta)=y'(x)-\delta y''(x)$. If we substitute it into (19), the problem turns into

$$(\varepsilon-\delta)y''(x)+y'(x)-y(x)=0,\quad 0\le x\le 1. \tag{20}$$

In order to obtain a uniformly valid approximation (UVA), we first seek an outer solution which is valid far from the boundary layer (the boundary layer is near the point $x=0$ for this problem) and then using SCEM we add complementary solution to it. Later, using the same idea, we will get more accurate approximations.

Outer Region Solutions. Let us take $\theta=\varepsilon-\delta$, assuming that δ depends on ε, and adopt a solution for the outer region in the form of $y_{\text{out}}(x)=y_1(x,\theta)+\theta y_2(x,\theta)$. Equation (20) turns into

$$\begin{aligned} &\theta\left(y_1''(x,\theta)+\theta y_2''(x,\theta)\right)+\left(y_1'(x,\theta)+\theta y_2'(x,\theta)\right)\\ &\quad-\left(y(x,\theta)+\theta y_2(x,\theta)\right)=0 \end{aligned} \tag{21}$$

and balancing the terms of the order $O(1)$ and $O(\theta)$, we reach the equations

$$\begin{aligned} y_1'(x,\theta)-y_1(x,\theta)&=0,\quad y_1(1,\theta)=1,\\ y_1''(x,\theta)+y_2'(x,\theta)-y_2(x,\theta)&=0,\quad y_2(1,\theta)=0. \end{aligned} \tag{22}$$

One can easily find the exact solutions of these equations as

$$\begin{aligned} y_1(x,\theta)&=e^{x-1},\\ y_2(x,\theta)&=e^{x-1}(1-x). \end{aligned} \tag{23}$$

It means that the outer solution is of the form

$$y_{\text{out}}(x,\theta)=e^{x-1}+\theta e^{x-1}(x-1). \tag{24}$$

Complementary Solutions. Now applying the stretching transformation $\overline{x} = x/\theta$ and adopting the complementary solution as $\Psi(\overline{x},\theta) = \Psi_1(\overline{x},\theta) + \theta\Psi_2(\overline{x},\theta)$, one can reach

$$\Psi_1''(\overline{x}) + \theta\Psi_2''(\overline{x},\theta) + \Psi_1'(\overline{x},\theta) + \theta\Psi_2'(\overline{x},\theta) - \theta\Psi_1(\overline{x},\theta) - \theta^2\Psi_2(\overline{x},\theta) = 0. \tag{25}$$

Balancing the terms of the order $O(1)$ and $O(\theta)$, we obtain

$$\Psi_1''(\overline{x},\theta) + \Psi_1'(\overline{x},\theta) = 0, \quad \Psi_1(0,\theta) = 1,\ \Psi_1\left(\frac{1}{\theta},\theta\right) = 0, \tag{26}$$

$$\Psi_2''(\overline{x},\theta) + \Psi_2'(\overline{x},\theta) - \Psi_1(\overline{x},\theta) = 0, \quad \Psi_2(0,\theta) = -e^{-1},\ \Psi_2\left(\frac{1}{\theta},\theta\right) = 0. \tag{27}$$

Here we are able to solve $\Psi_1(\overline{x},\theta)$ and $\Psi_2(\overline{x},\theta)$ exactly, but in many cases to obtain analytical solution to $\Psi_1(\overline{x},\theta)$ and $\Psi_2(\overline{x},\theta)$ is really tedious process, even for many problems it is impossible. The solutions may be given as

$$\Psi_1(\overline{x},\theta) = \frac{e-1}{e(1-e^{-1/\theta})}\left(e^{-\overline{x}} - e^{-1/\theta}\right) \tag{28}$$

and $\Psi_2(\overline{x},\theta)$ is given as the solution of (27). Since we solve (27) numerically (*MATLAB bvp4c* routine) and the others using an asymptotic scheme, our present method is a hybrid method. As a result, we obtain first two SCEM approximations to problem (19) as follows:

$$\begin{aligned} y_1^{\text{scem}}(x,\overline{x},\theta) &= e^{x-1} + \frac{e-1}{e(1-e^{-1/\theta})}\left(e^{-\overline{x}} - e^{-1/\theta}\right), \\ y_2^{\text{scem}}(x,\overline{x},\theta) &= y_1^{\text{scem}}(x,\overline{x},\theta) + \theta\left[e^{x-1}(1-x) + \Psi_2(\overline{x},\theta)\right]. \end{aligned} \tag{29}$$

3.2. Right Boundary Layer Problem. Consider singularly perturbed DDE that exhibits a boundary layer at the right end of the interval

$$\varepsilon y''(x) - y'(x-\delta) + y(x) = 0, \tag{30}$$

with the boundary and interval conditions

$$\begin{aligned} y(x) &= 1, \quad -\delta \le x \le 0, \\ y(1) &= -1. \end{aligned} \tag{31}$$

Using two-term Taylor expansion for the convection term, we reach $y'(x-\delta) = y'(x) - \delta y''(x)$ and applying it to (30) one can obtain

$$(\varepsilon+\delta)y''(x) - y'(x) + y(x) = 0, \quad 0 \le x \le 1 \tag{32}$$

with the boundary conditions $y(0) = 1$ and $y(1) = -1$.

In order to obtain a uniformly valid approximation, we first seek an outer solution which is valid for far from the boundary layer (the boundary layer is near the point $x = 1$ for this problem) and then using SCEM we add complementary solution to it. Later, using the same idea, we will get more accurate approximations.

Outer Region Solutions. Let us take $\theta = \varepsilon + \delta$ assuming that δ depends on ε and adopt an approximation for the outer region in the form of $y_{\text{out}}(x) = y_1(x,\theta) + \theta y_2(x,\theta)$. Equation (32) turns into

$$\theta\left(y_1''(x,\theta) + \theta y_2''(x,\theta)\right) - \left(y_1'(x,\theta) + \theta y_2'(x,\theta)\right) + \left(y(x,\theta) + \theta y_2(x,\theta)\right) = 0; \tag{33}$$

balancing the terms of the order $O(1)$ and $O(\theta)$, we reach the equations

$$\begin{aligned} y_1'(x,\theta) - y_1(x,\theta) &= 0, \quad y_1(0,\theta) = 1, \\ y_1''(x,\theta) - y_2'(x,\theta) + y_2(x,\theta) &= 0, \quad y_2(0,\theta) = 0. \end{aligned} \tag{34}$$

One can easily find the exact solutions of these equations as

$$\begin{aligned} y_1(x,\theta) &= e^x, \\ y_2(x,\theta) &= e^x x. \end{aligned} \tag{35}$$

It means that the outer solution is of the form

$$y_{\text{out}}(x,\theta) = e^x + \theta e^x x. \tag{36}$$

Complementary Solutions. Now applying the stretching transformation $\overline{x} = (x-1)/\theta$ and adopting the complementary solution as $\Psi(\overline{x},\theta) = \Psi_1(\overline{x},\theta) + \theta\Psi_2(\overline{x},\theta)$ one can reach

$$\Psi_1''(\overline{x}) + \theta\Psi_2''(\overline{x},\theta) - \Psi_1'(\overline{x},\theta) - \theta\Psi_2'(\overline{x},\theta) + \theta\Psi_1(\overline{x},\theta) - \theta^2\Psi_2(\overline{x},\theta) = 0. \tag{37}$$

Balancing the terms of the order $O(1)$ and $O(\theta)$ we obtain

$$\Psi_1''(\overline{x},\theta) - \Psi_1'(\overline{x},\theta) = 0, \quad \Psi_1\left(-\frac{1}{\theta},\theta\right) = 0,\ \Psi_1(0,\theta) = -1-e, \tag{38}$$

$$\Psi_2''(\overline{x},\theta) - \Psi_2'(\overline{x},\theta) + \Psi_1(\overline{x},\theta) = 0, \quad \Psi_2\left(-\frac{1}{\theta},\theta\right) = 0,\ \Psi_2(0,\theta) = -1-e. \tag{39}$$

The solutions are given as

$$\Psi_1(\overline{x},\theta) = \frac{e+1}{e^{-1/\theta}}\left(e^{\overline{x}} - 1\right) - e - 1 \tag{40}$$

and $\Psi_2(\overline{x},\theta)$ is given as the solution of (39). Thus, we reach uniformly valid SCEM approximations as

$$\begin{aligned} y_1^{\text{scem}}(x,\overline{x},\theta) &= e^x + \frac{e+1}{e^{-1/\theta}}\left(e^{\overline{x}} - 1\right) - e - 1, \\ y_2^{\text{scem}}(x,\overline{x},\theta) &= y_1^{\text{scem}}(x,\overline{x},\theta) + \theta\left[e^x x + \Psi_2(\overline{x},\theta)\right]. \end{aligned} \tag{41}$$

TABLE 1: Results of left layer problem for $\varepsilon = 10^{-3}$ and $\delta = 0.5\varepsilon$.

x	y_{exact}	y_1^{scem}	$\lvert y_{\text{exact}} - y_1^{\text{scem}}\rvert$	y_2^{scem}	$\lvert y_{\text{exact}} - y_2^{\text{scem}}\rvert$	Method [9]
0.0000	1.0000000	1.0000000	0.0000000	0.0000000	0.0000000	1.0000000
0.0010	0.4538692	0.45379572	$7.3488e-05$	0.4538692	$9.2048e-09$	0.3171426
0.0020	0.3803509	0.38019363	$1.5732e-04$	0.3803510	$9.4728e-08$	0.3603879
0.0030	0.3707303	0.37055161	$1.7865e-04$	0.3707303	$1.2683e-07$	0.3666117
0.0040	0.3697488	0.36956596	$1.8289e-04$	0.3697489	$1.3552e-07$	0.3678324
0.0050	0.3699358	0.36975214	$1.8364e-04$	0.3699359	$1.3753e-07$	0.3683816
0.0100	0.3717605	0.37157669	$1.8379e-04$	0.3717606	$1.3821e-07$	0.3708603
0.0150	0.3736230	0.37343923	$1.8378e-04$	0.3736231	$1.3848e-07$	0.3733556
0.0200	0.3754949	0.37531110	$1.8376e-04$	0.3754950	$1.3869e-07$	0.3758674
0.1000	0.4067525	0.40656966	$1.8281e-04$	0.4067526	$1.4163e-07$	0.4071563
0.2000	0.4495085	0.44932896	$1.7958e-04$	0.4495086	$1.4362e-07$	0.4499552
0.4000	0.5489761	0.54881164	$1.6450e-04$	0.5489762	$1.3978e-07$	0.5495225
0.6000	0.6704540	0.67032005	$1.3394e-04$	0.6704541	$1.2051e-07$	0.6711221
0.8000	0.8188125	0.81873075	$8.1795e-05$	0.8188126	$7.7685e-08$	0.8196300
0.9000	0.9048826	0.90483742	$4.5197e-05$	0.9048826	$4.4056e-08$	0.9057869
1.0000	1.0000000	1.0000000	0.0000000	0.0000000	0.0000000	1.0000000

TABLE 2: Results of left layer problem for $\varepsilon = 10^{-4}$ and $\delta = 0.5\varepsilon$.

x	Exact	y_1^{scem}	$\lvert y_{\text{exact}} - y_1^{\text{scem}}\rvert$	y_2^{scem}	$\lvert y_{\text{exact}} - y_2^{\text{scem}}\rvert$	Method [9]
0.0000	1.0000000	1.0000000	0.0000000	1.0000000	0.0000000	1.0000000
0.0001	0.4534718	0.4534644	$7.3497e-06$	0.4534717	$1.7565e-10$	0.3181581
0.0002	0.3795465	0.3795307	$1.5740e-05$	0.3795464	$1.1979e-09$	0.3612116
0.0003	0.3695746	0.3695566	$1.7877e-05$	0.3695745	$1.2808e-09$	0.3670360
0.0004	0.3682570	0.3682386	$1.8301e-05$	0.3682569	$1.3520e-09$	0.3678237
0.0005	0.3681105	0.3680921	$1.8377e-05$	0.3681105	$1.3718e-09$	0.3679338
0.0010	0.3682659	0.3682475	$1.8392e-05$	0.3682658	$1.3794e-09$	0.3682020
0.0015	0.3684501	0.3684316	$1.8392e-05$	0.3684500	$1.3806e-09$	0.3684702
0.0020	0.3686343	0.3686159	$1.8392e-05$	0.3686343	$1.3794e-09$	0.3687384
0.1000	0.4065880	0.4065696	$1.8294e-05$	0.4065879	$1.4170e-09$	0.4066914
0.2000	0.4493469	0.4493289	$1.7971e-05$	0.4493469	$1.4373e-09$	0.4494485
0.4000	0.5488281	0.5488116	$1.6462e-05$	0.5488281	$1.3990e-09$	0.5489215
0.6000	0.6703334	0.6703200	$1.3405e-05$	0.6703334	$1.2061e-09$	0.6704094
0.8000	0.8187389	0.8187307	$8.1865e-06$	0.8187389	$7.7755e-10$	0.8187855
0.9000	0.9048420	0.9048374	$4.5237e-06$	0.9048419	$4.4095e-10$	0.9048674
1.0000	1.0000000	1.0000000	0.0000000	1.0000000	0.0000000	1.0000000

4. Conclusion

In this paper, singularly perturbed second-order linear delay differential equations that have a delay in the convection term are considered. Firstly, the delayed terms are linearized using two-term Taylor series expansion. Later, an efficient asymptotic method so called Successive Complementary Expansion Method (SCEM) is employed so as to obtain a uniformly valid approximation scheme. At the last stage, the equations that come from the SCEM process are solved by a numerical procedure and so the present method is an asymptotic-numerical hybrid method. The method is easily applicable since it does not require any matching principle in contrast to the well-known method matched asymptotic expansions (MMAE). Highly accurate approximations are obtained in only few iterations and moreover boundary conditions are not satisfied asymptotically, but exactly. In Tables 1 and 2, exact solution, present method approximations, and approximations that are obtained by the method given in [9] are compared and to show the efficiency of present method, results are supported by Figures 1, 3, and 4. In Figures 2 and 5, the delay effects are compared and since the right

Table 3: Results of right layer problem for $\varepsilon = 10^{-3}$ and $\delta = 0.5\varepsilon$.

x	y_1^{scem}	y_2^{scem}	$\lvert y_2^{\text{scem}} - y_1^{\text{scem}} \rvert$	Numerical method
0.0000	1.0000000	1.0000000	0.0000000	1.0000000
0.1000	1.1051709	1.1052261	$5.5258e-05$	1.1052262
0.2000	1.2214027	1.2215248	$1.2214e-04$	1.2215250
0.4000	1.4918246	1.4921230	$2.9836e-04$	1.4921233
0.6000	1.8221188	1.8226654	$5.4663e-04$	1.8226660
0.8000	2.2255409	2.2264311	$8.9021e-04$	2.2264322
0.9000	2.4596031	2.4607099	0.0011068	2.4607112
0.9980	2.6447479	2.6460768	0.0013288	2.6458021
0.9985	2.5290851	2.5303725	0.0012873	2.5299480
0.9990	2.2123501	2.2135226	0.00117247	2.2128733
0.9995	1.3490435	1.3499013	$8.5777e-04$	1.3488799
0.9996	1.0464630	1.0472103	$7.4736e-04$	1.0456402
0.9997	0.6768301	0.6774425	$6.1241e-04$	0.6780147
0.9998	0.2252993	0.2257469	$4.4754e-04$	0.2279200
0.9999	−0.3262616	−0.3260155	$2.4612e-04$	−0.3247246
1.0000	−1.0000000	−1.0000000	0.0000000	−1.0000000

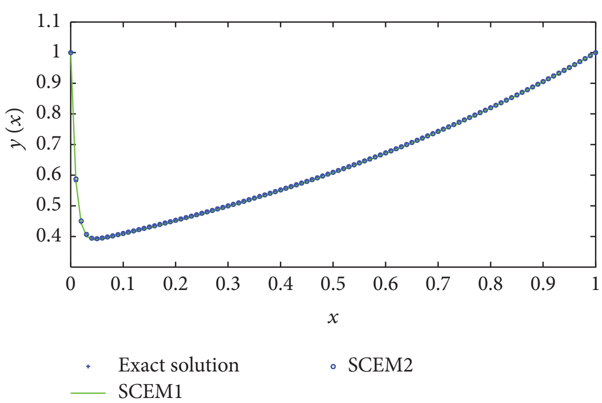

Figure 1: Left layer problem for $\varepsilon = 0.01$ and $\delta = 0.1\varepsilon$.

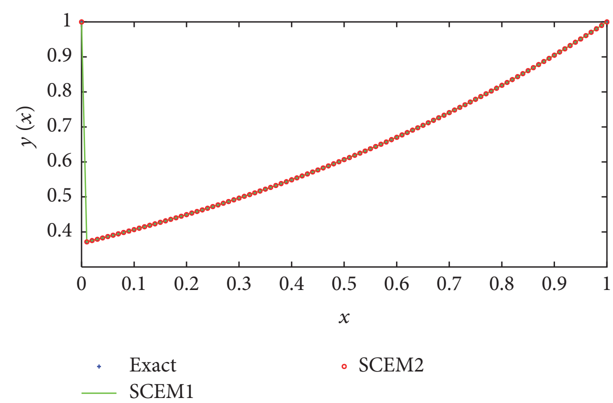

Figure 3: Left layer problem for $\varepsilon = 0.001$ and $\delta = 0.9\varepsilon$.

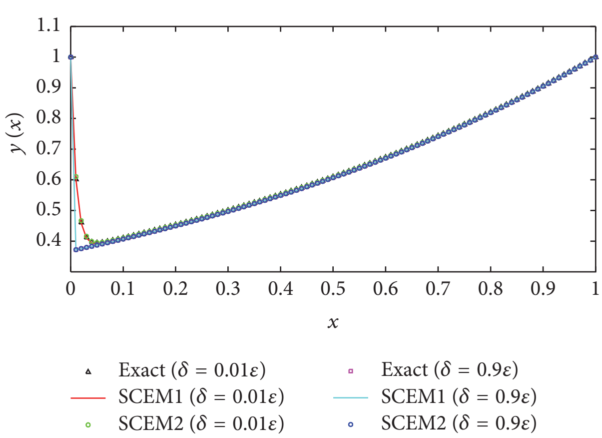

Figure 2: Delay effect on left layer problem for $\varepsilon = 0.01$, $\delta = 0.01\varepsilon$, and $\delta = 0.9\varepsilon$.

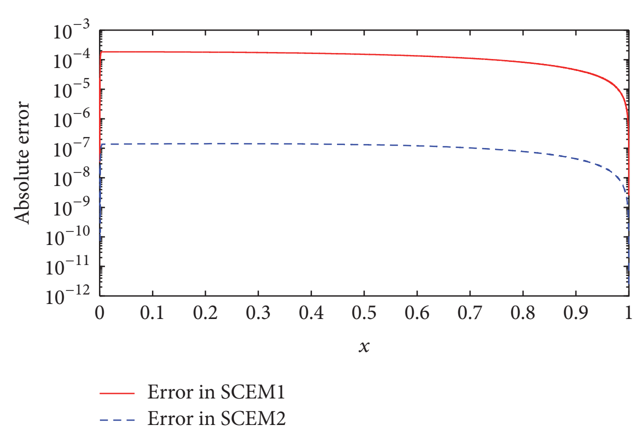

Figure 4: Absolute errors in SCEM approximations for left layer problem.

layer problem does not have an exact solution, the first two SCEM approximations are compared in Table 3. As a result, the present method is a simple and very efficient technique for solving singularly perturbed linear DDEs.

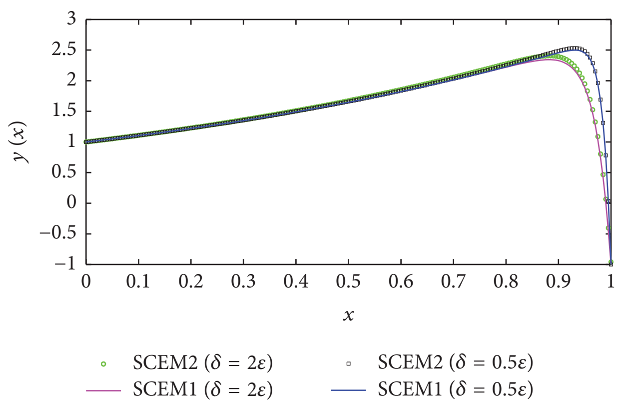

FIGURE 5: Delay effect on right layer problem for $\varepsilon = 0.01$, $\delta = 0.5\varepsilon$, and $\delta = 2\varepsilon$.

Competing Interests

The author declares that there is no conflict of interests.

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