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Research Article

Existence and Uniqueness of Solutions for BVP of Nonlinear Fractional Differential Equation

Cheng-Min Su, Jian-Ping Sun, and Ya-Hong Zhao

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, China

Correspondence should be addressed to Jian-Ping Sun; jpsun@lut.cn

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In this paper, we study the existence and uniqueness of solutions for the following boundary value problem of nonlinear fractional differential equation: $({}^CD_{0+}^q u)(t) = f(t, u(t))$, $t \in (0,1)$, $u(0) = u''(0) = 0$, $({}^CD_{0+}^{\sigma_1} u)(1) = \lambda (I_{0+}^{\sigma_2} u)(1)$, where $2 < q < 3$, $0 < \sigma_1 \le 1$, $\sigma_2 > 0$, and $\lambda \ne \Gamma(2+\sigma_2)/\Gamma(2-\sigma_1)$. The main tools used are nonlinear alternative of Leray-Schauder type and Banach contraction principle.

1. Introduction

Fractional calculus has wide applications in many fields of science and engineering, for example, fluid flow, biosciences, rheology, electrical networks, chemical physics, control theory of dynamical systems, and optics and signal processing [1].

Recently, nonlinear fractional differential equations have been discussed under the following boundary conditions (BCs for short):

(1) Integer derivative BCs:

$$u(0) = u(1) = 0,$$
$$u(0) + u'(0) = 0, u(1) + u'(1) = 0,$$
$$u(0) = u'(1) = u''(0) = 0,$$
$$u(0) = 0, u'(0) + u''(0) = 0, u'(1) + u''(1) = 0,$$
$$u(0) = u_0, u'(0) = u_0^*, u''(T) = u_T,$$
$$u(0) = u'(1) = u''(0) = \cdots = u^{(n-1)}(0) = 0;$$

see papers [2–7], respectively.

(2) Integer derivative and integral BCs:

$$\alpha u(0) - \beta u'(0) = \int_0^1 g(s)u(s)ds, \gamma u(1) + \delta u'(1) = \int_0^1 h(s)u(s)ds,$$
$$u(0) = u'(0) = u''(0) = 0, u(1) = \lambda \int_0^\eta u(s)ds;$$

see papers [8, 9], respectively.

(3) Integer and fractional derivative BCs:

$$u(0) = ({}^CD_{0+}^{\sigma_1} u)(1) = 0, u''(0) = ({}^CD_{0+}^{\sigma_2} u)(1) = 0,$$
$$u(0) = u''(0) = 0, u'(1) = ({}^CD_{0+}^{\sigma} u)(1),$$
$$u(0) = u'(0) = 0, u'(1) = ({}^CD_{0+}^{\sigma} u)(1),$$
$$u(0) = 0, (D_{0+}^{\beta} u)(1) = \sum_{i=1}^{m-2} \xi_i (D_{0+}^{\beta} u)(\eta_i),$$
$$u(0) = 0, u(1) + (D_{0+}^{\beta} u)(1) = ku(\xi) + l(D_{0+}^{\beta} u)(\eta),$$
$$u(0) = 0, (D_{0+}^{\beta} u)(1) = a(D_{0+}^{\beta} u)(\xi),$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, (D_{0+}^{\alpha} u)(1) = 0;$$

see papers [10–16], respectively.

(4) Integer derivative and fractional integral BCs:

$$u(0) = \alpha I_{0+}^{p} u(\eta),$$
$$u(0) = 0, u'(1) = I_{0+}^{\sigma} u(1);$$

see papers [17, 18], respectively.

Besides, there are some other BCs involved in fractional differential equations, such as nonlinear BCs; refer to [19, 20].

Motivated greatly by the above-mentioned works, in this paper, we study the following boundary value problem (BVP for short) of nonlinear fractional differential equation with fractional integral BCs as well as integer and fractional derivative

$$\left({}^{C}D_{0+}^{q}u\right)(t)=f(t,u(t)),\quad t\in(0,1),$$
$$u(0)=u''(0)=0, \tag{1}$$
$$\left({}^{C}D_{0+}^{\sigma_1}u\right)(1)=\lambda\left(I_{0+}^{\sigma_2}u\right)(1),$$

where ${}^{C}D_{0+}^{q}$ and ${}^{C}D_{0+}^{\sigma_1}$ denote the standard Caputo fractional derivatives and $I_{0+}^{\sigma_2}$ denotes the standard Riemann-Liouville fractional integral. Throughout this paper, we always assume that $2<q<3$, $0<\sigma_1\le 1$, $\sigma_2>0$, $\lambda\ne\Gamma(2+\sigma_2)/\Gamma(2-\sigma_1)$, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous.

In order to prove our main results, the following well-known fixed point theorems are needed.

Theorem 1 (nonlinear alternative of Leray-Schauder type [21]). *Let B be a Banach space with $E\subseteq B$ closed and convex. Assume Ω is a relatively open subset of E with $\theta\in\Omega$ and $T:\overline{\Omega}\to E$ is a continuous and compact map. Then either*

(a) *T has a fixed point in $\overline{\Omega}$ or*

(b) *there exists $u\in\partial\Omega$ and $\eta\in(0,1)$ such that $u=\eta Tu$.*

Theorem 2 (Banach contraction principle [22]). *Let (X,d) be a complete metric space and $T: X\to X$ be contractive. Then T has a unique fixed point in X.*

2. Preliminaries

In this section, we always assume that $\mathbb{N}=\{1,2,3,\dots\}$, $\alpha,\beta>0$, and $[\alpha]$ denotes the integer part of α. Now, for the convenience of the reader, we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives and the Caputo fractional derivatives on a finite interval of the real line, which may be found in [1].

Definition 3. The Riemann-Liouville fractional integrals $I_{0+}^{\alpha}u$ and $I_{1-}^{\alpha}u$ of order α on $[0,1]$ are defined by

$$\left(I_{0+}^{\alpha}u\right)(t):=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{u(s)\,ds}{(t-s)^{1-\alpha}},$$
$$\left(I_{1-}^{\alpha}u\right)(t):=\frac{1}{\Gamma(\alpha)}\int_t^1\frac{u(s)\,ds}{(s-t)^{1-\alpha}}, \tag{2}$$

respectively.

Definition 4. The Riemann-Liouville fractional derivatives $D_{0+}^{\alpha}u$ and $D_{1-}^{\alpha}u$ of order α on $[0,1]$ are defined by

$$\left(D_{0+}^{\alpha}u\right)(t):=\left(\frac{d}{dt}\right)^n\left(I_{0+}^{n-\alpha}u\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{u(s)\,ds}{(t-s)^{\alpha-n+1}},$$

$$\left(D_{1-}^{\alpha}u\right)(t):=\left(-\frac{d}{dt}\right)^n\left(I_{1-}^{n-\alpha}u\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n\int_t^1\frac{u(s)\,ds}{(s-t)^{\alpha-n+1}}, \tag{3}$$

respectively, where $n=[\alpha]+1$.

Definition 5. Let $D_{0+}^{\alpha}[u(s)](t)\equiv(D_{0+}^{\alpha}u)(t)$ and $D_{1-}^{\alpha}[u(s)](t)\equiv(D_{1-}^{\alpha}u)(t)$ be the Riemann-Liouville fractional derivatives of order α. Then the Caputo fractional derivatives ${}^{C}D_{0+}^{\alpha}u$ and ${}^{C}D_{1-}^{\alpha}u$ of order α on $[0,1]$ are defined by

$$\left({}^{C}D_{0+}^{\alpha}u\right)(t):=\left(D_{0+}^{\alpha}\left[u(s)-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}s^k\right]\right)(t),$$
$$\left({}^{C}D_{1-}^{\alpha}u\right)(t):=\left(D_{1-}^{\alpha}\left[u(s)-\sum_{k=0}^{n-1}\frac{u^{(k)}(1)}{k!}(1-s)^k\right]\right)(t), \tag{4}$$

respectively, where

$$n=\begin{cases}[\alpha]+1, & \alpha\notin\mathbb{N},\\ \alpha, & \alpha\in\mathbb{N}.\end{cases} \tag{5}$$

Lemma 6 (see [23]). *If $\alpha+\beta>1$, then the equation $(I_{0+}^{\alpha}I_{0+}^{\beta}u)(t)=(I_{0+}^{\alpha+\beta}u)(t)$, $t\in[0,1]$, is satisfied for $u\in L_1[0,1]$.*

Lemma 7 (see [23]). *Let $\beta>\alpha$. Then the equation $({}^{C}D_{0+}^{\alpha}I_{0+}^{\beta}u)(t)=(I_{0+}^{\beta-\alpha}u)(t)$, $t\in[0,1]$, is satisfied for $u\in C[0,1]$.*

Lemma 8 (see [1]). *Let n be given by (5). Then the following relations hold:*

(1) *For $k\in\{0,1,2,\dots,n-1\}$, ${}^{C}D_{0+}^{\alpha}t^k=0$.*

(2) *If $\beta>n$, then ${}^{C}D_{0+}^{\alpha}t^{\beta-1}=(\Gamma(\beta)/\Gamma(\beta-\alpha))t^{\beta-\alpha-1}$.*

Lemma 9 (see [1]). *Let n be given by (5) and $u\in C^n[0,1]$. Then*

$$\left(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}u\right)(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{6}$$

where $c_i\in\mathbb{R}$, $i=0,1,\dots,n-1$.

For any $x\in L_1[0,1]$, we define

$$\|x\|_{L_1}=\int_0^1|x(t)|\,dt. \tag{7}$$

Lemma 10. *Let $u\in L_1[0,1]$ be nonnegative. Then $(I_{0+}^{\alpha+1}u)(t)\le\|I_{0+}^{\alpha}u\|_{L_1}$, $t\in[0,1]$.*

Proof. For any $t \in [0,1]$, we have

$$\begin{aligned}\left(I_{0+}^{\alpha+1}u\right)(t) &= \frac{1}{\Gamma(\alpha+1)}\int_0^t \frac{u(s)}{(t-s)^{-\alpha}}ds \\ &= \frac{1}{\alpha\Gamma(\alpha)}\int_0^t u(s)(t-s)^{\alpha}\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t u(s)\int_s^t (r-s)^{\alpha-1}\,dr\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^r \frac{u(s)}{(r-s)^{1-\alpha}}ds\,dr \\ &\le \int_0^1 \frac{1}{\Gamma(\alpha)}\int_0^r \frac{u(s)}{(r-s)^{1-\alpha}}ds\,dr \\ &= \int_0^1 \left(I_{0+}^{\alpha}u\right)(r)\,dr = \left\|I_{0+}^{\alpha}u\right\|_{L_1}.\end{aligned} \tag{8}$$

□

3. Main Results

In the remainder of this paper, for any nonnegative function $g \in L_1[0,1]$, we denote

$$M_g = \left\|I_{0+}^{q-1}g\right\|_{L_1} + \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{\left|\lambda\Gamma(2-\sigma_1)-\Gamma(2+\sigma_2)\right|}\left[|\lambda|\left(I_{0+}^{q+\sigma_2}g\right)(1) + \left(I_{0+}^{q-\sigma_1}g\right)(1)\right] \tag{9}$$

and for any $y \in C[0,1]$, we use the norm

$$\|y\|_{\infty} = \max_{t\in[0,1]}|y(t)|. \tag{10}$$

Lemma 11. *Let $y \in C[0,1]$ be a given function. Then the BVP*

$$\begin{gathered}\left({}^{C}D_{0+}^{q}u\right)(t) = y(t), \quad t \in (0,1), \\ u(0) = u''(0) = 0, \\ \left({}^{C}D_{0+}^{\sigma_1}u\right)(1) = \lambda\left(I_{0+}^{\sigma_2}u\right)(1)\end{gathered} \tag{11}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\,y(s)\,ds, \quad t \in [0,1], \tag{12}$$

where

$$G(t,s) = -\frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{\lambda\Gamma(2-\sigma_1)-\Gamma(2+\sigma_2)} \cdot t\left[\frac{\lambda(1-s)^{q+\sigma_2-1}}{\Gamma(q+\sigma_2)} - \frac{(1-s)^{q-\sigma_1-1}}{\Gamma(q-\sigma_1)}\right] + \begin{cases}\dfrac{(t-s)^{q-1}}{\Gamma(q)}, & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1.\end{cases} \tag{13}$$

Proof. It follows from the equation in (11) and Lemma 9 that

$$u(t) = \left(I_{0+}^{q}y\right)(t) - c_0 - c_1 t - c_2 t^2, \quad t \in [0,1]. \tag{14}$$

So,

$$u'(t) = \left(I_{0+}^{q-1}y\right)(t) - c_1 - 2c_2 t, \quad t \in [0,1], \tag{15}$$

$$u''(t) = \left(I_{0+}^{q-2}y\right)(t) - 2c_2, \quad t \in [0,1]. \tag{16}$$

In view of (14), (16), and the BCs $u(0) = u''(0) = 0$, we get

$$c_0 = c_2 = 0, \tag{17}$$

and so,

$$u(t) = \left(I_{0+}^{q}y\right)(t) - c_1 t, \quad t \in [0,1]. \tag{18}$$

Then, by using Lemmas 6, 7, and 8, we may obtain

$$\begin{aligned}\left({}^{C}D_{0+}^{\sigma_1}u\right)(t) &= \left(I_{0+}^{q-\sigma_1}y\right)(t) - c_1\frac{\Gamma(2)}{\Gamma(2-\sigma_1)}t^{1-\sigma_1}, \quad t \in [0,1], \\ \left(I_{0+}^{\sigma_2}u\right)(t) &= \left(I_{0+}^{q+\sigma_2}y\right)(t) - c_1\frac{\Gamma(2)}{\Gamma(2+\sigma_2)}t^{1+\sigma_2}, \quad t \in [0,1],\end{aligned} \tag{19}$$

which together with the BC $({}^{C}D_{0+}^{\sigma_1}u)(1) = \lambda(I_{0+}^{\sigma_2}u)(1)$ implies that

$$c_1 = \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{\lambda\Gamma(2-\sigma_1)-\Gamma(2+\sigma_2)}\left[\lambda\left(I_{0+}^{q+\sigma_2}y\right)(1) - \left(I_{0+}^{q-\sigma_1}y\right)(1)\right]. \tag{20}$$

Therefore, the BVP (11) has a unique solution

$$\begin{aligned}u(t) &= \left(I_{0+}^{q}y\right)(t) - \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{\lambda\Gamma(2-\sigma_1)-\Gamma(2+\sigma_2)}\left[\lambda\left(I_{0+}^{q+\sigma_2}y\right)(1) - \left(I_{0+}^{q-\sigma_1}y\right)(1)\right]t \\ &= \int_0^t\left\{-\frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{\lambda\Gamma(2-\sigma_1)-\Gamma(2+\sigma_2)} \cdot t\left[\frac{\lambda(1-s)^{q+\sigma_2-1}}{\Gamma(q+\sigma_2)} - \frac{(1-s)^{q-\sigma_1-1}}{\Gamma(q-\sigma_1)}\right] + \frac{(t-s)^{q-1}}{\Gamma(q)}\right\} \\ &\quad \cdot y(s)\,ds + \int_t^1\left\{-\frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{\lambda\Gamma(2-\sigma_1)-\Gamma(2+\sigma_2)} \cdot t\left[\frac{\lambda(1-s)^{q+\sigma_2-1}}{\Gamma(q+\sigma_2)} - \frac{(1-s)^{q-\sigma_1-1}}{\Gamma(q-\sigma_1)}\right]\right\}y(s)\,ds \\ &= \int_0^1 G(t,s)\,y(s)\,ds, \quad t \in [0,1].\end{aligned} \tag{21}$$

□

Lemma 12. *Let $g \in L_1[0,1]$ be nonnegative. Then*

$$\int_0^1 |G(t,s)| g(s)\, ds \le M_g, \quad t \in [0,1]. \tag{22}$$

Proof. In view of Lemma 10, we have

$$\begin{aligned}
\int_0^1 |G(t,s)| g(s)\, ds &= \int_0^t |G(t,s)| g(s)\, ds \\
&+ \int_t^1 |G(t,s)| g(s)\, ds \\
&\le \int_0^t \Bigg\{ \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{|\lambda\Gamma(2-\sigma_1) - \Gamma(2+\sigma_2)|} \\
&\cdot t \left[\frac{|\lambda| (1-s)^{q+\sigma_2-1}}{\Gamma(q+\sigma_2)} + \frac{(1-s)^{q-\sigma_1-1}}{\Gamma(q-\sigma_1)} \right] \\
&+ \frac{(t-s)^{q-1}}{\Gamma(q)} \Bigg\} g(s)\, ds \\
&+ \int_t^1 \Bigg\{ \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{|\lambda\Gamma(2-\sigma_1) - \Gamma(2+\sigma_2)|} \\
&\cdot t \left[\frac{|\lambda| (1-s)^{q+\sigma_2-1}}{\Gamma(q+\sigma_2)} + \frac{(1-s)^{q-\sigma_1-1}}{\Gamma(q-\sigma_1)} \right] \Bigg\} g(s)\, ds \\
&= \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds \\
&+ \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{|\lambda\Gamma(2-\sigma_1) - \Gamma(2+\sigma_2)|} t \left[\frac{|\lambda|}{\Gamma(q+\sigma_2)} \right. \\
&\cdot \int_0^1 \frac{g(s)}{(1-s)^{1-q-\sigma_2}} ds + \frac{1}{\Gamma(q-\sigma_1)} \\
&\left. \cdot \int_0^1 \frac{g(s)}{(1-s)^{1-q+\sigma_1}} ds \right] = (I_{0+}^q g)(t) \\
&+ \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{|\lambda\Gamma(2-\sigma_1) - \Gamma(2+\sigma_2)|} t \left[|\lambda| \left(I_{0+}^{q+\sigma_2} g \right)(1) \right. \\
&\left. + \left(I_{0+}^{q-\sigma_1} g \right)(1) \right] \le \left\| I_{0+}^{q-1} g \right\|_{L_1} \\
&+ \frac{\Gamma(2+\sigma_2)\Gamma(2-\sigma_1)}{|\lambda\Gamma(2-\sigma_1) - \Gamma(2+\sigma_2)|} \left[|\lambda| \left(I_{0+}^{q+\sigma_2} g \right)(1) \right. \\
&\left. + \left(I_{0+}^{q-\sigma_1} g \right)(1) \right] = M_g, \quad t \in [0,1].
\end{aligned} \tag{23}$$

□

Now, we define an operator $T : C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad t \in [0,1]. \tag{24}$$

Obviously, u is a solution of the BVP (1) if and only if u is a fixed point of T.

Theorem 13. *Assume that $f(t,0) \not\equiv 0$, $t \in (0,1)$, and there exist nonnegative functions $g_1, g_2 \in L_1[0,1]$, nonnegative increasing continuous function ϕ defined on $[0,+\infty)$, and $r > 0$ such that*

$$\left| f(t,x) \right| \le g_1(t) + g_2(t)\phi(|x|), \quad (t,x) \in [0,1] \times \mathbb{R}, \tag{25}$$

$$M_{g_1} + \phi(r) M_{g_2} < r. \tag{26}$$

Then the BVP (1) has one nontrivial solution.

Proof. Let $\Omega = \{u \in C[0,1] : \|u\|_\infty < r\}$. Since $G(t,s)$ and $f(t,x)$ are continuous on $[0,1] \times [0,1]$ and $[0,1] \times \mathbb{R}$, respectively, we may denote

$$L = \max_{(t,s)\in[0,1]\times[0,1]} |G(t,s)|, \tag{27}$$

$$H = \max_{(t,x)\in[0,1]\times[-r,r]} \left| f(t,x) \right|. \tag{28}$$

First, we prove that $T: \overline{\Omega} \to C[0,1]$ is continuous. Suppose that u_n $(n = 1, 2, \ldots)$, $u_0 \in \overline{\Omega}$, and $\|u_n - u_0\|_\infty \to 0$ $(n \to \infty)$. Then for any n and $s \in [0,1]$, we have $|u_n(s)| \le r$. This together with (27) and (28) implies that, for any n and $t \in [0,1]$,

$$\left| G(t,s) f(s, u_n(s)) \right| \le LH, \quad s \in [0,1]. \tag{29}$$

By applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\lim_{n\to\infty} (Tu_n)(t) &= \lim_{n\to\infty} \int_0^1 G(t,s) f(s, u_n(s))\, ds \\
&= \int_0^1 G(t,s) f(s, u_0(s))\, ds \\
&= (Tu_0)(t), \quad t \in [0,1],
\end{aligned} \tag{30}$$

which indicates that $T: \overline{\Omega} \to C[0,1]$ is continuous.

Next, we show that $T: \overline{\Omega} \to C[0,1]$ is compact. Assume that K is a subset of $\overline{\Omega}$. Then for any $u \in K$, we have

$$|u(s)| \le r, \quad s \in [0,1]. \tag{31}$$

In what follows, we will prove that $T(K)$ is relatively compact. On the one hand, for any $y \in T(K)$, there exists $u \in K$ such that $y = Tu$, and so, it follows from (27), (28), and (31) that

$$\begin{aligned}
\left| y(t) \right| = |(Tu)(t)| &= \left| \int_0^1 G(t,s) f(s, u(s))\, ds \right| \\
&\le \int_0^1 |G(t,s)| \left| f(s, u(s)) \right| ds \le LH, \\
&\qquad t \in [0,1],
\end{aligned} \tag{32}$$

which shows that $T(K)$ is uniformly bounded. On the other hand, for any $\varepsilon > 0$, since $G(t,s)$ is uniformly continuous on

$[0,1]\times[0,1]$, there exists $\delta > 0$ such that, for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$,

$$\left|G(t_1, s) - G(t_2, s)\right| < \frac{\varepsilon}{H}, \quad s \in [0,1]. \tag{33}$$

For any $y \in T(K)$, there exists $u \in K$ such that $y = Tu$, and so, for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, it follows from (28), (31), and (33) that

$$\begin{aligned} \left|y(t_1) - y(t_2)\right| &= \left|(Tu)(t_1) - (Tu)(t_2)\right| \\ &= \left|\int_0^1 \left[G(t_1, s) - G(t_2, s)\right] f(s, u(s))\, ds\right| \\ &\le \int_0^1 \left|G(t_1, s) - G(t_2, s)\right| \left|f(s, u(s))\right| ds \\ &\le H \int_0^1 \left|G(t_1, s) - G(t_2, s)\right| ds < \varepsilon, \end{aligned} \tag{34}$$

which indicates that $T(K)$ is equicontinuous. By Arzela-Ascoli theorem, we know that $T(K)$ is relatively compact. Therefore, $T\colon \overline{\Omega} \to C[0,1]$ is compact.

Now, we will prove that (a) of Theorem 1 is fulfilled. Suppose on the contrary that (b) of Theorem 1 is satisfied; that is, there exists $u \in \partial\Omega$ and $\eta \in (0,1)$ such that $u = \eta T u$. Then, in view of (25), (26), and Lemma 12, we have

$$\begin{aligned} |u(t)| &= \left|\eta (Tu)(t)\right| \le |(Tu)(t)| \\ &= \left|\int_0^1 G(t, s) f(s, u(s))\, ds\right| \\ &\le \int_0^1 |G(t, s)| \left|f(s, u(s))\right| ds \\ &\le \int_0^1 |G(t, s)| \left[g_1(s) + g_2(s)\phi(|u(s)|)\right] ds \\ &\le \int_0^1 |G(t, s)|\, g_1(s)\, ds \\ &\quad + \phi(r) \int_0^1 |G(t, s)|\, g_2(s)\, ds \\ &\le M_{g_1} + \phi(r) M_{g_2} < r, \quad t \in [0,1], \end{aligned} \tag{35}$$

which shows that

$$\|u\|_\infty < r. \tag{36}$$

This contradicts the fact $u \in \partial\Omega$.

So, it follows from Theorem 1 that T has a fixed point u^*, which is a desired solution of the BVP (1). At the same time, since $f(t, 0) \not\equiv 0$, $t \in (0,1)$, we know that the zero function is not a solution of the BVP (1). Therefore, u^* is a nontrivial solution of the BVP (1). □

Theorem 14. *Assume that there exists a nonnegative function $g_3 \in L_1[0,1]$ such that*

$$\left|f(t, x) - f(t, y)\right| \le g_3(t) \left|x - y\right|, \quad t \in [0,1],\ x, y \in \mathbb{R}, \tag{37}$$

$$M_{g_3} < 1. \tag{38}$$

Then the BVP (1) has a unique solution.

Proof. For any $u, v \in C[0,1]$, in view of (37) and Lemma 12, we have

$$\begin{aligned} &|(Tu)(t) - (Tv)(t)| \\ &\quad = \left|\int_0^1 G(t, s)\left[f(s, u(s)) - f(s, v(s))\right] ds\right| \\ &\quad \le \int_0^1 |G(t, s)| \left|f(s, u(s)) - f(s, v(s))\right| ds \\ &\quad \le \int_0^1 |G(t, s)|\, g_3(s)\, |u(s) - v(s)|\, ds \\ &\quad \le \|u - v\|_\infty \int_0^1 |G(t, s)|\, g_3(s)\, ds \le M_{g_3} \|u - v\|_\infty, \\ &\qquad t \in [0,1]. \end{aligned} \tag{39}$$

This indicates that

$$\|Tu - Tv\|_\infty \le M_{g_3} \|u - v\|_\infty, \tag{40}$$

which together with (38) implies that T is contractive. So, it follows from Theorem 2 that T has a unique fixed point, and so, the BVP (1) has a unique solution. □

Example 15. We consider the BVP

$$\begin{gathered} \left({}^{C}D_{0+}^{5/2} u\right)(t) = t - \frac{t}{2}\sqrt{|u(t)|}, \quad t \in (0,1), \\ u(0) = u''(0) = 0, \\ \left({}^{C}D_{0+}^{1/2} u\right)(1) = \frac{1}{2}\left(I_{0+}^{3/2} u\right)(1). \end{gathered} \tag{41}$$

Let $f(t, x) = t - (t/2)\sqrt{|x|}$, $(t, x) \in [0,1] \times \mathbb{R}$. Then $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t, 0) \ne 0$, $t \in (0,1)$.

If we choose $g_1(t) = t$, $g_2(t) = t/2$, $t \in [0,1]$, and $\phi(y) = \sqrt{y}$, $y \in [0, +\infty)$, then it is easy to verify that (25) is satisfied.

Since $q = 5/2$, $\sigma_1 = 1/2$, $\sigma_2 = 3/2$, and $\lambda = 1/2$, a direct calculation shows that

$$\begin{aligned} \frac{\Gamma(2 + \sigma_2)}{\Gamma(2 - \sigma_1)} &= \frac{15}{4}, \\ M_{g_1} &= \frac{6656 + 4305\pi}{43680\sqrt{\pi}}, \\ M_{g_2} &= \frac{6656 + 4305\pi}{87360\sqrt{\pi}}. \end{aligned} \tag{42}$$

If we choose $r = 1$, then (26) is fulfilled.

Therefore, it follows from Theorem 13 that the BVP (41) has one nontrivial solution.

Example 16. We consider the BVP

$$\left({}^{C}D_{0+}^{5/2}u\right)(t) = \frac{t}{\pi}\left\{u(t)\arctan u(t) - \frac{1}{2}\ln\left[1+u^{2}(t)\right]\right\}, \quad t \in (0,1), \quad (43)$$

$$u(0) = u''(0) = 0,$$

$$\left({}^{C}D_{0+}^{1/2}u\right)(1) = \frac{1}{2}\left(I_{0+}^{3/2}u\right)(1).$$

Let $f(t,x) = (t/\pi)[x\arctan x - (1/2)\ln(1+x^2)]$, $(t,x) \in [0,1]\times\mathbb{R}$. Then $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous.

If we choose $g_3(t) = t/2$, $t \in [0,1]$, then we may assert that (37) is satisfied. In fact, for any $t \in [0,1]$, if $x = y$, then (37) is obvious. When $x \neq y$, we may suppose that $x < y$. In this case, by Lagrange mean value theorem, there exists $\xi \in (x,y)$ such that, for any $t \in [0,1]$,

$$\left|f(t,x) - f(t,y)\right| = \frac{t}{\pi}\left|x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) - y\arctan y + \frac{1}{2}\ln\left(1+y^2\right)\right| = \frac{t}{\pi}\left|\arctan\xi\right|\left|x - y\right| \le g_3(t)\left|x-y\right|; \quad (44)$$

that is, (37) is satisfied.

On the other hand, in view of $M_{g_3} = M_{g_2} = (6656 + 4305\pi)/87360\sqrt{\pi}$, we know that (38) is fulfilled.

Therefore, it follows from Theorem 14 that the BVP (43) has a unique solution.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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