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Research Article

Bifurcation and Global Dynamics of a Leslie-Gower Type Competitive System of Rational Difference Equations with Quadratic Terms

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We investigate global dynamics of the following systems of difference equations $x_{n+1} = x_n/(A_1 + B_1 x_n + C_1 y_n)$, $y_{n+1} = y_n^2/(A_2 + B_2 x_n + C_2 y_n^2)$, $n = 0, 1, \ldots$, where the parameters A_1, A_2, B_1, B_2, C_1, and C_2 are positive numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers. This system is a version of the Leslie-Gower competition model for two species. We show that this system has rich dynamics which depends on the part of parametric space.

1. Introduction

In this paper we study the global dynamics of the following rational system of difference equations:

$$x_{n+1} = \frac{x_n}{A_1 + B_1 x_n + C_1 y_n}, \quad y_{n+1} = \frac{y_n^2}{A_2 + B_2 x_n + C_2 y_n^2}, \quad n = 0, 1, \ldots, \tag{1}$$

where the parameters A_1, A_2, B_1, B_2, C_1, and C_2 are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

System (1) is a competitive system, and our results are based on recent results about competitive systems in the plane; see [1]. System (1) can be used as a mathematical model for competition in population dynamics. System (1) is related to Leslie-Gower competition model

$$x_{n+1} = \frac{x_n}{A_1 + B_1 x_n + C_1 y_n}, \quad y_{n+1} = \frac{y_n}{A_2 + B_2 x_n + C_2 y_n}, \quad n = 0, 1, \ldots, \tag{2}$$

where the parameters A_1, A_2, B_1, B_2, C_1, and C_2 are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers, considered in [2]. System (2) globally exhibits three dynamic scenarios in five parametric regions which are competitive exclusion, competitive coexistence, and existence of an infinite number of equilibrium solutions; see [1–3]. System (2) does not exhibit the Allee effect, which is desirable from modeling point of view. The simplest variation of system (2) which exhibits the Allee effect is probably system

$$x_{n+1} = \frac{x_n^2}{A_1 + B_1 x_n^2 + C_1 y_n}, \quad y_{n+1} = \frac{y_n^2}{A_2 + B_2 x_n + C_2 y_n^2}, \quad n = 0, 1, \ldots, \tag{3}$$

where the parameters A_1, A_2, B_1, B_2, C_1, and C_2 are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers, considered in [4]. System (3) has between 1 and 9 equilibrium points and exhibits nine dynamics scenarios part of each is the Allee effect. In the case of the dynamic scenario with nine equilibrium points system (3) exhibits both competitive exclusion and competitive coexistence as well as the Allee effect. Another system with quadratic terms is

$$x_{n+1} = \frac{x_n^2}{B_1 x_n^2 + C_1 y_n^2},$$
$$y_{n+1} = \frac{y_n^2}{A_2 + B_2 x_n^2 + C_2 y_n^2}, \tag{4}$$
$$n = 0, 1, \ldots,$$

where the parameters A_2, B_1, B_2, C_1, and C_2 are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$, considered in [5]. System (4) exhibits seven scenarios part of each is singular Allee's effect, which means that the origin as the singular point of this system still has some basin of attraction. First systematic study for a system with quadratic terms was performed in [6] for system

$$x_{n+1} = \frac{x_n}{A_1 + y_n^2},$$
$$y_{n+1} = \frac{y_n}{A_2 + x_n^2}, \tag{5}$$
$$n = 0, 1, \ldots$$

which exhibits nine dynamic scenarios and whose dynamics is very similar to the corresponding system without quadratic terms considered in [7].

In general, it seems that an introduction of quadratic terms in equations of the Leslie-Gower model (2) generates the Allee effect. We will test this hypothesis in this paper by introducing the quadratic terms only in the second equation. System (1) can be considered as the competitive version of the decoupled system

$$x_{n+1} = \frac{x_n}{A_1 + B_1 x_n},$$
$$y_{n+1} = \frac{y_n^2}{A_2 + C_2 y_n^2}, \tag{6}$$
$$n = 0, 1, \ldots,$$

where the parameters A_1, A_2, B_1, and C_2 are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers, whose dynamics can be directly obtained from two separate equations. Unlike system (2) which has five regions of parameters with distinct local behavior system (1) has eighteen regions of parameters with distinct local behavior, which is caused by the geometry of the problem, that is, by the geometry of equilibrium curves. More precisely, the equilibrium curves of system (2) are lines while the equilibrium curves of system (1) are a line and a parabola. In the case when $A_1 > 1$, all equilibrium points are hyperbolic and all solutions are attracted to the three equilibrium points on the y-axis and we can describe this situation as competitive exclusion case. When $A_1 = 1$, the equilibrium point E_1 is nonhyperbolic and dynamics is analogous to the case when $A_1 > 1$. In both cases the Allee effect is present. When $A_1 < 1$, there exist 11 regions of parameters with different global dynamics. In nine of these regions the global dynamics is in competitive exclusion case, which means that all solutions converge to one of the equilibrium points on the axes and in only two situations we have competitive coexistence case, which means that the interior equilibrium points have substantial basin of attraction. In all 11 cases, the zero equilibrium has some basin of attraction which is a part of y-axis so we can say that in these cases system (1) exhibits weak Allee's effect. Figure 3 gives the bifurcation diagram showing the transition from different global dynamics situations when $A_1 < 1$, since the cases $A_1 \geq 1$ are simple and do not need graphical interpretation.

The paper is organized as follows. Section 2 contains some necessary results on competitive systems in the plane. Section 3 provides some basic information about the number of equilibrium points. Section 4 contains local stability analysis of all equilibrium solutions. Section 5 contains some global results on injectivity of the map associated with system (1). Section 6 gives global dynamics of system (1) in all regions of the parameters.

2. Preliminaries

A first-order system of difference equations

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n), \tag{7}$$
$$n = 0, 1, \ldots,$$

where $\mathcal{S} \subset \mathbb{R}^2$, $(f, g) : \mathcal{S} \to \mathcal{S}$, f, g are continuous functions is *competitive* if $f(x, y)$ is nondecreasing in x and nonincreasing in y, and $g(x, y)$ is nonincreasing in x and nondecreasing in y. If both f and g are nondecreasing in x and y, system (7) is *cooperative*. Competitive and cooperative maps are defined similarly. *Strongly competitive* systems of difference equations or strongly competitive maps are those for which the functions f and g are coordinate-wise strictly monotone.

Competitive and cooperative systems have been investigated by many authors; see [1–3, 7–16]. Special attention to discrete competitive and cooperative systems in the plane was given in [1–3, 16, 17]. One of the reasons for paying special attention to two-dimensional discrete competitive and cooperative systems is their applicability and the fact that many examples of mathematical models in biology and economy which involve competition or cooperation are models which involve two species. Another reason is that the theory of two-dimensional discrete competitive and cooperative systems is very well developed, unlike such theory for three-dimensional and higher systems. Part of the reason for this situation is de Mottoni-Schiaffino theorem given below, which provides relatively simple scenarios for possible behavior of many two-dimensional discrete competitive and cooperative systems. However, this does not mean that one can not encounter chaos in such systems as has been shown by Smith; see [16].

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1,2,3,4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, that is, $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \geq u,\ y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \leq u,\ y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{\text{se}}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{\text{se}} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{\text{ne}}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{\text{ne}} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, define the *distance from x to $\mathcal{A}$* as $\operatorname{dist}(x, \mathcal{A}) := \inf\{\|x - y\| : y \in \mathcal{A}\}$. By $\operatorname{int}\mathcal{A}$ we denote the interior of a set $\mathcal{A}$.

It is easy to show that a map F is competitive if it is nondecreasing with respect to the South-East partial order, that is, if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{\text{se}} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Longrightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{\text{se}} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{8}$$

For standard definitions of attracting fixed point, saddle point, stable manifold, and related notions see [10].

We now state three results for competitive maps in the plane. The following definition is from [16].

Definition 1. Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition (O+) if for every x, y in $\mathcal{S}$, $T(x) \preceq_{\text{ne}} T(y)$ implies $x \preceq_{\text{ne}} y$, and T is said to satisfy condition (O−) if for every x, y in $\mathcal{S}$, $T(x) \preceq_{\text{ne}} T(y)$ implies $y \preceq_{\text{ne}} x$.

The following theorem was proved by de Mottoni-Schiaffino [17] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [13, 14].

Theorem 2. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which (O+) holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead (O−) holds, then for all $x \in \mathcal{S}$, $\{T^{2n}(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [16], with the domain of the map specialized to be the Cartesian product of intervals of real numbers. It gives a sufficient condition for conditions (O+) and (O−).

Theorem 3. *Let $\mathcal{R} \subset \mathbb{R}^2$ be the Cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$ then T satisfies (O+). If T is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$ then T satisfies (O−).*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess (see [18]) and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 4. *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

Next result is well known global attractivity result which holds in partially ordered Banach spaces as well; see [18].

Theorem 5. *Let T be a monotone map on a closed and bounded rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Suppose that T has a unique fixed point $\overline{\mathbf{e}}$ in $\mathcal{R}$. Then $\overline{\mathbf{e}}$ is a global attractor of T on $\mathcal{R}$.*

The following theorems were proved by Kulenović and Merino [1] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or nonhyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. These results are useful for determining basins of attraction of fixed points of competitive maps.

Theorem 6. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}(\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x}))$ is nonempty (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true:*

(a) *The map T has a C^1 extension to a neighborhood of $\overline{x}$.*

(b) *The Jacobian $J_T(\overline{x})$ of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{x}$ that is invariant and a subset of the basin of attraction of $\overline{x}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{x}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

The situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 7. *For the curve $\mathcal{C}$ of Theorem 6 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied*

(i) *The map T has no fixed points or periodic points of minimal period-two in Δ.*

(ii) *The map T has no fixed points in Δ, $\det J_T(\overline{x}) > 0$, and $T(x) = \overline{x}$ has no solutions $x \in \Delta$.*

(iii) *The map T has no points of minimal period-two in Δ, $\det J_T(\overline{x}) < 0$, and $T(x) = \overline{x}$ has no solutions $x \in \Delta$.*

The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 8. *(A) Assume the hypotheses of Theorem 6, and let $\mathscr{C}$ be the curve whose existence is guaranteed by Theorem 6. If the endpoints of $\mathscr{C}$ belong to $\partial\mathscr{R}$, then $\mathscr{C}$ separates $\mathscr{R}$ into two connected components, namely,*

$$\mathscr{W}_- := \{x \in \mathscr{R} \setminus \mathscr{C} : \exists y \in \mathscr{C} \text{ with } x \preceq_{se} y\},$$
$$\mathscr{W}_+ := \{x \in \mathscr{R} \setminus \mathscr{C} : \exists y \in \mathscr{C} \text{ with } y \preceq_{se} x\}, \tag{9}$$

such that the following statements are true:

(i) *$\mathscr{W}_-$ is invariant, and* $\text{dist}(T^n(x), \mathcal{Q}_2(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathscr{W}_-$.

(ii) *$\mathscr{W}_+$ is invariant, and* $\text{dist}(T^n(x), \mathcal{Q}_4(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathscr{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{x}$ is an interior point of $\mathscr{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{x}$, then T has no periodic points in the boundary of $\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})$ except for $\overline{x}$, and the following statements are true:

(iii) *For every $x \in \mathscr{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}\, \mathcal{Q}_2(\overline{x})$ for $n \geq n_0$.*

(iv) *For every $x \in \mathscr{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}\, \mathcal{Q}_4(\overline{x})$ for $n \geq n_0$.*

If T is a map on a set $\mathscr{R}$ and if $\overline{x}$ is a fixed point of T, the *stable set* $\mathscr{W}^s(\overline{x})$ of $\overline{x}$ is the set $\{x \in \mathscr{R} : T^n(x) \to \overline{x}\}$ and *unstable set* $\mathscr{W}^u(\overline{x})$ of $\overline{x}$ is the set

$$\Big\{x \in \mathscr{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathscr{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x,\ \lim_{n\to-\infty} x_n = \overline{x}\Big\}. \tag{10}$$

When T is noninvertible, the set $\mathscr{W}^s(\overline{x})$ may not be connected and made up of infinitely many curves, or $\mathscr{W}^u(\overline{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathscr{R}$, the sets $\mathscr{W}^s(\overline{x})$ and $\mathscr{W}^u(\overline{x})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 9. *In addition to the hypotheses of part (B) of Theorem 8, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathscr{C}$ of Theorem 6 has endpoints in $\partial\mathscr{R}$, then $\mathscr{C}$ is the stable set $\mathscr{W}^s(\overline{x})$ of $\overline{x}$, and the unstable set $\mathscr{W}^u(\overline{x})$ of $\overline{x}$ is a curve in $\mathscr{R}$ that is tangential to E^μ at $\overline{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathscr{W}^u(\overline{x})$ in $\mathscr{R}$ are fixed points of T.*

3. Number of Equilibria

In this section we give some basic facts which are used later. Let T be the map associated with system (1) given by

$$T(x, y) = (f(x, y), g(x, y)) = \left(\frac{x}{A_1 + B_1 x + C_1 y}, \frac{y^2}{A_2 + B_2 x + C_2 y^2}\right). \tag{11}$$

Let $\mathscr{R} = \mathbb{R}_+^2$. The equilibrium points $(\overline{x}, \overline{y})$ of system (1) satisfy equations

$$\frac{\overline{x}}{A_1 + B_1\overline{x} + C_1\overline{y}} = \overline{x},$$
$$\frac{\overline{y}^2}{A_2 + B_2\overline{x} + C_2\overline{y}^2} = \overline{y}. \tag{12}$$

For $\overline{x} = 0$ we have

$$\overline{y} = \overline{y}^2 - A_2\overline{y} - C_2\overline{y}^3 \tag{13}$$

from which we obtain three equilibrium points

$$E_1 = (0, 0),$$
$$E_2 = \left(0, \frac{1 - \sqrt{\Delta_1}}{2C_2}\right),$$
$$E_3 = \left(0, \frac{1 + \sqrt{\Delta_1}}{2C_2}\right), \tag{14}$$

where $\Delta_1 = 1 - 4A_2C_2$.

Assume that $\overline{x} \neq 0$. Then, from the first equation of system (12) we have

$$\overline{y} = -\frac{A_1 + B_1\overline{x} - 1}{C_1}. \tag{15}$$

By substituting this into the second equation we obtain

$$A_1 + B_1\overline{x} - 1 = 0 \tag{16}$$

or

$$\widetilde{g}(x) \equiv B_1^2 C_2 x^2 + x\left(B_1\left(2\left(A_1 - 1\right)C_2 + C_1\right) + B_2 C_1^2\right) + \left(A_1 - 1\right)^2 C_2 + \left(A_1 - 1\right)C_1 + A_2 C_1^2 = 0, \tag{17}$$

from which we obtain the other three equilibrium points

$$E_4 = \left(\frac{1-A_1}{B_1}, 0\right),$$
$$E_5 = \left(\frac{-2A_1B_1C_2 - B_2C_1^2 - B_1C_1 + 2B_1C_2 + C_1\sqrt{\Delta_2}}{2B_1^2C_2}, \frac{B_1 + B_2C_1 - \sqrt{\Delta_2}}{2B_1C_2}\right), \quad (18)$$
$$E_6 = \left(\frac{-2A_1B_1C_2 - B_2C_1^2 - B_1C_1 + 2B_1C_2 - C_1\sqrt{\Delta_2}}{2B_1^2C_2}, \frac{B_1 + B_2C_1 + \sqrt{\Delta_2}}{2B_1C_2}\right),$$

where

$$\Delta_2 = (B_1 + B_2C_1)^2 - 4B_1(A_2B_1 - (A_1 - 1)B_2)C_2. \quad (19)$$

Lemma 10. *The following hold:*

(i) *The equilibrium points E_2 and E_3 exist if and only if $\Delta_1 \geq 0$ and $E_2 = E_3$ if and only if $\Delta_1 = 0$.*

(ii) *The equilibrium point E_4 exists if and only if $A_1 \leq 1$ and $E_4 = E_1$ if and only if $A_1 = 1$.*

(iii) *Assume that $\Delta_2 \geq 0$. The equilibrium point E_5 exists if and only if $A_1 < 1$ and*

$$C_1 \leq \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, \quad C_2 \leq \frac{(B_2C_1 + B_1)^2}{4B_1(A_2B_1 + (1-A_1)B_2)} \quad (20)$$

or

$$C_1 > \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, \quad C_2 \leq \frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}. \quad (21)$$

(iv) *Assume that $\Delta_2 \geq 0$. The equilibrium point E_6 exists if and only if $A_1 < 1$ and*

$$C_1 \leq \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$$
$$\frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} \leq C_2 \leq \frac{(B_2C_1 + B_1)^2}{4B_1(A_2B_1 + (1-A_1)B_2)}. \quad (22)$$

Proof. The proof of the statements (i) and (ii) is trivial and we skip it. Now we prove the statement (iii). In view of Descartes' rule of signs we obtain that (17) has no positive solutions if $A_1 \geq 1$. Now, we suppose that $A_1 < 1$. One can see that $\overline{y}_5 > 0$ for all values of parameters. We consider two cases:

(1) Assume that

$$-2A_1B_1C_2 - B_2C_1^2 - B_1C_1 + 2B_1C_2 \geq 0, \quad (23)$$

which is equivalent to

$$C_2 \geq \frac{C_1(B_2C_1 + B_1)}{2(1-A_1)B_1}. \quad (24)$$

Since

$$\Delta_2 \geq 0 \Longleftrightarrow C_2 \leq \frac{(B_2C_1 + B_1)^2}{4B_1(A_2B_1 + (1-A_1)B_2)} \quad (25)$$

we have that $\overline{x}_5 \geq 0$ if and only if

$$\frac{C_1(B_2C_1 + B_1)}{2(1-A_1)B_1} \leq C_2 \leq \frac{(B_2C_1 + B_1)^2}{4B_1(A_2B_1 + (1-A_1)B_2)}, \quad (26)$$

$$\frac{(B_2C_1 + B_1)^2}{4B_1(A_2B_1 - (A_1 - 1)B_2)} - \frac{C_1(B_2C_1 + B_1)}{2(1-A_1)B_1} = -\frac{(B_2C_1 + B_1)((1-A_1)B_2C_1 + B_1(2A_2C_1 + A_1 - 1))}{4(1-A_1)B_1(A_2B_1 + (1-A_1)B_2)} \geq 0 \quad (27)$$

which is equivalent to

$$C_1 \leq \frac{(1-A_1)B_1}{2A_2B_1 + (1-A_1)B_2}. \quad (28)$$

From (27) and (28) it follows $\overline{x}_5 \geq 0$ if and only if

$$C_1 \leq \frac{(1-A_1)B_1}{2A_2B_1 + (1-A_1)B_2}, \quad \frac{C_1(B_2C_1 + B_1)}{2(1-A_1)B_1} \leq C_2 \leq \frac{(B_2C_1 + B_1)^2}{4B_1(A_2B_1 + (1-A_1)B_2)}. \quad (29)$$

(2) Assume that

$$-2A_1B_1C_2 - B_2C_1^2 - B_1C_1 + 2B_1C_2 < 0 \quad (30)$$

which is equivalent to

$$C_2 < \frac{C_1(B_2C_1 + B_1)}{2(1-A_1)B_1}. \quad (31)$$

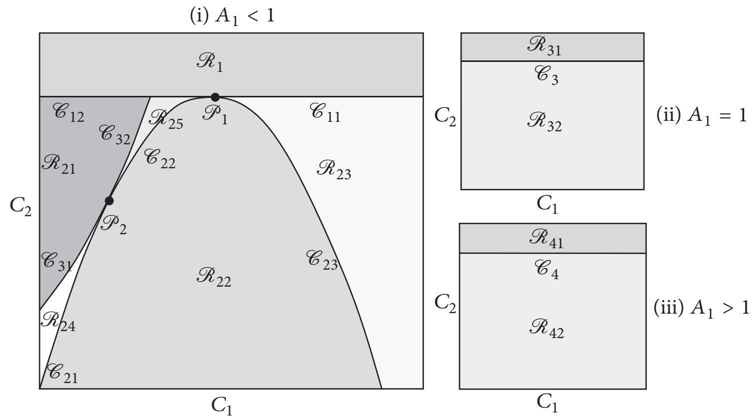


FIGURE 1: Parameter regions in the (C_1, C_2)-plane. The curves $\mathscr{C}_{21}, C_{22}$, and $\mathscr{C}_{23}$ are defined as part of the parabola $C_2 = C_1(1-A_1-A_2C_1)/(1-A_1)^2$ and the curves $\mathscr{C}_{32}$ and $\mathscr{C}_{31}$ are defined as part of the parabola $C_2 = (B_2C_1 + B_1)^2/(4B_1(A_2B_1 + (1 - A_1)B_2))$.

Then $\overline{x}_5 \geq 0$ if and only if

$$\Delta_2 C_1^2 - \left(-B_1\left(2\left(A_1-1\right)C_2+C_1\right)-B_2C_1^2\right)^2 = -4B_1^2C_2\left(\left(A_1-1\right)^2C_2+C_1\left(A_2C_1+A_1-1\right)\right) \geq 0, \tag{32}$$

which is equivalent to

$$C_2 \leq \frac{C_1\left(-A_2C_1-A_1+1\right)}{\left(1-A_1\right)^2}, \tag{33}$$

$$\Delta_2 \geq 0 \Longleftrightarrow C_2 \leq \frac{\left(B_2C_1+B_1\right)^2}{4B_1\left(A_2B_1+\left(1-A_1\right)B_2\right)}. \tag{34}$$

Since

$$\frac{\left(B_2C_1+B_1\right)^2}{4B_1\left(A_2B_1-\left(A_1-1\right)B_2\right)} - \frac{C_1\left(-A_2C_1-A_1+1\right)}{\left(1-A_1\right)^2} = \frac{\left(\left(1-A_1\right)B_2C_1+B_1\left(-2A_2C_1-A_1+1\right)\right)^2}{4\left(1-A_1\right)^2B_1\left(A_2B_1+\left(1-A_1\right)B_2\right)} \geq 0 \tag{35}$$

then from (33) and $\Delta_2 \geq 0$ we have

$$C_2 \leq \frac{C_1\left(1-A_2C_1-A_1\right)}{\left(1-A_1\right)^2}. \tag{36}$$

Since

$$\frac{C_1\left(1-A_2C_1-A_1\right)}{\left(1-A_1\right)^2} - \frac{C_1\left(B_2C_1+B_1\right)}{2\left(1-A_1\right)B_1} = -\frac{C_1\left(\left(1-A_1\right)B_2C_1+B_1\left(2A_2C_1+A_1-1\right)\right)}{2\left(A_1-1\right)^2B_1} \tag{37}$$

we have that (31) and (36) are equivalent to

$$C_1 > \frac{\left(1-A_1\right)B_1}{2A_2B_1+\left(1-A_1\right)B_2}, \quad C_2 \leq \frac{C_1\left(1-A_2C_1-A_1\right)}{\left(1-A_1\right)^2} \tag{38}$$

or

$$C_1 \leq \frac{\left(1-A_1\right)B_1}{2A_2B_1+\left(1-A_1\right)B_2}, \quad C_2 < \frac{C_1\left(B_2C_1+B_1\right)}{2\left(1-A_1\right)B_1}. \tag{39}$$

Now, the proof of the statement (iii) follows from (28), (38), and (39). The proof of the statement (iv) is similar and we skip it. □

We now introduce the following notation for regions in parameter space (C_1, C_2) (see Figure 1):

$$\mathscr{R}_{24} = \left\{\left(C_1,C_2\right): A_1 < 1,\ \Delta_1 > 0,\ C_1 < \frac{\left(1-A_1\right)B_1}{\left(1-A_1\right)B_2+2A_2B_1},\ \frac{C_1\left(1-A_1-A_2C_1\right)}{\left(1-A_1\right)^2} < C_2 < \frac{\left(B_2C_1+B_1\right)^2}{4B_1\left(A_2B_1+\left(1-A_1\right)B_2\right)}\right\},$$

$$\mathscr{R}_{22} = \left\{\left(C_1,C_2\right): A_1 < 1,\ \Delta_1 > 0,\ C_2 < \frac{C_1\left(1-A_1-A_2C_1\right)}{\left(1-A_1\right)^2}\right\},$$

$$\mathscr{R}_{21} = \left\{\left(C_1,C_2\right): A_1 < 1,\ \Delta_1 > 0,\ C_2 > \frac{\left(B_2C_1+B_1\right)^2}{4B_1\left(A_2B_1+\left(1-A_1\right)B_2\right)}\right\},$$

$$\mathscr{R}_{25} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ \frac{1-A_1}{2A_2} > C_1 > \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},\ \frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \frac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}\right\},$$

$$\mathscr{R}_{23} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ C_1 > \frac{1-A_1}{2A_2},\ \frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \frac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}\right\},$$

$$\mathscr{R}_{1} = \{(C_1, C_2) : A_1 < 1,\ \Delta_1 < 0\},$$

$$\mathscr{C}_{31} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ C_1 < \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},\ C_2 = \frac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}\right\},$$

$$\mathscr{C}_{32} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ \frac{1-A_1}{2A_2} > C_1 > \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},\ C_2 = \frac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}\right\},$$

$$\mathscr{C}_{21} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ C_1 < \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},\ C_2 = \frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}\right\},$$

$$\mathscr{C}_{22} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ \frac{1-A_1}{2A_2} > C_1 > \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},\ C_2 = \frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}\right\},$$

$$\mathscr{C}_{23} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 > 0,\ C_1 > \frac{1-A_1}{2A_2},\ C_2 = \frac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}\right\},$$

$$\mathscr{C}_{11} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 = 0,\ C_1 > \frac{1-A_1}{2A_2}\right\},$$

$$\mathscr{C}_{12} = \left\{(C_1, C_2) : A_1 < 1,\ \Delta_1 = 0,\ C_1 < \frac{1-A_1}{2A_2}\right\},$$

$$\mathscr{P}_{1} = \left\{(C_1, C_2) : A_1 < 1,\ C_1 = \frac{1-A_1}{2A_2},\ C_2 = \frac{1}{4A_2}\right\},$$

$$\mathscr{P}_{2} = \left\{(C_1, C_2) : A_1 < 1,\ C_1 = \frac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},\ C_2 = \frac{B_1(A_2B_1-(A_1-1)B_2)}{((A_1-1)B_2-2A_2B_1)^2}\right\},$$

$$\mathscr{R}_{31} = \{(C_1, C_2) : A_1 = 1,\ \Delta_1 < 0\},$$

$$\mathscr{R}_{32} = \{(C_1, C_2) : A_1 = 1,\ \Delta_1 > 0\},$$

$$\mathscr{C}_{3} = \{(C_1, C_2) : A_1 = 1,\ \Delta_1 = 0\},$$

$$\mathscr{R}_{41} = \{(C_1, C_2) : A_1 > 1,\ \Delta_1 < 0\},$$

$$\mathscr{R}_{42} = \{(C_1, C_2) : A_1 > 1,\ \Delta_1 > 0\},$$

$$\mathscr{C}_{4} = \{(C_1, C_2) : A_1 > 1,\ \Delta_1 = 0\}. \tag{40}$$

Figure 1 gives a graphical representation of above sets. The following result gives a complete classification for the number of equilibrium solutions of system (1).

Proposition 11. *Let A_1, A_2, B_1, B_2, C_1, and C_2 be positive real numbers. Then, the number of positive equilibrium solutions of system (1) with parameters A_1, A_2, B_1, B_2, C_1, and C_2 can be from 1 to 6. The different cases are given in Table 1.*

Proof. The proof follows from Lemma 10. □

4. Linearized Stability Analysis

The Jacobian matrix of the map T has the form

$$J_T = \begin{pmatrix} \dfrac{A_1 + yC_1}{(A_1 + xB_1 + yC_1)^2} & -\dfrac{xC_1}{(A_1 + xB_1 + yC_1)^2} \\ -\dfrac{y^2B_2}{(A_2 + xB_2 + y^2C_2)^2} & \dfrac{2y(A_2 + xB_2)}{(A_2 + xB_2 + y^2C_2)^2} \end{pmatrix}. \tag{41}$$

Table 1: The criteria for the existence of the equilibrium points.

Case	Equilibria	Region	The criteria for the existence
(i)	$E_1, E_2, E_3, E_4, E_5, E_6$	$\mathscr{R}_{24}$	$A_1 < 1, \Delta_1 > 0, C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(ii)	$E_1, E_2, E_3, E_4, E_5 = E_6$	$\mathscr{C}_{31}$	$A_1 < 1, \Delta_1 > 0, C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $C_2 = \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(iii)	E_1, E_2, E_3, E_4, E_5	$\mathscr{R}_{22}$	$A_1 < 1, \Delta_1 > 0, C_2 < \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(iv)	E_1, E_2, E_3, E_4	$\mathscr{R}_{21} \cup \mathscr{R}_{25} \cup \mathscr{C}_{32}$	$A_1 < 1, \Delta_1 > 0, C_2 > \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$ or $A_1 < 1, \Delta_1 > 0, \dfrac{1-A_1}{2A_2} > C_1 > \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 \leq \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(v)	E_1, E_2, E_3, E_4	$\mathscr{R}_{23}$	$A_1 < 1, \Delta_1 > 0, C_1 > \dfrac{1-A_1}{2A_2}$, $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 \leq \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(vi)	$E_1, E_2, E_3 = E_6, E_4, E_5$	$\mathscr{C}_{21}$	$A_1 < 1, \Delta_1 > 0, C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(vii)	$E_1, E_2, E_3 = E_5 = E_6, E_4$	$\mathscr{P}_2$	$A_1 < 1, \Delta_1 > 0, C_1 = \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(viii)	$E_1, E_2, E_3 = E_5, E_4$	$\mathscr{C}_{22}$	$A_1 < 1, \Delta_1 > 0, \dfrac{1-A_1}{2A_2} > C_1 > \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(ix)	$E_1, E_3, E_2 = E_5, E_4$	$\mathscr{C}_{23}$	$A_1 < 1, \Delta_1 > 0, C_1 > \dfrac{1-A_1}{2A_2}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(x)	$E_1, E_2 = E_3 = E_5, E_4$	$\mathscr{P}_1$	$A_1 < 1, \Delta_1 = 0, C_1 = \dfrac{1-A_1}{2A_2}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(xi)	$E_1, E_2 = E_3, E_4$	$\mathscr{C}_{11} \cup \mathscr{C}_{12}$	$A_1 < 1, \Delta_1 = 0, C_1 \neq \dfrac{1-A_1}{2A_2}$
(xii)	E_1, E_4	$\mathscr{R}_1$	$A_1 < 1, \Delta_1 < 0$
(xiii)	$E_1 = E_4, E_2, E_3$	$\mathscr{R}_{32}$	$A_1 = 1, \Delta_1 > 0$
(xiv)	$E_1 = E_4, E_2 = E_3$	$\mathscr{C}_3$	$A_1 = 1, \Delta_1 = 0$
(xv)	$E_1 = E_4$	$\mathscr{R}_{31}$	$A_1 = 1, \Delta_1 < 0$
(xvi)	E_1, E_2, E_3	$\mathscr{R}_{42}$	$A_1 > 1, \Delta_1 > 0$
(xvii)	$E_1, E_2 = E_3$	$\mathscr{C}_4$	$A_1 > 1, \Delta_1 = 0$
(xviii)	E_1	$\mathscr{R}_{41}$	$A_1 > 1, \Delta_1 < 0$

The determinant of (41) at the equilibrium point is given by

$$\det J_T(\overline{x}, \overline{y}) = \frac{\overline{y}\left(2A_1\left(A_2 + B_2\overline{x}\right) + \overline{y}\left(2A_2 + B_2\overline{x}\right)C_1\right)}{\left(A_1 + B_1\overline{x} + C_1\overline{y}\right)^2\left(A_2 + B_2\overline{x} + C_2\overline{y}^2\right)^2} \tag{42}$$

and the trace of (41) at the equilibrium point is given by

$$\operatorname{tr} J_T(\overline{x}, \overline{y}) = \frac{A_1 + \overline{y}C_1}{\left(A_1 + B_1\overline{x} + C_1\overline{y}\right)^2} + \frac{2\overline{y}\left(A_2 + B_2\overline{x}\right)}{\left(A_2 + B_2\overline{x} + C_2\overline{y}^2\right)^2}. \tag{43}$$

The characteristic equation has the form

$$\lambda^2 - \lambda\left(\frac{A_1+\overline{y}C_1}{\left(A_1+B_1\overline{x}+C_1\overline{y}\right)^2}+\frac{2\overline{y}\left(A_2+B_2\overline{x}\right)}{\left(A_2+B_2\overline{x}+C_2\overline{y}^2\right)^2}\right) + \frac{\overline{y}\left(2A_1\left(A_2+B_2\overline{x}\right)+\overline{y}\left(2A_2+B_2\overline{x}\right)C_1\right)}{\left(A_1+B_1\overline{x}+C_1\overline{y}\right)^2\left(A_2+B_2\overline{x}+C_2\overline{y}^2\right)^2}=0. \tag{44}$$

Lemma 12. *The following statements hold:*

(a) E_1 *is locally asymptotically stable if* $A_1 > 1$.

(b) E_1 *is a saddle point if* $A_1 < 1$.

(c) E_1 *is a nonhyperbolic equilibrium point if* $A_1 = 1$.

Proof. We have that, for the equilibrium point E_1, $\operatorname{tr} J_T(E_1) = 1/A_1$ and $\det J_T(E_1) = 0$. The characteristic equation of (50) at E_1 has the form $\lambda^2 - (1/A_1)\lambda = 0$, from which the proof follows. □

Lemma 13. *The following statements hold:*

(a) E_4 *is locally asymptotically stable if* $A_1 < 1$.

(b) E_4 *is a nonhyperbolic equilibrium point if* $A_1 = 1$.

Proof. We have that, for the equilibrium point E_4, $\operatorname{tr} J_T(E_4) = A_1$ and $\det J_T(E_4) = 0$. The characteristic equation of (50) at E_4 has the form $\lambda^2 - A_1\lambda = 0$, from which the proof follows. □

The equilibrium points E_5 and E_6 are intersection points of the curves

$$x_f(y) = \frac{1-A_1-C_1y}{B_1}, \qquad x_g(y) = \frac{y-A_2-C_2y^2}{B_2}. \tag{45}$$

Let $\tilde{x}(y) = x_f(y) - x_g(y)$ for $y \in [(1-\sqrt{1-4A_2C_2})/2C_2, (1+\sqrt{1-4A_2C_2})/2C_2]$.

Lemma 14. *Let* $T = (f, g)$ *be the map defined by (11). Then* $f'_x(E_5) < 1$, $f'_x(E_6) < 1$, $g'_y(E_6) < 1$. *Let*

$$\tilde{f}(y) = B_1C_2y^2 - \left(B_2C_1 + B_1\right)y + A_2B_1 + B_2\left(1 - A_1\right). \tag{46}$$

Then, $\overline{y}_5$ *and* $\overline{y}_6$ *are zeros of* $\tilde{f}(y)$ *and* $\operatorname{sign}(\tilde{x}(y)) = \operatorname{sign}(\tilde{f}(y))$ *for* $y \in [(1-\sqrt{1-4A_2C_2})/2C_2, (1+\sqrt{1-4A_2C_2})/2C_2]$.

Proof. The first derivative of $x_f(\overline{y}_6)$ is given by

$$x'_f\left(\overline{y}_i\right) = \frac{f'_y\left(E_i\right)}{1-f'_x\left(E_i\right)} = -\frac{C_1}{B_1} < 0, \quad i = 5, 6. \tag{47}$$

Since $f'_y(E_i) < 0, i = 5, 6$, we get $f'_x(E_i) < 1, i = 5, 6$. Similarly, one can see that

$$x'_g\left(\overline{y}_6\right) = \frac{1-g'_y\left(E_6\right)}{g'_x\left(E_6\right)} = -\frac{\sqrt{\left(B_2C_1+B_1\right)^2 - 4B_1C_2\left(A_2B_1-\left(A_1-1\right)B_2\right)} + B_2C_1}{B_1B_2} < 0. \tag{48}$$

Since $g'_x(E_6) < 0$, we get $g'_y(E_6) < 1$. Further,

$$\tilde{x}(y) = \frac{1-A_1-C_1y}{B_1} - \frac{y-A_2-C_2y^2}{B_2} = \frac{B_1C_2y^2 - \left(B_2C_1+B_1\right)y + A_2B_1 - A_1B_2 + B_2}{B_1B_2} = \frac{\tilde{f}(y)}{B_1B_2}, \tag{49}$$

from which the proof follows. □

Lemma 15. *Let* T *be the map associated with system (1) and*

$$J_T\left(\overline{x}_i, \overline{y}_i\right) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \tag{50}$$

be the Jacobian matrix of T *at fixed point* $E_i (i = 5, 6)$. *Then the Jacobian matrix (50) has real and distinct eigenvalues* λ_1 *and* λ_2 *such that* $0 < \lambda_1 \le \lambda_2$. *Furthermore, the following hold:*

$$\operatorname{sign}\left(\tilde{x}'\left(\overline{y}_6\right)\right) = \operatorname{sign}\left(1-\lambda_2^{(6)}\right), \quad \lambda_2^{(6)} < 1, \tag{51}$$

$$\operatorname{sign}\left(\tilde{x}'\left(\overline{y}_5\right)\right) = \operatorname{sign}\left(1-\lambda_2^{(5)}\right)\left(1-\lambda_2^{(5)}\right). \tag{52}$$

Proof. Implicit differentiation of the equations defining C_f and C_g at E_i gives

$$x'_f\left(\overline{y}_i\right) = \frac{f'_y\left(E_i\right)}{1-f'_x\left(E_i\right)}, \qquad x'_g\left(\overline{y}_i\right) = \frac{1-g'_y\left(E_i\right)}{g'_x\left(E_i\right)}. \tag{53}$$

Characteristic equations associated with the Jacobian matrix of T at E_i are given by

$$p(\lambda) = \lambda^2 - \left[f'_x\left(E_i\right) + g'_y\left(E_i\right)\right]\lambda + \left[f'_x\left(E_i\right)g'_y\left(E_i\right) - f'_y\left(E_i\right)g'_x\left(E_i\right)\right] = \lambda^2 - \left(a_i + d_i\right)\lambda + \left(a_id_i - b_ic_i\right). \tag{54}$$

Since the map T is competitive, then the eigenvalues of the Jacobian matrix of the map T, at the equilibrium E_i, are real and distinct and furthermore $0 < \lambda_1^{(i)} < \lambda_2^{(i)}$. By (53), we have

$$\begin{aligned}\tilde{x}'(\overline{y}_i) &= x'_f(\overline{y}_i) - x'_g(\overline{y}_i)\\ &= \frac{f'_y(E_i)}{1-f'_x(E_i)} - \frac{1-g'_y(E_i)}{g'_x(E_i)}\\ &= \frac{b_i}{1-a_i} - \frac{1-d_i}{c_i}\\ &= \frac{-1+(a_i+d_i)-(a_id_i-b_ic_i)}{c_i(1-a_i)} = \frac{-p(1)}{c_i(1-a_i)}\\ &= \frac{(1-\lambda_1^{(i)})(1-\lambda_2^{(i)})}{c_i(a_i-1)}.\end{aligned} \quad (55)$$

In view of Lemma 14 and from $\operatorname{tr} J_T(E_6) = a_6 + d_6 = f'_x(E_6) + g'_y(E_6) = \lambda_1^{(6)} + \lambda_2^{(6)} < 2$ we get $\lambda_1^{(6)} < 1$. The map T is competitive, which implies $c_6 = g'_x(E_6) < 0$. In view of Lemma 14 we get $a_6 = f'_x(E_6) < 1$ from which it follows (51). Similarly, from $c_5 = g'_x(E_5) < 0$ and $a_5 = f'_x(E_5) < 1$ we obtain (52). □

The following lemma describes the local stability of the equilibrium points E_5 and E_6.

Lemma 16. *Assume that $A_1 < 1$ and $\Delta_2 \geq 0$. Then the following hold:*

(i) *If $\Delta_2 > 0$ and E_6 exists then it is locally asymptotically stable.*

(ii) *If $\Delta_2 > 0$ and E_5 exists then it is a saddle point.*

(iii) *If $\Delta_2 = 0$ then $E_5 = E_6$. Furthermore, if $E_5 = E_6$ exists then it is nonhyperbolic equilibrium point. The eigenvalues of $J_T(E_5 = E_6)$ are given by $\lambda_1 = 1$ and $\lambda_2 < 1$.*

Proof. Assuming that $\Delta_2 > 0$, then $\overline{y}_5$ and $\overline{y}_6$ are zeros of multiplicity one of

$$\begin{aligned}\tilde{f}(y) = B_1C_2y^2 - (B_2C_1+B_1)\,y + A_2B_1\\ + B_2(1-A_1)\end{aligned} \quad (56)$$

and $\overline{y}_6 > \overline{y}_5 > 0$. From this we have $\tilde{f}(y) > 0$ for $y \in (0,\overline{y}_5) \cup (\overline{y}_6, +\infty)$ and $\tilde{f}(y) < 0$ for $y \in (\overline{y}_5, \overline{y}_5)$.

By Lemmas 6 and 7 from [19] the equilibrium curves $x_f(y)$ and $x_g(y)$ intersect transversally at E_5 and E_6, that is, $\tilde{x}'(\overline{y}_i) \neq 0$ $i = 5, 6$. By this and Lemma 14 and by continuity of function $\tilde{x}(y)$ there exists a neighborhood $U_{\overline{y}_i}^{(i)}$ of $\overline{y}_i$ such that $\tilde{x}'(y) > 0$ for $y \in U_{\overline{y}_6}$ and $\tilde{x}'(y) < 0$ for $y \in U_{\overline{y}_5}^{(5)}$. This implies that $\tilde{x}'(\overline{y}_6) > 0$ and $\tilde{x}'(\overline{y}_5) < 0$. By Lemma 15 we have that E_6 is locally asymptotically stable and E_5 is a saddle point whenever equilibrium points E_5 and E_6 exist.

Assume that $\Delta_2 = 0$. Then $\overline{y}_5 = \overline{y}_6$ is zero of $\tilde{f}(y)$ of multiplicity two. In view of Lemmas 6 and 7 from [19] we have that $\tilde{x}'(\overline{y}_5) = 0$. The rest of the proof follows from the proof of Lemma 15. □

Lemma 17. *Assume that $\Delta_1 \geq 0$. The following statements are true:*

(a) *E_2 is a saddle point if $\Delta_1 > 0$, and $A_1 \geq 1$ or $A_1 < 1$ and $C_1 > (1-A_1)(1+\sqrt{1-4A_2C_2})/2A_2$.*

(b) *E_2 is a repeller if $\Delta_1 > 0$, $A_1 < 1$, and $C_1 < (1-A_1)(1+\sqrt{1-4A_2C_2})/2A_2$.*

(c) *E_2 is a nonhyperbolic equilibrium point if $\Delta_1 = 0$ or*

$$\begin{aligned}&\Delta_1 > 0,\\ &A_1 < 1,\\ &C_1 = \frac{(1-A_1)(1+\sqrt{1-4A_2C_2})}{2A_2}.\end{aligned} \quad (57)$$

If $\Delta_1 = 0$ then the eigenvalues of $J_T(E_2)$ are given by

$$\begin{aligned}\lambda_1 &= 1,\\ \lambda_2 &= \frac{1}{2A_2C_1 + A_1}\end{aligned} \quad (58)$$

with corresponding eigenvectors

$$\begin{aligned}\mathbf{v}_1 &= (0,1)^T,\\ \mathbf{v}_2 &= \left(\frac{2A_2C_1 + A_1 - 1}{B_2(2A_2C_1+A_1)}, 1\right)^T.\end{aligned} \quad (59)$$

If (57) holds then the eigenvalues of $J_T(E_2)$ are given by

$$\begin{aligned}\lambda_1 &= 1,\\ \lambda_2 &= 1 + \sqrt{1-4A_2C_2} > 1\end{aligned} \quad (60)$$

with corresponding eigenvectors

$$\begin{aligned}\mathbf{v}_1 &= \left(\frac{\sqrt{1-4A_2C_2}}{B_2}, 1\right)^T,\\ \mathbf{v}_2 &= (0,1)^T.\end{aligned} \quad (61)$$

Proof. One can see that

$$\begin{aligned}&1 + \det J_T(E_2) - \operatorname{tr} J_T(E_2)\\ &\quad = \frac{2(1-A_1)\sqrt{1-4A_2C_2}C_2 + C_1(1-4A_2C_2 - \sqrt{1-4A_2C_2})}{2A_1C_2 + C_1(1-\sqrt{1-4A_2C_2})},\\ &1 - \det J_T(E_2)\\ &\quad = \frac{C_1(1-\sqrt{1-4A_2C_2}) + 2C_2(A_1 - 1 - \sqrt{1-4A_2C_2})}{2A_1C_2 + C_1(1-\sqrt{1-4A_2C_2})}.\end{aligned} \quad (62)$$

(a) Since $\det J_T(E_2) > 0$ and $\operatorname{tr} J_T(E_2) > 0$, the equilibrium E_2 is a saddle point if and only if $1+\det J_T(E_2) - \operatorname{tr} J_T(E_2) < 0$. If $A_1 \geq 1$ it is obvious that $1 + \det J_T(E_2) - \operatorname{tr} J_T(E_2) < 0$. Assume that $A_1 < 1$. Then $1 + \det J_T(E_2) - \operatorname{tr} J_T(E_2) < 0$ if and only if

$$C_1 > \frac{2(1-A_1)C_2\sqrt{1-4A_2C_2}}{4A_2C_2 - 1 + \sqrt{1-4A_2C_2}} = \frac{(1-A_1)(\sqrt{1-4A_2C_2}+1)}{2A_2}, \quad (63)$$

from which the proof of the statement follows.

(b) Since $\det J_T(E_2) > 0$ and $\operatorname{tr} J_T(E_2) > 0$, the equilibrium E_2 is repeller if and only if $1+\det J_T(E_2) - \operatorname{tr} J_T(E_2) > 0$ and $1 - \det J_T(E_2) < 0$. The proof of the statement follows from the facts

$$\begin{gathered} 1 - \det J_T(E_2) < 0 \Longleftrightarrow \\ C_1 < \frac{2C_2(\sqrt{1-4A_2C_2} - A_1 + 1)}{1-\sqrt{1-4A_2C_2}}, \\ \frac{2(1-A_1)C_2\sqrt{1-4A_2C_2}}{4A_2C_2 - 1 + \sqrt{1-4A_2C_2}} - \frac{2C_2(\sqrt{1-4A_2C_2} - A_1 + 1)}{1-\sqrt{1-4A_2C_2}} \\ = -\frac{-4A_2C_2 + \sqrt{1-4A_2C_2}+1}{2A_2} < 0. \end{gathered} \quad (64)$$

(c) Since $\det J_T(E_2) > 0$ and $\operatorname{tr} J_T(E_2) > 0$, the equilibrium E_2 is nonhyperbolic if and only if $1+\det J_T(E_2) - \operatorname{tr} J_T(E_2) = 0$ or $\det J_T(E_2) = 1$ and $\operatorname{tr} J_T(E_2) \leq 2$. From the proof of the statements (a) and (b) if

$$\begin{gathered} \Delta_1 = 0 \\ \text{or } \Delta_1 > 0, \\ A_1 < 1, \\ C_1 = \frac{(1-A_1)(\sqrt{1-4A_2C_2}+1)}{2A_2} \end{gathered} \quad (65)$$

we obtain $1+\det J_T(E_2) - \operatorname{tr} J_T(E_2) = 0$. Now, assume that $\Delta_1 > 0$ and

$$\det J_T(E_2) = 1 \Longleftrightarrow C_1 = \frac{2C_2(\sqrt{1-4A_2C_2} - A_1 + 1)}{1-\sqrt{1-4A_2C_2}}. \quad (66)$$

This implies $\operatorname{tr} J_T(E_2) - 2 = (1 - 4A_2C_2)/(\sqrt{1-4A_2C_2}+1) > 0$. The rest of the proof follows from the fact that if $\Delta_1 = 0$ then

$$J_T(E_2) = \begin{pmatrix} \dfrac{1}{A_1 + 2A_2C_1} & 0 \\ 0 & 1 \end{pmatrix} \quad (67)$$

and if (57) holds then

$$J_T(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-4A_2C_2}+1 \end{pmatrix}. \quad (68)$$

□

Lemma 18. *Assume that $\Delta_1 \geq 0$. The following statements are true:*

(a) *E_3 is locally asymptotically stable if $\Delta_1 > 0$, and $A_1 \geq 1$ or $A_1 < 1$ and $C_1 > (1-A_1)(1-\sqrt{1-4A_2C_2})/2A_2$.*

(b) *E_3 is a saddle point if $\Delta_1 > 0$, and $A_1 < 1$, $C_1 < (1-A_1)(1-\sqrt{1-4A_2C_2})/2A_2$.*

(c) *E_3 is a nonhyperbolic equilibrium point if $\Delta_1 = 0$ or*

$$\begin{gathered} \Delta_1 > 0, \\ A_1 < 1, \\ C_1 = \frac{(1-A_1)(1-\sqrt{1-4A_2C_2})}{2A_2}. \end{gathered} \quad (69)$$

If $\Delta_1 = 0$ then the eigenvalues of $J_T(E_3)$ are given by

$$\begin{gathered} \lambda_1 = 1, \\ \lambda_2 = \frac{1}{2A_2C_1 + A_1} \end{gathered} \quad (70)$$

with corresponding eigenvectors

$$\begin{gathered} \mathbf{v}_1 = (0,1)^T, \\ \mathbf{v}_2 = \left(\frac{2A_2C_1 + A_1 - 1}{B_2(2A_2C_1 + A_1)}, 1\right)^T. \end{gathered} \quad (71)$$

If (57) holds then the eigenvalues of $J_T(E_3)$ are given by

$$\begin{gathered} \lambda_1 = 1, \\ \lambda_2 = 1 - \sqrt{1-4A_2C_2} < 1 \end{gathered} \quad (72)$$

with corresponding eigenvectors

$$\begin{gathered} \mathbf{v}_1 = \left(-\frac{\sqrt{1-4A_2C_2}}{B_2}, 1\right)^T \\ \mathbf{v}_2 = (0,1)^T. \end{gathered} \quad (73)$$

Proof. Since the proof of this lemma is similar to the proof of Lemma 17, it is omitted. □

We summarize results about local stability in the following theorem.

Theorem 19. *Let A_1, A_2, B_1, B_2, C_1, and C_2 be positive real numbers. Then, local stability of the equilibrium points for different parameter regions is given by Table 2.*

Proof. The proof follows from Theorem 8 and Lemmas 17 and 18. □

Figure 2 illustrates visually local stability of all equilibrium points of system (1).

Table 2: The local stability of the equilibrium points.

Case	The local stability	Parameter region
(i)	Saddle: E_1, E_3, E_5; LAS: E_4, E_6; repeller: E_2	$A_1 < 1, \Delta_1 > 0, C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(ii)	Saddle: E_1, E_3; LAS: E_4; nonhyperbolic: $E_5 = E_6$; repeller: E_2	$A_1 < 1, \Delta_1 > 0, C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, C_2 = \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(iii)	Saddle: E_1, E_5; LAS: E_3, E_4; repeller: E_2	$A_1 < 1, \Delta_1 > 0, C_2 < \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(iv)	Saddle: E_1, E_3; LAS: E_4; repeller: E_2	$A_1 < 1, \Delta_1 > 0, C_2 > \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$ or $A_1 < 1, \Delta_1 > 0$, $\dfrac{1-A_1}{2A_2} > C_1 > \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 \leq \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(v)	Saddle: E_1, E_2; LAS: E_3, E_4	$A_1 < 1, \Delta_1 > 0, C_1 > \dfrac{1-A_1}{2A_2}, \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 \leq \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$
(vi)	Saddle: E_1, E_5; LAS: E_4; repeller: E_2 Nonhyperbolic: $E_3 = E_6$;	$A_1 < 1, \Delta_1 > 0, C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}$, $C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(vii)	Saddle: E_1; LAS: E_4; repeller: E_2 Nonhyperbolic: $E_3 = E_5 = E_6$;	$A_1 < 1, \Delta_1 > 0, C_1 = \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(viii)	Saddle: E_1; LAS: E_4; repeller: E_2 Nonhyperbolic: $E_3 = E_5$	$A_1 < 1, \Delta_1 > 0, \dfrac{1-A_1}{2A_2} > C_1 > \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(ix)	Saddle: E_1; LAS: E_3, E_4; nonhyperbolic: $E_2 = E_5$	$A_1 < 1, \Delta_1 > 0, C_1 > \dfrac{1-A_1}{2A_2}$, $C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(x)	Saddle: E_1; LAS: E_4; nonhyperbolic: $E_2 = E_3 = E_5$	$A_1 < 1, \Delta_1 = 0, C_1 = \dfrac{1-A_1}{2A_2}, C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$
(xi)	Saddle: E_1; LAS: E_4; nonhyperbolic: $E_2 = E_3$	$A_1 < 1, \Delta_1 = 0, C_1 \neq \dfrac{1-A_1}{2A_2}$
(xii)	Saddle: E_1; LAS: E_4	$A_1 < 1, \Delta_1 < 0$
(xiii)	Saddle: E_2; LAS: E_3; nonhyperbolic: $E_1 = E_4$	$A_1 = 1, \Delta_1 > 0$
(xiv)	Nonhyperbolic: $E_1 = E_4$, $E_2 = E_3$	$A_1 = 1, \Delta_1 = 0$
(xv)	Nonhyperbolic: $E_1 = E_4$	$A_1 = 1, \Delta_1 < 0$
(xvi)	Saddle: E_1, E_3; LAS: E_2;	$A_1 > 1, \Delta_1 > 0$
(xvii)	LAS: E_1; nonhyperbolic: $E_2 = E_3$	$A_1 > 1, \Delta_1 = 0$
(xviii)	LAS: E_1	$A_1 > 1, \Delta_1 < 0$

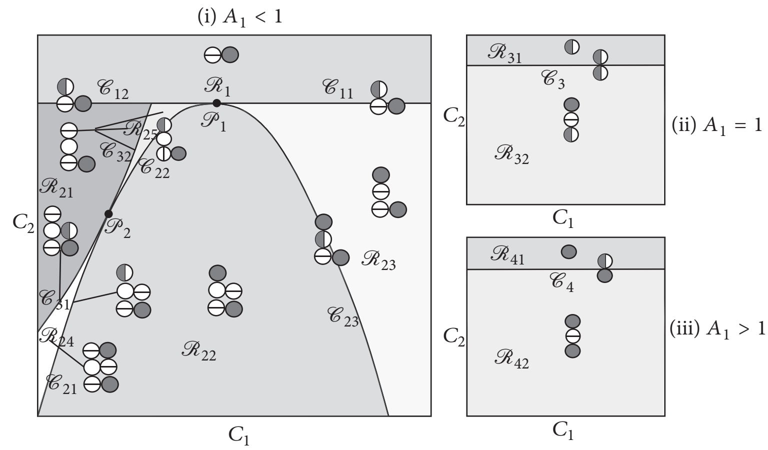


Figure 2: Equilibria in different parameter regions for the number of equilibria of system (1) when A_1, A_2, B_1, and B_2 are fixed positive real numbers, as given by Proposition 11. Each circle in the parameters (C_1, C_2)-plane indicates the existence of an isolated equilibrium point of system (1) in the nonnegative quadrant of the xy-plane. Local stability character of equilibria as given in Theorem 19 is indicated as follows: •, locally asymptotically stable equilibrium; ⊖, saddle; ◯, repelling equilibrium point: two-colored circle, semistable nonhyperbolic equilibrium.

5. Injectivity and Convergence to Equilibrium Points

In this section we prove some global properties of the map T such as injectivity and $(O+)$ property and give global behavior on the coordinate axes.

Lemma 20. *The map T is injective.*

Proof. Assume that $T(x_1, y_1) = T(x_2, y_2)$. Then, we have

$$\left(\frac{A_1x_1 - A_1x_2 - C_1x_2y_1 + C_1x_1y_2}{(A_1 + B_1x_1 + C_1y_1)(A_1 + B_1x_2 + C_1y_2)}, \frac{A_2{y_1}^2 + B_2x_2{y_1}^2 - A_2{y_2}^2 - B_2x_1{y_2}^2}{(A_2 + B_2x_1 + C_2{y_1}^2)(A_2 + B_2x_2 + C_2{y_2}^2)}\right) = (0,0). \tag{74}$$

Equation (74) is equivalent to

$$A_1x_1 - A_1x_2 - C_1x_2y_1 + C_1x_1y_2 = 0, \tag{75}$$

$$A_2{y_1}^2 + B_2x_2{y_1}^2 - A_2{y_2}^2 - B_2x_1{y_2}^2 = 0. \tag{76}$$

Equation (75) implies

$$x_1 = \frac{A_1x_2 + C_1x_2y_1}{A_1 + C_1y_2}. \tag{77}$$

By substituting this into (76) we obtain

$$\frac{(y_1 - y_2)\left(A_1A_2y_1 + A_1B_2x_2y_1 + A_1A_2y_2 + A_1B_2x_2y_2 + A_2C_1y_1y_2 + B_2C_1x_2y_1y_2 + A_2C_1{y_2}^2\right)}{A_1 + C_1y_2} = 0, \tag{78}$$

from which it follows that $y_1 = y_2$. From (77) we have $x_1 = x_2$, which complete the proof. □

The global behavior of T on the coordinate axes is described with the following result.

Lemma 21. *The following statements hold:*

(i) *If $\Delta_1 \geq 0$ and $(x_n, y_n) = T^n(0, y_0)$ then $x_n = 0$ and $(x_n, y_n) \to E_3$ for $y_0 \in ((1 - \sqrt{1 - 4A_2C_2})/2C_2, \infty)$ and $(x_n, y_n) \to E_1$ for $y_0 \in (0, (1 + \sqrt{1 - 4A_2C_2})/2C_2)$.*

(ii) *If $\Delta_1 < 0$ and $(x_n, y_n) = T^n(0, y_0)$ then $x_n = 0$ and $(x_n, y_n) \to E_1$ for $y_0 \in (0, \infty)$.*

(iii) *If $A_1 < 1$ and $(x'_n, y'_n) = T^n(x_0, 0)$ then $y'_n = 0$ and $(x'_n, y'_n) \to E_4$ for $x_0 \in (0, \infty)$.*

(iv) *If $A_1 \geq 1$ and $(x'_n, y'_n) = T^n(x_0, 0)$ then $y'_n = 0$ and $(x'_n, y'_n) \to E_1$ for $x_0 \in (0, \infty)$.*

(v) *$T([0, \infty) \times [0, \infty)) \subset [0, 1/B_1] \times [0, 1/C_2]$.*

Proof.

(i) From (11) it is easy to see that if $x_0 = 0$ then $x_n = 0$ for $n > 0$. Since

$$T(0, y_0) - (0, y_0) = \left(0, -\frac{y_0 (C_2 y_0^2 - y_0 + A_2)}{A_2 + C_2 y_0^2}\right) \quad (79)$$

we obtain that

$$\begin{aligned} &(0, y_0) \preceq_{se} T(0, y_0) \\ &\text{for } y_0 \in \left(0, \frac{1 - \sqrt{1 - 4A_2C_2}}{2C_2}\right) \cup \left(\frac{1 + \sqrt{1 - 4A_2C_2}}{2C_2}, \infty\right), \\ &T(0, y_0) \preceq_{se} (0, y_0) \\ &\quad \text{for } y_0 \in \left(\frac{1 - \sqrt{1 - 4A_2C_2}}{2C_2}, \frac{1 + \sqrt{1 - 4A_2C_2}}{2C_2}\right). \end{aligned} \quad (80)$$

Take $y_0 > (1 + \sqrt{1 - 4A_2C_2})/2C_2$. Then $T^n(0, y_0) \preceq_{se} T^{n+1}(0, y_0) \preceq_{se} E_3$. Since $T^n(0, y_0) \preceq_{se} E_3 \prec_{se} E_2 \prec_{se} E_1$ we obtain $T^n(0, y_0) \to E_3$ as $n \to \infty$. Similarly, if $y_0 \in ((1 - \sqrt{1 - 4A_2C_2})/2C_2, (1 + \sqrt{1 - 4A_2C_2})/2C_2)$ then $E_3 \preceq_{se} T^{n+1}(0, y_0) \preceq_{se} T^n(0, y_0) \preceq_{se} E_2 \prec_{se} E_1$ which implies $T^n(0, y_0) \to E_3$ as $n \to \infty$. If $y_0 < (1 - \sqrt{1 - 4A_2C_2})/2C_2$ then $E_3 \prec_{se} E_2 \prec_{se} T^n(0, y_0) \preceq_{se} T^{n+1}(0, y_0) \preceq_{se} E_1$ which implies $T^n(0, y_0) \to E_1$ as $n \to \infty$.

(ii) If $\Delta_1 < 0$ then T has only equilibrium E_1 on y-axis and $T(0, y_0) \preceq_{se} (0, y_0)$ for all $y_0 \geq 0$. Since T is monotone map we get $T^n(0, y_0) \preceq_{se} T^{n+1}(0, y_0) \preceq_{se} E_1$ which implies $T^n(0, y_0) \to E_1$ as $n \to \infty$ from which the proof follows.

(iii) The proof of the statements (iii) and (iv) is similar to the proof of statements (i) and (ii) and follows from the fact that

$$T(x_0, 0) - (x_0, 0) = \left(-\frac{x_0 (A_1 + B_1 x_0 - 1)}{A_1 + B_1 x_0}, 0\right) \quad (81)$$

and will be omitted.

(v) The proof follows from the facts that $f(x, y) \leq 1/B_1$ and $g(x, y) \leq 1/C_2$.

□

Lemma 22. *Let $M(t) \equiv (t, (1 - A_1 - B_1 t)/C_1)$. Then $M(\overline{x}_6) = E_6$, $M(\overline{x}_5) = E_5$, and $M(\overline{x}_4) = E_4$ and the following hold:*

(i) *If $\Delta_2 \geq 0$ then $M(t) \preceq_{se} T(M(t))$ for $t \in (0, \overline{x}_6) \cup (\overline{x}_5, \overline{x}_4)$ and $T(M(t)) \preceq_{se} M(t)$ for $t \in (\overline{x}_6, \overline{x}_5)$.*

(ii) *If $\Delta_2 < 0$ then $M(t) \preceq_{se} T(M(t))$ for $t \in (0, \overline{x}_4)$ and $T(M(t)) \preceq_{se} M(t)$ for $t \in (\overline{x}_6, \overline{x}_5)$.*

Proof. The proof follows from the fact

$$\begin{aligned} &T(M(t)) - M(t) \\ &= \left(0, \frac{(A_1 + B_1 t - 1)\,\tilde{g}(x)}{C_1^3 (A_2 + B_2 t) + C_2 C_1 (A_1 + B_1 t - 1)^2}\right), \end{aligned} \quad (82)$$

where $\tilde{g}(x)$ is given by (17) and $\tilde{g}(\overline{x}_5) = \tilde{g}(\overline{x}_6) = 0$ and $\overline{x}_4 = (1 - A_1)/B_1$. □

Theorem 23. *Every solution of system (1) converges to an equilibrium point.*

Proof. The map T associated with the system is injective. Relation (42) implies that determinant of Jacobian (41) is positive for all $x \in [0, \infty) \times [0, \infty)$. By using Lemma 20 we have that condition $(O+)$ of Theorem 3 is satisfied for the map T (T is competitive). Theorem 2 implies that $T^n(x)$ is eventually componentwise monotone for all $x \in [0, \infty)^2$. The statement (v) of Lemma 21 implies that every solution enters in compact set $[0, 1/B_1] \times [0, 1/C_2]$, from which the proof follows. □

Remark 24. In view of Theorem 23 the main objective in determining the global dynamics of system (1) is to characterize the basins of attractions of all equilibrium points. As we will see in Theorem 25 the boundaries of these basins of attractions will be the global stable manifolds of the saddle or nonhyperbolic equilibrium points, whose existence is guaranteed by Theorems 7, 8, and 9.

6. Global Behavior

In this section we give results which precisely describe global dynamics of system (1) including precise characterization of basins of attraction of different equilibrium points. The main result of this paper is the following.

Theorem 25. *The global behavior of system (1) is given by Table 3. See Figure 3 for visual illustration of dynamic scenarios.*

Proof. We will prove statements (i)–(x) listed in the second column of Table 3 in the given order. The proof of other statements is similar. Let $\mathscr{R} = [0, \infty) \times [0, \infty)$.

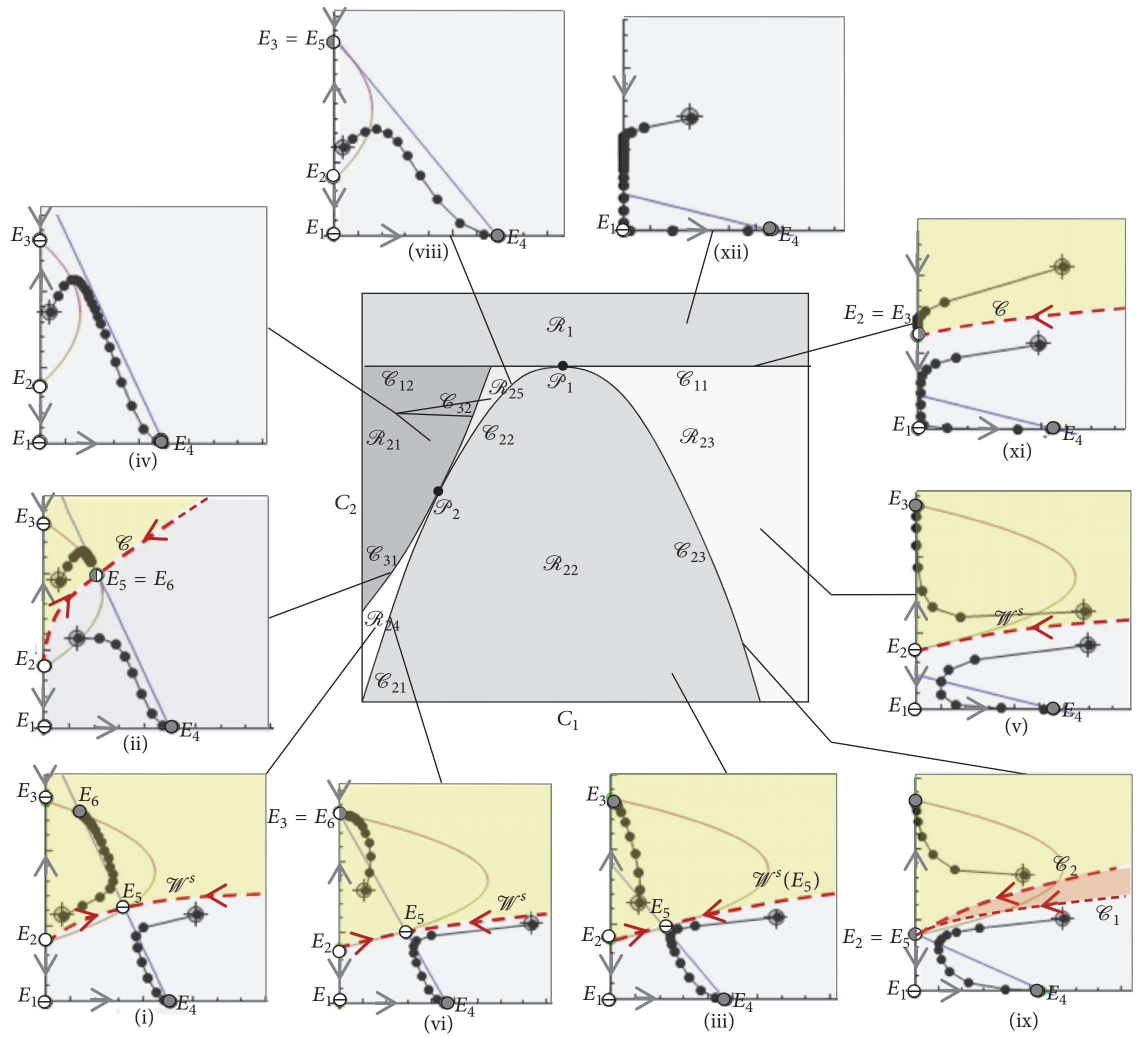
(i) Suppose $(C_1, C_2) \in \mathscr{R}_{24}$. By Proposition 11, in $Q_1(0, 0)$ there exist six equilibria E_1, E_2, E_3, E_4, E_5, and E_6. By Theorem 19 equilibria E_4 and E_5 are locally asymptotically stable; E_3, E_4, and E_5 are the saddle points and E_2 is repeller. In view of (41) the map T is competitive on $\mathscr{R}$ and strongly competitive on $\text{int}(\mathscr{R})$. It follows from the Perron-Frobenius Theorem and a change of variables [16] that, at each point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly competitive then no eigenvector is aligned with a coordinate axis. Hence, all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the global stable manifold $\mathscr{W}^s(E_5)$, with endpoint at point E_2, which is graph of an increasing function. Let $\mathscr{W}^- = \{(x, y) \mid (x, y) \preceq_{se} (\tilde{x}_0, \tilde{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathscr{W}^s(E_5)\}$ and $\mathscr{W}^+ = \{(x, y) \mid (\tilde{x}_1, \tilde{y}_1) \preceq_{se} (x, y)$ for some $(\tilde{x}_1, \tilde{y}_1) \in \mathscr{W}^s(E_5)\}$. By Lemma 21 and uniqueness of the global stable manifold we have $\mathscr{W}^s(E_1) = \{(0, y) : 0 \leq$

Table 3: The global behavior of system (1).

Case	Parameter region	Global behavior
(i)	$\mathscr{R}_{24}: A_1 < 1, \Delta_1 > 0,$ $C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$ $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$	There exist six equilibrium points E_1, E_2, E_3, E_4, E_5, and E_6, where E_4 and E_5 are locally asymptotically stable, E_3, E_4, and E_6 are saddle points, and E_2 is repeller. The stable manifold $\mathscr{W}^s(E_6)$ of the saddle point E_6 is an increasing separatrix with endpoint at E_2, and solutions with initial point above the $\mathscr{W}^s(E_6)$ converge to E_5, while solutions with initial point below the $\mathscr{W}^s(E_6)$ converge to E_4. All orbits that start on $\mathscr{W}^s(E_6)$ are attracted to E_6. The basins of attraction of E_1 and E_3 are, respectively, $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$ and $\mathscr{B}(E_3) = \{(0,y) : y > (1-\sqrt{1-4A_2C_2})/(2A_2)\}$
(ii)	$\mathscr{C}_{31}: A_1 < 1, \Delta_1 > 0,$ $C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$ $C_2 = \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$	There exist five equilibrium points E_1, E_2, E_3, E_4, and $E_5 = E_6$, where E_4 is locally asymptotically stable, E_1 and E_3 are saddle points, E_2 is repeller, and $E_5 = E_6$ is nonhyperbolic. There exists a continuous increasing curve $\mathscr{C}$ with endpoint at E_2, which is a subset of the basin of attraction of $E_5 = E_6$. All solutions with initial point above $\mathscr{C}$ converge to $E_5 = E_6$, while solutions with initial point below $\mathscr{C}$ converge to E_4. The basins of attraction of E_1 and E_3 are, respectively, $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$ and $\mathscr{B}(E_3) = \{(0,y) : y > (1-\sqrt{1-4A_2C_2})/(2A_2)\}$
(iii)	$\mathscr{R}_{22}: A_1 < 1, \Delta_1 > 0,$ $C_2 < \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$	There exist five equilibrium points E_1, E_2, E_3, E_4, and E_5, where E_3 and E_4 are locally asymptotically stable, E_1 and E_5 are saddle points, and E_2 is repeller. The stable manifold $\mathscr{W}^s(E_5)$ of the saddle point E_5 is an increasing separatrix with endpoint at E_2, and solutions with initial point above $\mathscr{W}^s(E_5)$ converge to E_3, while solutions with initial point below $\mathscr{W}^s(E_5)$ converge to E_4. All orbits that start on $\mathscr{W}^s(E_5)$ are attracted to E_5. The basin of attraction of E_1 is $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$
(iv)	$\mathscr{R}_{21} \cup \mathscr{R}_{25} \cup \mathscr{C}_{32}: A_1 < 1, \Delta_1 > 0,$ $C_2 > \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$ or $A_1 < 1,$ $\Delta_1 > 0,$ $\dfrac{1-A_1}{2A_2} > C_1 > \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$ $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$	There exist four equilibrium points E_1, E_2, E_3, and E_4, where E_4 is globally asymptotically stable, E_1 and E_3 are saddle points, and E_2 is a repeller. The basins of attraction of E_1, E_3, and E_4 are, respectively, $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$, $\mathscr{B}(E_3) = \{(0,y) : y > (1-\sqrt{1-4A_2C_2})/(2A_2)\}$, and $\mathscr{B}(E_4) = (0,\infty) \times [0,\infty)$.
(v)	$\mathscr{R}_{23}: A_1 < 1, \Delta_1 > 0,$ $C_1 > \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$ $\dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2} < C_2 < \dfrac{(B_2C_1+B_1)^2}{4B_1(A_2B_1+(1-A_1)B_2)}$	There exist four equilibrium points E_1, E_2, E_3, and E_4, where E_3 and E_4 are locally asymptotically stable and E_1 and E_2 are saddle points. The stable manifold $\mathscr{W}^s(E_2)$ of the saddle point E_2 is an increasing separatrix, and solutions with initial point above $\mathscr{W}^s(E_1)$ converge to E_3, while solutions with initial point below $\mathscr{W}^s(E_2)$ converge to E_4. All orbits that start on $\mathscr{W}^s(E_2)$ are attracted to E_2. The basin of attraction of E_1 is given by $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$.
(vi)	$\mathscr{C}_{21}: A_1 < 1, \Delta_1 > 0,$ $C_1 < \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$ $C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$	There exist five equilibrium points E_1, E_2, $E_3 = E_6$, E_4, and E_5, where E_4 is locally asymptotically stable, E_1 and E_5 are saddle points, $E_3 = E_6$ is nonhyperbolic, and E_2 is a repeller. The stable manifold $\mathscr{W}^s(E_5)$ of the saddle point E_5 is an increasing separatrix with endpoint at E_2, and solutions with initial point above $\mathscr{W}^s(E_5)$ converge to $E_3 = E_6$, while solutions with initial point below $\mathscr{W}^s(E_5)$ converge to E_4. All orbits that start on $\mathscr{W}^s(E_5)$ are attracted to E_5. The basin of attraction of E_1 is given by $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$.
(vii)	$\mathscr{P}_2: A_1 < 1, \Delta_1 > 0,$ $C_1 = \dfrac{(1-A_1)B_1}{(1-A_1)B_2 + 2A_2B_1},$ $C_2 = \dfrac{C_1(1-A_1-A_2C_1)}{(1-A_1)^2}$	There exist four equilibrium points E_1, E_2, $E_3 = E_5 = E_6$, and E_4, where E_4 is locally asymptotically stable, E_1 is a saddle point, $E_3 = E_5 = E_6$ is nonhyperbolic, and E_2 is a repeller. There exists a strictly increasing curve $\mathscr{C}$ with endpoint at $E_3 = E_5 = E_6$, which is the subset of the basin of attraction of $E_3 = E_5 = E_6$. All solutions with initial point above $\mathscr{C}$ converge to $E_3 = E_5 = E_6$, while solutions with initial point below $\mathscr{C}$ converge to E_4. The basin of attraction of E_1 is given by $\mathscr{B}(E_1) = \{(0,y) : 0 \le y < (1-\sqrt{1-4A_2C_2})/(2A_2)\}$.

Table 3: Continued.

Case	Parameter region	Global behavior
(viii)	$\mathscr{C}_{22} : A_1 < 1, \Delta_1 > 0$, $C_2 = \dfrac{C_1(1 - A_1 - A_2C_1)}{(1 - A_1)^2}$, $\dfrac{1 - A_1}{2A_2} > C_1 > \dfrac{(1 - A_1)B_1}{(1 - A_1)B_2 + 2A_2B_1}$	There exist five equilibrium points E_1, E_2, $E_3 = E_5$, and E_4, where E_4 is locally asymptotically stable, E_1 is a saddle point, $E_3 = E_5$ is nonhyperbolic, and E_2 is a repeller. The basins of attraction of E_1 and E_3 are given by $\mathscr{B}(E_1) = \{(0, y) : 0 \le y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and $\mathscr{B}(E_3) = \{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and the basin of attraction of E_4 is $\mathscr{B}(E_4) = \{(x, y) : x > 0, y \ge 0\}$.
(ix)	$\mathscr{C}_{23} : A_1 < 1, \Delta_1 > 0$, $C_2 = \dfrac{C_1(1 - A_1 - A_2C_1)}{(1 - A_1)^2}$, $\dfrac{1 - A_1}{2A_2} < C_1$	There exist four equilibrium points E_1, E_3, $E_2 = E_5$, and E_4, where E_3 and E_4 are locally asymptotically stable, E_1 is a saddle point, and $E_2 = E_5$ is nonhyperbolic. There exist continuous increasing curves $\mathscr{C}_1$ and $\mathscr{C}_2$ with endpoint at $E_2 = E_5$, which are the subsets of the basin of attraction of $E_2 = E_5$. Further, all solutions with initial point above $\mathscr{C}_2$ converge to E_3 and all solutions with initial point above $\mathscr{C}_1$ and below $\mathscr{C}_2$ converge to E_2, while solutions with initial point below $\mathscr{C}_1$ converge to E_4. The basin of attraction of E_1 is $\mathscr{B}(E_1) = \{(0, y) : 0 \le y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$.
(x)	$A_1 < 1, \Delta_1 = 0, C_1 \ne \dfrac{1 - A_1}{2A_2}$	There exist three equilibrium points E_1, $E_2 = E_3$, and E_4, where E_4 is locally asymptotically stable, E_1 is a saddle point, and $E_2 = E_3$ is nonhyperbolic. There exists a continuous increasing curve $\mathscr{C}$ with endpoint at $E_2 = E_3$ which is a subset of the basin of attraction of $E_2 = E_3$. All solutions with initial point above $\mathscr{C}$ converge to $E_2 = E_3$, while solutions with initial point below $\mathscr{C}$ converge to E_4. The basin of attraction of E_1 is given by $\mathscr{B}(E_1) = \{(0, y) : 0 \le y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$.
(xi)	$A_1 < 1, \Delta_1 < 0$	There exist two equilibrium points E_1, which is a saddle point, and E_4, which is globally asymptotically stable, where $\mathscr{B}(E_4) = (0, \infty) \times [0, \infty)$ and $\mathscr{B}(E_1) = \{(0, y) : y \ge 0\}$.
(xii)	$A_1 = 1, \Delta_1 > 0$	There exist three equilibrium points $E_1 = E_4$, E_2, and E_3, where $E_1 = E_4$ is nonhyperbolic, E_3 is locally asymptotically stable, and E_2 is a saddle point. The stable manifold $\mathscr{W}^s(E_2)$ of the saddle point E_2 is an increasing separatrix with endpoint at E_2, and solutions with initial point above $\mathscr{W}^s(E_2)$ converge to E_3, while solutions with initial point below $\mathscr{W}^s(E_2)$ converge to E_4. All orbits that start on $\mathscr{W}^s(E_2)$ are attracted to E_2. The basin of attraction of E_1 is given by $\mathscr{B}(E_1) = \{(0, y) : 0 \le y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$.
(xiii)	$A_1 = 1, \Delta_1 = 0$	There exist two equilibrium points $E_1 = E_4$ and $E_2 = E_3$ which are nonhyperbolic. There exists a continuous increasing curve $\mathscr{C}$ with endpoint at $E_2 = E_3$ which is a subset of the basin of attraction of $E_2 = E_3$. All solutions with initial point above $\mathscr{C}$ converge to $E_2 = E_3$, while solutions with initial point below $\mathscr{C}$ converge to $E_1 = E_4$.
(xiv)	$A_1 = 1, \Delta_1 < 0$	There exists one equilibrium point $E_1 = E_4$ which is nonhyperbolic and global attractor. The basin of attraction of E_1 is $\mathscr{B}(E_1) = [0, \infty) \times [0, \infty)$
(xv)	$A_1 > 1, \Delta_1 > 0$	There exist three equilibrium points E_1, E_2, and E_3, where E_1 and E_3 are locally asymptotically stable and E_2 is a saddle point. The stable manifold $\mathscr{W}^s(E_2)$ of the saddle point E_2 is an increasing separatrix with endpoint at E_2, and solutions with initial point above $\mathscr{W}^s(E_2)$ converge to E_3, while solutions with initial point below $\mathscr{W}^s(E_2)$ converge to E_1. All orbits that start on $\mathscr{W}^s(E_2)$ are attracted to E_2.
(xvi)	$A_1 > 1, \Delta_1 < 0$	There exists one equilibrium point E_1 which is globally asymptotically stable. The basin of attraction of E_1 is $\mathscr{B}(E_1) = [0, \infty) \times [0, \infty)$
(xvii)	$A_1 > 1, \Delta_1 = 0$	There exist two equilibrium points E_1 and $E_2 = E_3$, where E_1 is locally asymptotically stable and $E_2 = E_3$ is nonhyperbolic. There exists a continuous increasing curve $\mathscr{C}$ with endpoint at $E_2 = E_3$ which is a subset of the basin of attraction of $E_2 = E_3$. All solutions with initial point above $\mathscr{C}$ converge to $E_2 = E_3$, while solutions with initial point below $\mathscr{C}$ converge to E_1.



(i) $A_1 < 1$

FIGURE 3: Parameter regions in terms of parameters C_1 and C_2 and corresponding dynamic scenarios for system (1) if $A_1 < 1$.

$y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and $\mathcal{W}^s(E_3) = \{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$. Take $(x_0, y_0) \in \mathcal{W}^- \cap \operatorname{int}(Q_1(0,0))$. By Theorem 8 and Lemma 21 we have that there exists $n_0 > 0$ such that $T^n(x_0, y_0) \in \operatorname{int}(Q_4(E_3) \cap Q_2(E_5))$, $n > n_0$. In view of Corollary 4 $[[E_1, E_6]] \subseteq \mathcal{B}(E_6)$ and $[[E_6, E_5]] \subseteq \mathcal{B}(E_6)$. Since T is competitive, this implies $\operatorname{int}(Q_4(E_3) \cap Q_2(E_5)) = [[E_1, E_5]] \subseteq \mathcal{B}(E_6)$. Take $(x_0, y_0) \in \mathcal{W}^+ \cap \operatorname{int}(Q_1(0,0))$. By Theorem 8 and Lemma 21 we have that there exists $n_0' > 0$ such that $T^n(x_0, y_0) \in \operatorname{int}(Q_4(E_5) \cap Q_2(E_4))$, $n > n_0'$. Since T is competitive, in view of Corollary 4 $\operatorname{int}(Q_4(E_5) \cap Q_2(E_4)) = [[E_5, E_4]] \subseteq \mathcal{B}(E_4)$. This completes the proof of statement (i).

(ii) Suppose $(C_1, C_2) \in \mathcal{C}_{31}$. By Proposition 11, in $Q_1(0,0)$ there exist five equilibria E_1, E_2, E_3, E_4, and $E_5 = E_6$. By Theorem 19 E_1 and E_3 are the saddle points, E_2 is repeller, and $E_5 = E_6$ is nonhyperbolic. Similarly as in the proof of the statement (i), all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the invariant curve $\mathcal{C}$ with one endpoint at E_2 and which is passing through $E_5 = E_6$, and it is graph of an increasing function. Let $\mathcal{W}^- = \{(x, y) \mid (x, y) \preceq_{se} (\tilde{x}_0, \tilde{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{C}\}$ and $\mathcal{W}^+ = \{(x, y) \mid (\tilde{x}_1, \tilde{y}_1) \preceq_{se} (x, y)$ for some $(\tilde{x}_1, \tilde{y}_1) \in \mathcal{C}\}$. By Lemma 21 and uniqueness of the global stable manifold we have $\mathcal{W}^s(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and $\mathcal{W}^s(E_3) = \{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$. Take $(x_0, y_0) \in \mathcal{W}^- \cap \operatorname{int}(Q_1(0,0))$. By Theorem 8 we have that there exists $n_0 > 0$ such that $T^n(x_0, y_0) \in \operatorname{int}(Q_4(E_3) \cap Q_2(E_5 = E_6))$, $n > n_0$. In view of Corollary 4 $\operatorname{int}(Q_4(E_3) \cap Q_2(E_5 = E_6)) = [[E_1, E_5 = E_6]] \subseteq \mathcal{B}(E_5 = E_6)$. Take $(x_0, y_0) \in \mathcal{W}^+ \cap \operatorname{int}(Q_1(0,0))$. By Theorem 8 and Lemma 21 we have that there exists $n_0' > 0$ such that $T^n(x_0, y_0) \in \operatorname{int}(Q_4(E_5) \cap Q_2(E_4))$, $n > n_0'$. Since

T is competitive, in view of Corollary 4 $\text{int}(Q_4(E_5) \cap Q_2(E_4)) = [[E_5, E_4]] \subseteq \mathscr{B}(E_4)$. This completes the proof of statement (ii)

(iii) The proof is similar to the proof of case (i) and we skip it.

(iv) Suppose $(C_1, C_2) \in \mathscr{R}_{21} \cup \mathscr{R}_{25} \cup \mathscr{C}_{32}$. By Proposition 11, in $Q_1(0,0)$ there exist four equilibria E_1, E_2, E_3, and E_4. By Theorem 19 E_1 and E_3 are saddle points; E_2 is repeller; E_4 is locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have $\mathscr{B}(E_1) = \mathscr{W}^s(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and $\mathscr{B}(E_3) = \mathscr{W}^s(E_3) = \{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$. Since, by Theorem 23, every solution of system (1) converges to an equilibrium point, we have that $\mathscr{B}(E_4) = (0, \infty) \times [0, \infty)$.

(v) Suppose $(C_1, C_2) \in \mathscr{R}_{23}$. By Proposition 11, in $Q_1(0,0)$ there exist four equilibrium points E_1, E_2, E_3, and E_4. By Theorem 19 E_1 and E_2 are the saddle points; E_3 and E_4 are locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have $\mathscr{B}(E_1) = \mathscr{W}^s(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$, and $\{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\} \subseteq \mathscr{B}(E_3)$ and $\{(x, 0) : x > 0\} \subseteq \mathscr{B}(E_4)$. Similarly as in the proof of case (i), all conditions of Theorems 7, 8, and 9 are satisfied, which yields the existence of the global stable manifold $\mathscr{W}^s(E_2)$, which is a graph of an increasing function. The rest of the proof follows from the facts that $\mathscr{W}^- = \{(x, y) \mid (x, y) \preceq_{se} (\tilde{x}_0, \tilde{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathscr{W}^s(E_2)\}$ and $\mathscr{W}^+ = \{(x, y) \mid (\tilde{x}_1, \tilde{y}_1) \preceq_{se} (x, y)$ for some $(\tilde{x}_1, \tilde{y}_1) \in \mathscr{W}^s(E_2)\}$ are invariant sets, $E_4 \in \mathscr{W}^-$, $E_3 \in \mathscr{W}^+$, uniqueness of the global stable manifold $\mathscr{W}^s(E_2)$ and Theorem 23.

(vi) The proof is similar to the proof of case (i) and we skip it.

(vii) The proof is the same as the proof of case (viii) and we skip it.

(viii) Suppose $(C_1, C_2) \in \mathscr{C}_{22}$. By Proposition 11, in $Q_1(0,0)$ there exist four equilibrium points E_1, $E_3 = E_5$, E_2, and E_4. By Theorem 19 we have that E_4 is locally asymptotically stable; E_1 is a saddle point; $E_3 = E_5$ is nonhyperbolic; and E_2 is repeller. By Lemma 21 and uniqueness of the global stable manifold we have $\mathscr{B}(E_1) = \mathscr{W}^s(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$. In view of Lemma 18, for $E_3 = E_5$ we have that the eigenspace E_2^{λ} associated with λ_2 is a coordinate axis, so we can not use Theorem 6. By Lemma 21 we obtain $\{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\} \subseteq \mathscr{B}(E_3 = E_5)$. Similarly as in the proof of Theorem 8 (see [1] for more details) one can prove that for every $x \in \text{int}(Q_1(E_3))$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}(Q_4(E_3))$ for $n \geq n_0$. By Lemmas 21 and 22 for $(x_0, y_0) \in \text{int}(Q_4(E_3))$, there exists t_0' and t_0'' such that $E_3 \preceq_{se} M(t_0') \preceq_{se} (x_0, y_0) \preceq_{se} (t_0'', 0)$. Since $E_3 \preceq_{se} T^n(M(t_0')) \preceq_{se} T^{n+1}(M(t_0')) \preceq E_4$ we have that $T^n(M(t_0')) \to E_4$. Since $T^n(t_0'', 0) \to E_4$ we obtain $T^n(x_0, y_0) \to E_4$. This implies $\text{int}(Q_4(E_3)) \subseteq \mathscr{B}(E_4)$, which completes the proof.

(ix) Suppose $(C_1, C_2) \in \mathscr{C}_{23}$. By Proposition 11, there exist four equilibrium points E_1, $E_2 = E_5$, E_3, and E_4. By Theorem 19 we have that E_3 and E_4 are locally asymptotically stable; E_1 is a saddle point; $E_2 = E_5$ is nonhyperbolic. By Lemma 21 and uniqueness of the global stable manifold we have $\mathscr{B}(E_1) = \mathscr{W}^s(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$. In view of Lemma 18, for E_3, we have that $\lambda_1 = 1$ and $\lambda_2 > 1$, so we can not use Theorem 6. By Lemmas 21 and 22 for $(x_0, y_0) \in \text{int}(Q_4(E_3))$, there exists t_0' and t_0'' such that $E_3 \preceq_{se} M(t_0') \preceq_{se} (x_0, y_0) \preceq_{se} (t_0'', 0)$. Since $E_3 \preceq_{se} T^n(M(t_0')) \preceq_{se} T^{n+1}(M(t_0')) \preceq E_4$ we have that $T^n(M(t_0')) \to E_4$. Since $T^n(t_0'', 0) \to E_4$ we obtain $T^n(x_0, y_0) \to E_4$. This implies $\text{int}(Q_4(E_3)) \subseteq \mathscr{B}(E_4)$. Let $\mathscr{C}_1$ denote the boundary of $\mathscr{B}(E_4)$ considered as a subset of $\text{int}(Q_1(E_2))$ and let $\mathscr{C}_2$ denote the boundary of $\mathscr{B}(E_3)$ considered as a subset of $\text{int}(Q_1(E_2))$. It is easy to see by using Lemmas 21 and 22 that $E_2 \in \mathscr{C}_1 \cap \mathscr{C}_2$. Since $T(\text{int}(\mathscr{R})) \subset \text{int}(\mathscr{R})$, following the proof of Claims 1 and 2 [20], one can see that $T(\mathscr{C}_i) \subseteq \mathscr{C}_i$ and $T^n(x_0, y_0) \to E_2 \in \mathscr{C}_i$ for $(x_0, y_0) \in \mathscr{C}_i$ for $i = 1, 2$. Further, $\mathscr{C}_i$ are graphs of the continuous strictly increasing functions. If (x_0, y_0) is point above the curve $\mathscr{C}_1$ and below the curve $\mathscr{C}_2$ then there exists $(x_0', y_0') \in \mathscr{C}_1$ and $(x_0'', y_0'') \in \mathscr{C}_2$ such that $(x_0'', y_0'') \preceq_{se} (x_0, y_0) \preceq_{se} (x_0', y_0')$. Since $T^n(x_0'', y_0'') \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0', y_0')$ and $T^n(x_0'', y_0'') \to E_2$ and $T^n(x_0', y_0') \to E_2$ as $n \to \infty$ we have $T^n(x_0, y_0) \to E_2$ as $n \to \infty$.

(x) Suppose $(C_1, C_2) \in \mathscr{C}_{23}$. By Proposition 11, in $Q_1(0,0)$ there exist three equilibrium points E_1, $E_2 = E_3$, and E_4. By Theorem 19 $E_2 = E_3$ is nonhyperbolic, E_1 is a saddle point, and E_4 is locally asymptotically stable. By Lemma 21 and uniqueness of the global stable manifold we have $\mathscr{B}(E_1) = \mathscr{W}^s(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$. In view of Lemma 18 we have that $\lambda_1 = 1$ and $\lambda_2 < 1$ if $C_1 > (1 - A_1)/(2A_2)$ and $\lambda_2 > 1$ if $C_1 < (1 - A_1)/(2A_2)$, so we can use Theorem 6 if $C_1 > (1 - A_1)/(2A_2)$. In this case there exists strictly increasing curve $\mathscr{C}$ with endpoint at $E_2 = E_3$. The rest of the proof follows from Theorems 7, 8, and 9 and Lemma 21. Now, we assume that $C_1 < (1 - A_1)/(2A_2)$. By Lemmas 21 and 22 for $(x_0, y_0) \in \text{int}(Q_4(E_3))$, there exists t_0' and t_0'' such that $E_3 \preceq_{se} M(t_0') \preceq_{se} (x_0, y_0) \preceq_{se} (t_0'', 0)$. Since $E_3 \preceq_{se} T^n(M(t_0')) \preceq_{se} T^{n+1}(M(t_0')) \preceq E_4$ we have that $T^n(M(t_0')) \to E_4$. Since $T^n(t_0'', 0) \to E_4$ we obtain $T^n(x_0, y_0) \to E_4$. This implies $\text{int}(Q_4(E_3)) \subseteq \mathscr{B}(E_4)$. Let $\mathscr{C}$ denote the boundary of $\mathscr{B}(E_4)$ considered as a subset of $Q_1(E_3)$. It is easy to see by using Lemmas 21 and 22 that $E_3 \in \mathscr{C}$. Since $T(\text{int}(\mathscr{R})) \subset \text{int}(\mathscr{R})$, following the proof of Claims 1 and 2 [20], one can see that $T(\mathscr{C}) \subseteq \mathscr{C}$. Further, $\mathscr{C}$ is graph of strictly increasing function. By Theorem 23 if $(x_0, y_0) \in \mathscr{C}$ then $T^n(x_0, y_0) \rightarrow E_3 \in \mathscr{C}$. If (x_0, y_0) is point above the curve $\mathscr{C}$ then there exists (x_0', y_0') and $y_0'' > \overline{y}_3$ such that $(0, y_0'') \preceq_{\text{se}} (x_0, y) \preceq_{\text{se}} (x_0', y_0')$. Since $T^n(0, y_0'') \preceq_{\text{se}} T^n(x_0, y_0) \preceq_{\text{se}} T^n(x_0', y_0')$, $T^n(0, y_0'') \rightarrow E_3$, and $T^n(x_0', y_0') \rightarrow E_3$ as $n \rightarrow \infty$ we have $T^n(x_0, y_0) \rightarrow E_3$ as $n \rightarrow \infty$.

□

Based on a series of numerical simulations we propose the following conjectures.

Conjecture 26. *Suppose that all assumptions of the statement (ix) of Theorem 25 are satisfied; then the following holds:*

$$\mathscr{B}(E_2) = \mathscr{C}_1 = \mathscr{C}_2. \tag{83}$$

Conjecture 27. *Assume that $C_1, C_2 \in \mathscr{P}_1$. There exist three equilibrium points E_1, $E_3 = E_2 = E_5$, and E_4, where E_4 is locally asymptotically stable; E_1 is a saddle point; and $E_3 = E_2 = E_5$ is nonhyperbolic. The basins of attraction of E_1 and $E_3 = E_2 = E_5$ are given by $\mathscr{B}(E_1) = \{(0, y) : 0 \leq y < (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and $\mathscr{B}(E_3) = \{(0, y) : y > (1 - \sqrt{1 - 4A_2C_2})/(2A_2)\}$ and the basin of attraction of E_4 is $\mathscr{B}(E_4) = \{(x, y) : x > 0,\ y \geq 0\}$.*

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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