Research Article

Improving Fourier Partial Sum Approximation for Discontinuous Functions Using a Weight Function

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We introduce a generalized sigmoidal transformation $w_m(r; x)$ on a given interval $[a, b]$ with a threshold at $x = r \in (a, b)$. Using $w_m(r; x)$, we develop a weighted averaging method in order to improve Fourier partial sum approximation for a function having a jump-discontinuity. The method is based on the decomposition of the target function into the left-hand and the right-hand part extensions. The resultant approximate function is composed of the Fourier partial sums of each part extension. The pointwise convergence of the presented method and its availability for resolving Gibbs phenomenon are proved. The efficiency of the method is shown by some numerical examples.

1. Introduction

For a function f having a jump-discontinuity, every traditional spectral partial sum approximation will not converge uniformly on any interval containing the discontinuity. This deficiency of the spectral approximation results in the so-called Gibbs phenomenon which shows nonvanishing spikes near the discontinuity [1, 2]. There are lots of methods to overcome the problem such as the Fourier-Gegenbauer method [3–5], the inverse reconstruction [6, 7], and the adaptive filtering method [8–11]. But most existing methods need a large number of terms to support high accuracy.

In this work, focusing on the Fourier partial sum approximation for a piecewise smooth function f having a jump-discontinuity ξ, we aim to develop a constructive approximation procedure which is available for eliminating the Gibbs phenomenon near the discontinuity. First, in the following section, we introduce the so-called generalized sigmoidal transformation $w_m(r; x)$ with a threshold $r = \xi$. Using w_m, we decompose the target function f into the left-hand part extension $\widetilde{f}_L$ and the right-hand part extension $\widetilde{f}_R$ as described in Section 3. Then we combine Fourier partial sums of $\widetilde{f}_L$ and $\widetilde{f}_R$ by the form of a weighted average, F_n, as given in (26) in Section 4. We prove the pointwise convergence of the presented approximation F_n to the discontinuous function f over the whole interval. Moreover, it is shown that the asymptotic version of F_n which is composed of

uniform convergent partial sums will overcome the Gibbs phenomenon. This means that F_n can sufficiently resolve the problem of inevitable wiggles of the traditional Fourier partial sum approximation near the jump-discontinuity. In addition, numerical results for some examples show the availability of the presented method.

2. A Generalized Sigmoidal Transformation

For a given interval $[a, b]$ and some interior point $a < r < b$, referring to the literature [12], we introduce the real valued function

$$w_m(r; x) = \frac{(x - a)^m}{(x - a)^m + \{((r - a)/(b - r))(b - x)\}^m}, \quad (1)$$
$$a \leq x \leq b$$

for an integer $m \geq 1$. It was used for cumulative averaging method for piecewise polynomial interpolations in [12]. We call $w_m(x) = w_m(r; x)$ a generalized sigmoidal transformation of order m with a threshold r.

We can observe the basic properties of w_m as follows:

(i) The special case of $r = (a + b)/2$, with $a = 0$ and $b = 1$, is

$$w_m(x) = \frac{x^m}{x^m + (1 - x)^m}, \quad (2)$$
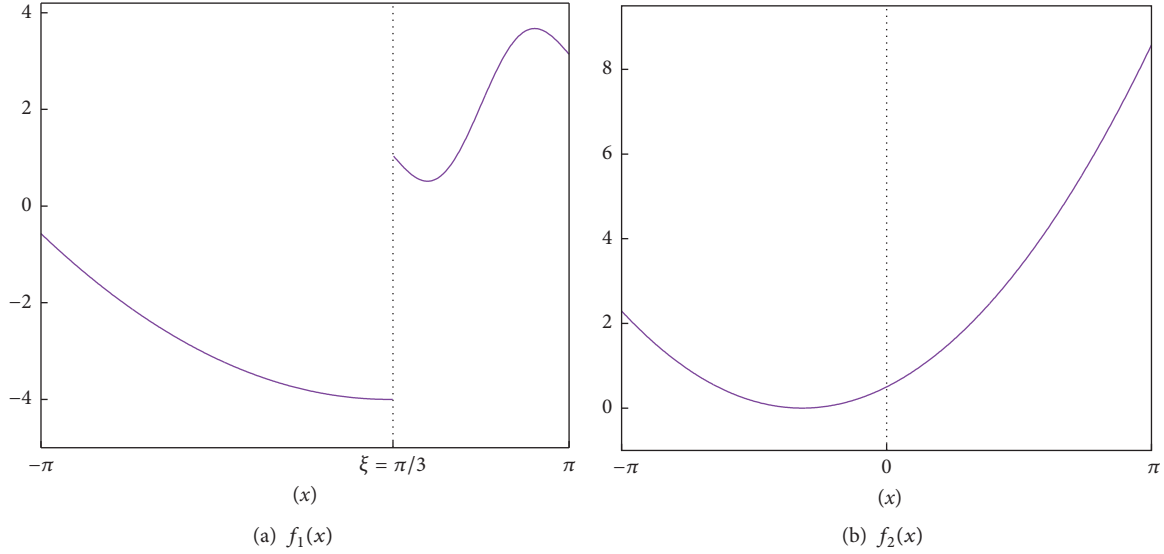
(a) $f_1(x)$

(b) $f_2(x)$

Figure 1: Graphs of test functions.

which is the same with the elementary sigmoidal transformation proposed in [13, 14].

(ii) Values of $w_m(x)$ at the points $x = a$, r, and b are

$$
w_m(a) = 0,
$$
$$
w_m(r) = \frac{1}{2}, \tag{3}
$$
$$
w_m(b) = 1
$$

independently of the parameters a, b, r, and m. In addition, $w_m(x)$ is strictly increasing on the interval $[a, b]$ because the derivative of $w_m(x)$ with respect to x satisfies

$$
\begin{aligned}
& w_m'(x) \\
& = \frac{m(b-a)(b-r)^m(r-a)^m(b-x)^{m-1}(x-a)^{m-1}}{\{(r-a)^m(b-x)^m + (b-r)^m(x-a)^m\}^2} \\
& > 0
\end{aligned} \tag{4}
$$

for all $a < x < b$.

(iii) Asymptotic behavior of $w_m(x)$ near the end points a and b is

$$
w_m(x) = \begin{cases} O\left(\left[\dfrac{x-a}{r-a}\right]^m\right), & x < r \\ 1 + O\left(\left[\dfrac{b-x}{b-r}\right]^m\right), & x > r \end{cases} \tag{5}
$$

as m goes to the infinity. Moreover, w_m is sufficiently smooth over the interval (a, b); that is, $w_m(x) \in C^\infty(a, b)$.

The generalized sigmoidal transformation w_m plays an important role in developing a new approximation method as a weight function in this work.

3. Decomposition of a Discontinuous Function

From now on we suppose that f is a piecewise smooth function containing a jump-discontinuity ξ in an interval $[a, b]$. We assume that the location of ξ or its accurate approximation is known and that the value $f(\xi)$ is defined to be the average of the left- and right-hand limits of f at ξ; that is,

$$
f(\xi) = \frac{\{f(\xi-) + f(\xi+)\}}{2}. \tag{6}
$$

On the other side, taking $r = \xi$ in formula (1) or $w_m = w_m(\xi; x)$, we will use it as a weight function for the proposed approximation method in this work.

We choose the test functions below whose graphs are given in Figure 1:

$$
f_1(x) = \begin{cases} \dfrac{1}{5}(x-1)^2 - 4, & -\pi \le x < \dfrac{\pi}{3} \\ x + \sin(3x), & \dfrac{\pi}{3} < x \le \pi \end{cases} \tag{7}
$$

which have a jump-discontinuity $\xi = \pi/3$:

$$
f_2(x) = \frac{1}{2}(x+1)^2, \quad -\pi \le x \le \pi \tag{8}
$$

which is continuous on the interval $[-\pi, \pi]$. We notice that $f_i(-\pi) \ne f_i(\pi)$, $i = 1, 2$, and thus both f_1 and f_2 have jump-discontinuities at $\pm\pi$ when we extend these functions to the 2π periodic functions over the real line.

Let the piecewise smooth function $f(x)$, containing a jump-discontinuity ξ, be defined as

$$
f(x) = \begin{cases} f_L(x), & a \le x < \xi \\ f_R(x), & \xi < x \le b, \end{cases} \tag{9}
$$

where f_L and f_R are continuous on $[a, \xi]$ and $(\xi, b]$, respectively. We assume that the order m of $w_m = w_m(\xi; x)$ is large

enough throughout this paper. First, we define two quadratic polynomials Q_L and Q_R as

$$Q_L(x) = \frac{a - \xi}{(b - \xi)^2}(x - \xi)^2 + \xi \tag{10}$$

satisfying

$$\begin{aligned} Q_L(\xi) &= \xi, \\ Q_L(b) &= a, \\ Q_R(x) &= \frac{b - \xi}{(a - \xi)^2}(x - \xi)^2 + \xi \end{aligned} \tag{11}$$

satisfying

$$\begin{aligned} Q_R(\xi) &= \xi, \\ Q_R(a) &= b. \end{aligned} \tag{12}$$

It should be noted that the left- and right-hand limits of Q_L and Q_R at ξ are

$$\begin{aligned} Q_L(\xi-) &= Q_L(\xi+) = \xi-, \\ Q_R(\xi-) &= Q_R(\xi+) = \xi+. \end{aligned} \tag{13}$$

Then we construct extensions of f_L and f_R onto the whole interval $[a, b]$ as

$$\widetilde{f}_L(x) = \{1 - w_m(x)\} f(x) + w_m(x) f(Q_L(x)), \tag{14}$$

$$\widetilde{f}_R(x) = w_m(x) f(x) + \{1 - w_m(x)\} f(Q_R(x)), \tag{15}$$

respectively, for $a \le x \le b$. It is seen that

$$\widetilde{f}_L(\xi) = \widetilde{f}_R(\xi) = f(\xi) = \frac{(f(\xi-) + f(\xi+))}{2} \tag{16}$$

as $w_m(\xi) = 1/2$.

One can surmise that, for sufficiently large m, $\widetilde{f}_L$ has the effect of reflecting the left part f_L of f on $[a, \xi]$ into the opposite side $[\xi, b]$. So does $\widetilde{f}_R$, symmetrically. In addition, these extended functions $\widetilde{f}_L$ and $\widetilde{f}_R$ defined in (14) and (15) have some particular properties as shown in the following lemmas.

Lemma 1. *Let f be a piecewise smooth function on $[a, b]$ with a jump-discontinuity ξ, and suppose that the order m of w_m is fixed and finite. Then we have the one-sided limits of $\widetilde{f}_L$ and $\widetilde{f}_R$ as follows:*

$$\begin{aligned} \widetilde{f}_L(\xi-) &= f(\xi-), \\ \widetilde{f}_L(\xi+) &= \frac{(f(\xi-) + f(\xi+))}{2}, \\ \widetilde{f}_R(\xi-) &= \frac{(f(\xi-) + f(\xi+))}{2}, \\ \widetilde{f}_R(\xi+) &= f(\xi+). \end{aligned} \tag{17}$$

Furthermore,

$$\begin{aligned} \widetilde{f}_L(a+) &= \widetilde{f}_L(b-) \quad (= f(a)), \\ \widetilde{f}_R(a+) &= \widetilde{f}_R(b-) \quad (= f(b)). \end{aligned} \tag{18}$$

Proof. Since $\lim_{x \to \xi} w_m(x) = 1/2$ and $f(Q_L(\xi-)) = f(Q_L(\xi+)) = f(\xi-)$ for some m fixed, from (14) we have

$$\begin{aligned} \widetilde{f}_L(\xi-) &= \frac{1}{2}\{f(\xi-) + f(Q_L(\xi-))\} = f(\xi-), \\ \widetilde{f}_L(\xi+) &= \frac{1}{2}\{f(\xi+) + f(\xi-)\}. \end{aligned} \tag{19}$$

By the same way, from (15) we have $\widetilde{f}_R(\xi-) = (1/2)\{f(\xi-) + f(\xi+)\}$ and $\widetilde{f}_R(\xi+) = f(\xi+)$. The equations in (18) directly result from the properties of Q_L, Q_R, and w_m. □

Properties (17) in Lemma 1 imply that both $\widetilde{f}_L$ and $\widetilde{f}_R$ have the jump-discontinuity at ξ if the original function f has a jump-discontinuity such as $f(\xi-) \ne f(\xi+)$. The properties in (18) may resolve the troublesome problem in Fourier series approximation resulting from the mismatch at the end points.

In Figure 2, graphs of $\widetilde{f}_L$ and $\widetilde{f}_R$ for the test function $f(x) = f_1(x)$ with $m = 40$, for example, illustrate the results in Lemma 1. Therein, thick lines indicate principal part $f_L(x)$ of the extended functions $\widetilde{f}_L$ in (a) and $f_R(x)$ of $\widetilde{f}_R$ in (b). Thin lines indicate reflected parts of $f_L(x)$ and $f_R(x)$ in (a) and (b), respectively, and dotted lines show the original graph of $f(x)$.

For sufficiently large m, however, we can see that the jump-discontinuities of $\widetilde{f}_L$ and $\widetilde{f}_R$ at ξ vanish as shown in the following lemma.

Lemma 2. *For a function f assumed in Lemma 1 both*

$$\begin{aligned} &\widetilde{f}_L(\xi+) - \widetilde{f}_L(\xi-), \\ &\widetilde{f}_R(\xi+) - \widetilde{f}_R(\xi-) \end{aligned} \tag{20}$$

vanish as m goes to the infinity.

Proof. It follows that $w_m(\xi-) = 0$ and $w_m(\xi+) = 1$ as $m \to \infty$. Thus from (14) and (15) we have $\widetilde{f}_L(\xi-) = \widetilde{f}_L(\xi+) = f(\xi-)$ and $\widetilde{f}_R(\xi-) = \widetilde{f}_R(\xi+) = f(\xi+)$. The proof is completed. □

Lemma 2 indicates the asymptotic behavior of $\widetilde{f}_L$ and $\widetilde{f}_R$ below:

$$\widetilde{f}_L(x) \sim \widetilde{f}_{L,\infty}(x) := \begin{cases} f_L(x), & a \le x \le \xi \\ f_L(Q_L(x)), & \xi < x \le b, \end{cases} \tag{21}$$

$$\widetilde{f}_R(x) \sim \widetilde{f}_{R,\infty}(x) := \begin{cases} f_R(Q_R(x)), & a \le x < \xi \\ f_R(x), & \xi \le x \le b \end{cases} \tag{22}$$

for m large enough. It should be noted that

$$\begin{aligned} \lim_{x \to \xi} \widetilde{f}_{L,\infty}(x) &= f(\xi-), \\ \lim_{x \to \xi} \widetilde{f}_{R,\infty}(x) &= f(\xi+). \end{aligned} \tag{23}$$

Thus, if we replace the values of f_L and f_R at ξ as $f_L(\xi) = f(\xi-)$ and $f_R(\xi) = f(\xi+)$, then both $\widetilde{f}_{L,\infty}$ and $\widetilde{f}_{R,\infty}$ are continuous on the whole interval $[a, b]$.

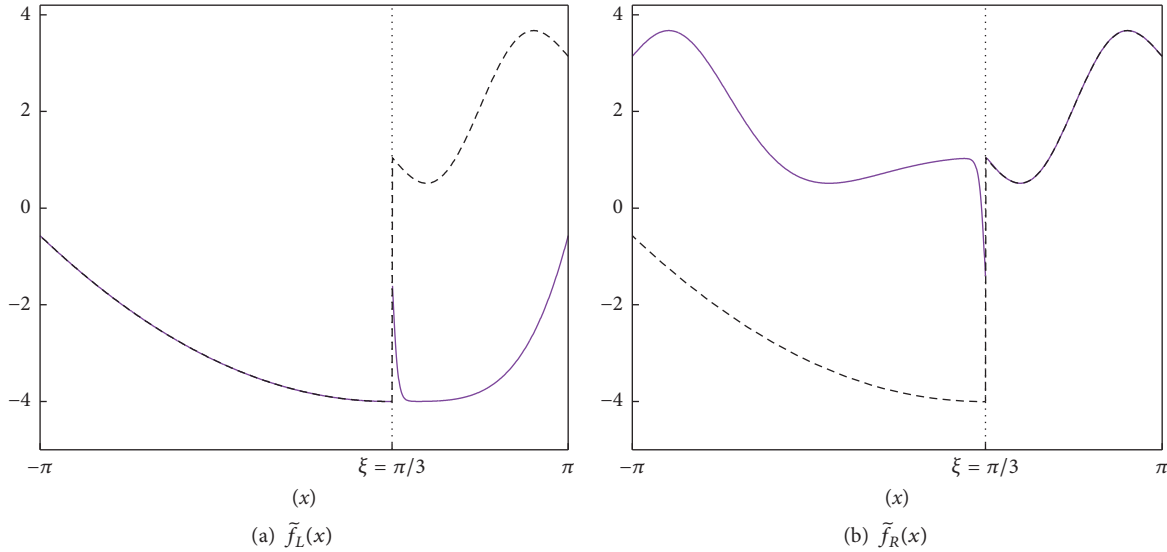(a) $\widetilde{f}_L(x)$

(b) $\widetilde{f}_R(x)$

FIGURE 2: Graphs of the extended functions $\widetilde{f}_L(x)$ and $\widetilde{f}_R(x)$ with $m = 40$ for the test function $f_1(x)$.

4. Improving Fourier Partial Sum Approximation

In this section we assume that the piecewise smooth function f is defined on $[a, b] = [-\pi, \pi]$ with a jump-discontinuity $-\pi < \xi < \pi$. We consider Fourier series of $\widetilde{f}_L$ and $\widetilde{f}_R$ in the form of

$$
S_n \widetilde{f}_L (x) = \sum_{k=-n}^{n} c_k^L e^{ikx},
$$

$$
S_n \widetilde{f}_R (x) = \sum_{k=-n}^{n} c_k^R e^{ikx},
$$
(24)

where c_k^L and c_k^R are Fourier coefficients defined as

$$
c_k^L = \int_{-\pi}^{\pi} \widetilde{f}_L (t) e^{-ikt} dt,
$$

$$
c_k^R = \int_{-\pi}^{\pi} \widetilde{f}_R (t) e^{-ikt} dt.
$$
(25)

Then we propose a weighted average of $S_n \widetilde{f}_L$ and $S_n \widetilde{f}_R$ as follows:

$$
F_n (x) = \left\{ 1 - w_m (x) \right\} S_n \widetilde{f}_L (x) + w_m (x) S_n \widetilde{f}_R (x)
$$

$$
= \sum_{k=-n}^{n} \left\{ c_k^L + \left(c_k^R - c_k^L \right) w_m (x) \right\} e^{ikx}
$$
(26)

for $-\pi \leq x \leq \pi$. It is noted that, like $\widetilde{f}_L$ and $\widetilde{f}_R$, the weighted average F_n is discontinuous at ξ if $f(\xi-) \neq f(\xi+)$. Nevertheless, F_n has the meaningful convergence properties shown in the following theorem.

Theorem 3. *Let f be a function assumed in Lemma 1 with $[a, b] = [-\pi, \pi]$. Then we have the following:*

(1) *For the order m of w_m fixed, the weighted average $F_n(x)$ converges to*

$$
f (x) + O \left(h (x)^m \right),
$$

$$
h (x) = \min \left\{ \frac{x + \pi}{\xi + \pi}, \frac{\pi - x}{\pi - \xi} \right\}
$$
(27)

pointwise over the interval $[-\pi, \pi]$, provided that the value of F_n at the jump-discontinuity ξ is defined as $F_n(\xi) := (F_n(\xi-) + F_n(\xi+))/2$.

(2) *Let $F_{n,\infty}$ be a modified formula of F_n, in (26), obtained by replacing $\widetilde{f}_L$ and $\widetilde{f}_R$ by their asymptotic versions $\widetilde{f}_{L,\infty}$ and $\widetilde{f}_{R,\infty}$ defined in (21) and (22), respectively, with the assumptions $f_L(\xi) = f(\xi-)$ and $f_R(\xi) = f(\xi+)$. Then $F_{n,\infty}$ converges to f pointwise over the interval $[-\pi, \pi]$, getting out of the Gibbs phenomenon, as $m, n \to \infty$.*

Proof. It is noted that $S_n \widetilde{f}_L$ and $S_n \widetilde{f}_R$, respectively, converge to $\widetilde{f}_L$ and $\widetilde{f}_R$ pointwise on the interval $[-\pi, \pi]$ because $\widetilde{f}_L$ and $\widetilde{f}_R$ are both piecewise smooth [1].

Let $x < \xi$. Then from (26) and (5) we have

$$
\lim_{n \to \infty} F_n (x)
$$

$$
= \lim_{n \to \infty} \left\{ S_n \widetilde{f}_L (x) + \left[S_n \widetilde{f}_R (x) - S_n \widetilde{f}_L (x) \right] w_m (x) \right\}
$$

$$
= f (x) + O \left(\left[\frac{x + \pi}{\xi + \pi} \right]^m \right).
$$
(28)

Similarly, for $x > \xi$,

$$
\lim_{n \to \infty} F_n (x) = \lim_{n \to \infty} \left\{ S_n \widetilde{f}_R (x) \right.
$$

$$
\left. + \left[S_n \widetilde{f}_L (x) - S_n \widetilde{f}_R (x) \right] (1 - w_m (x)) \right\} = f (x)
$$

$$
+ O \left(\left[\frac{\pi - x}{\pi - \xi} \right]^m \right).
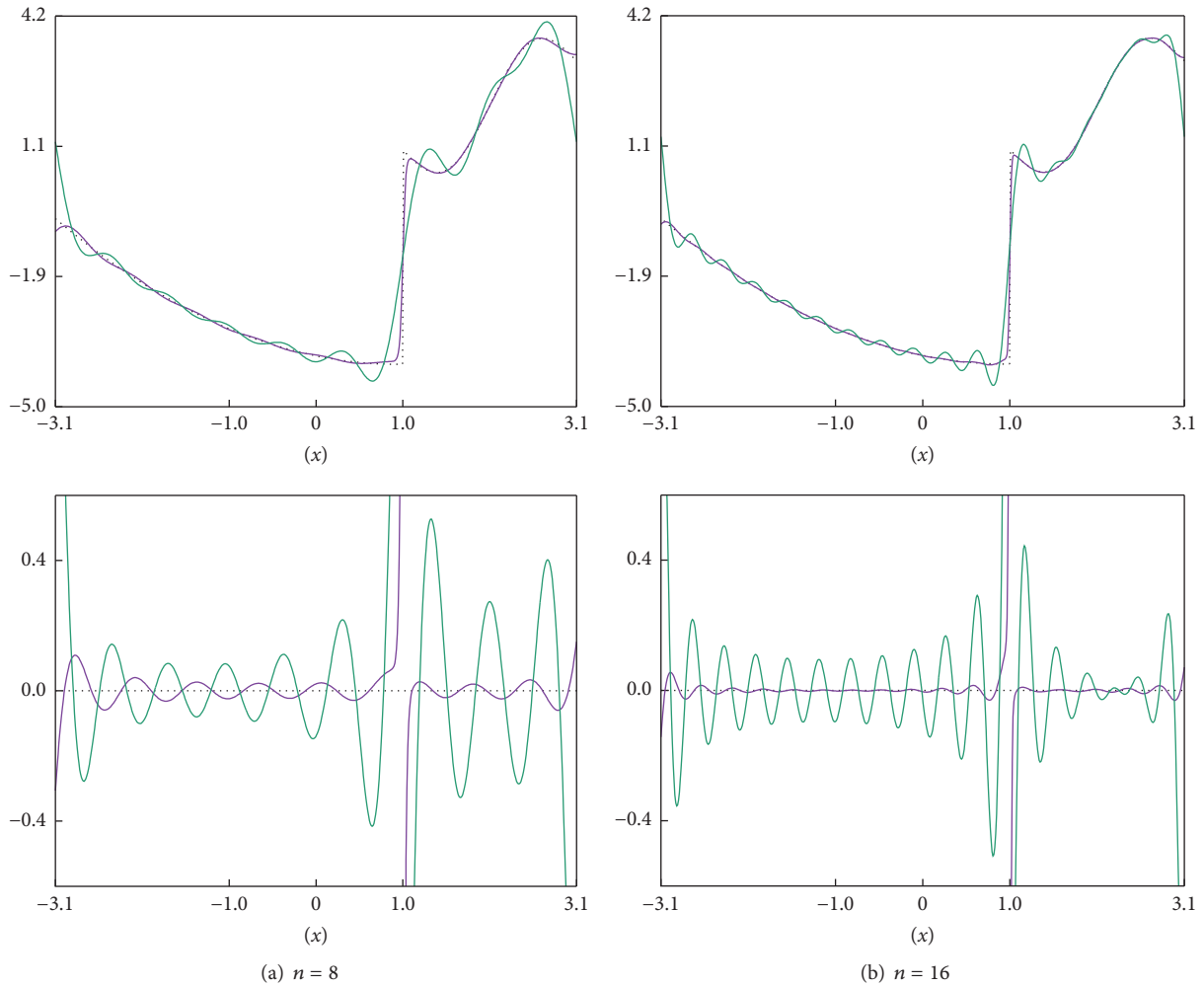$$
(29)

(a) $n = 8$

(b) $n = 16$

FIGURE 3: Approximations of the weighted average $F_n(x)$ in upper rows and the corresponding errors in lower rows for the test function $f_1(x)$. Thin lines indicate approximations and errors of the Fourier partial sum $S_n f(x)$.

For $x = \xi$, from definition (26) of F_n and results (17) in Lemma 1, we have the equations

$$\lim_{n \to \infty} F_n(\xi-) = \frac{1}{2} \left\{ \tilde{f}_L(\xi-) + \tilde{f}_R(\xi-) \right\}$$

$$= \frac{(3f(\xi-) + f(\xi+))}{4},$$

$$\lim_{n \to \infty} F_n(\xi+) = \frac{1}{2} \left\{ \tilde{f}_L(\xi+) + \tilde{f}_R(\xi+) \right\}$$

$$= \frac{(f(\xi-) + 3f(\xi+))}{4}.$$

(30)

This implies that

$$\lim_{n \to \infty} F_n(\xi) = \lim_{n \to \infty} \frac{(F_n(\xi-) + F_n(\xi+))}{2}$$

$$= \frac{f(\xi-) + f(\xi+)}{2} = f(\xi).$$

(31)

Therefore, the proof of the assertion that F_n converges to f pointwise over the interval $[-\pi, \pi]$ is completed.

For (2), it is clear from assertion (1) that $F_{n,\infty}$ converges to f pointwise over the interval $[-\pi, \pi]$ as $m, n \to \infty$. On the other hand, the definitions of $\tilde{f}_{L,\infty}$ and $\tilde{f}_{R,\infty}$ in (21) and (22), respectively, and the assumptions $f_L(\xi) = f(\xi-)$ and $f_R(\xi) = f(\xi+)$ imply that $\tilde{f}_{L,\infty}$ and $\tilde{f}_{R,\infty}$ are continuous at the original discontinuity ξ with $\tilde{f}_{L,\infty}(-\pi) = \tilde{f}_{L,\infty}(\pi)$ and $\tilde{f}_{R,\infty}(-\pi) = \tilde{f}_{R,\infty}(\pi)$. That is, $\tilde{f}_{L,\infty}$ and $\tilde{f}_{R,\infty}$ are free of jump-discontinuity at $x = \xi, \pm\pi$. Thus, the Fourier series $S_n \tilde{f}_{L,\infty}$ and $S_n \tilde{f}_{R,\infty}$ uniformly converge to $\tilde{f}_{L,\infty}$ and $\tilde{f}_{R,\infty}$, respectively. As a result, we can see that the weighted combination $F_{n,\infty}$ of $S_n \tilde{f}_{L,\infty}$ and $S_n \tilde{f}_{R,\infty}$ will get out of the Gibbs phenomenon as $m, n \to \infty$. This completes the proof. □

Results of the approximations and errors of $F_n(x)$ with $n = 8, 16$, for the test functions f_1 and f_2, are illustrated in Figures 3 and 4, respectively. Therein, we took the order of weight function as $m = 10n$, for example. The results
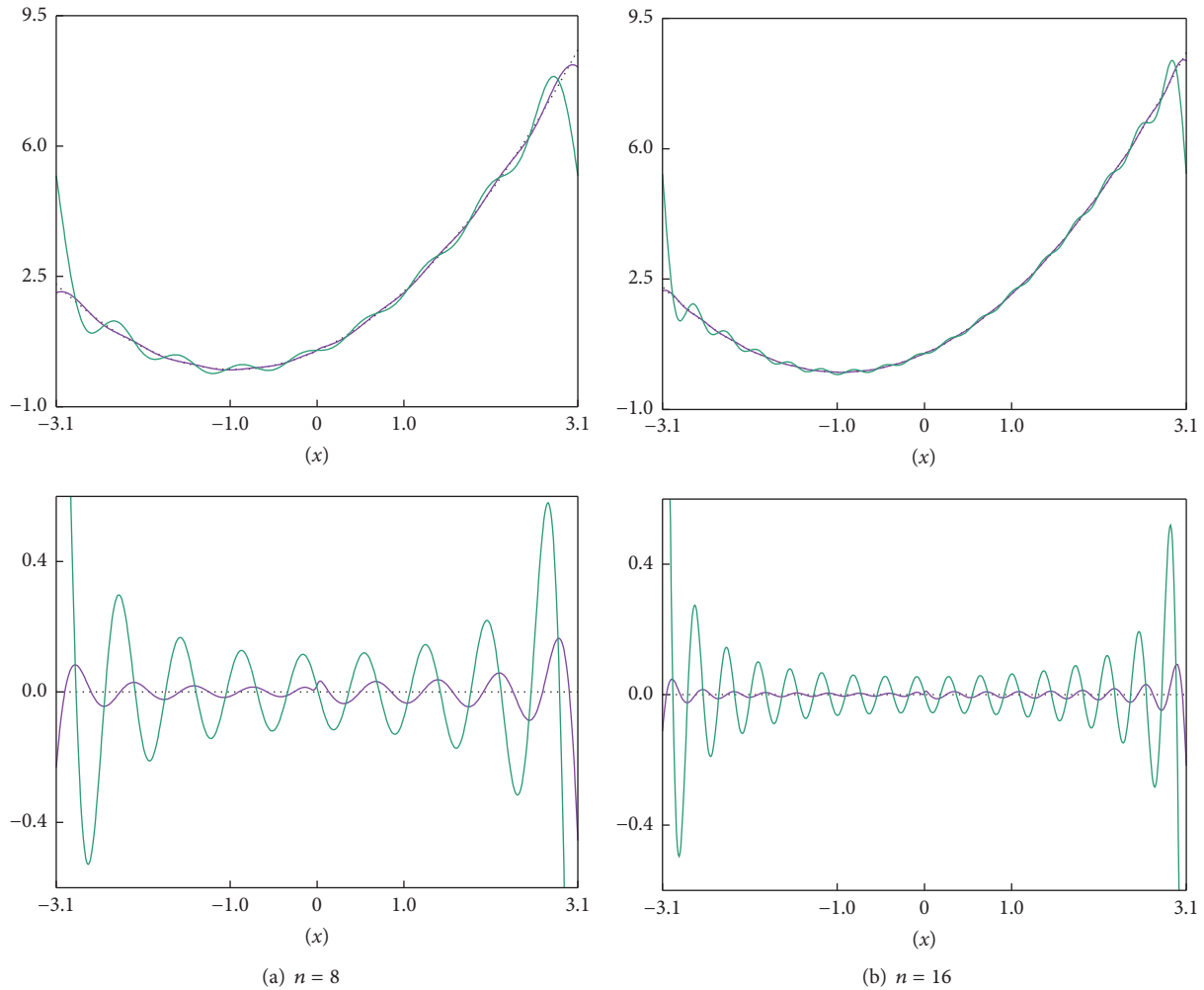
(a) $n = 8$ (b) $n = 16$

FIGURE 4: Approximations of the weighted average $F_n(x)$ in upper rows and the corresponding errors in lower rows for the test function $f_2(x)$. Thin lines indicate approximations and errors of the Fourier partial sum $S_n f(x)$.

of $F_n(x)$, indicated by the thick lines, are compared with those of the traditional Fourier partial sum approximation $S_n f(x)$ which are indicated by the thin lines. The figures show that the proposed approximation $F_n(x)$ highly improves the Fourier partial sum approximation over the whole interval. In particular, the Gibbs phenomenon resulting from the interior jump-discontinuity or the mismatch at the end points has been resolved by $F_n(x)$.

Conflicts of Interest

The author declares that he has no conflicts of interest regarding the publication of this paper.

Acknowledgments

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