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Research Article

The Order Classes of 2-Generator p-Groups

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In order to classify a finite group using its elements orders, the order classes are defined. This partition determines the number of elements for each order. The aim of this paper is to find the order classes of 2-generator p-groups of class 2. The results obtained here are supported by Groups, Algorithm and Programming (GAP).

1. Introduction

One of the major partitions for finite groups is the order classes. A basic concept in group theory is that the order of an element $x \in G$ denoted by $o(x)$ is the smallest positive integer k, such that x^k is the identity. The relation "x is of the same order as y" is an equivalence relation, which induces a partition for the group G, which is called the order classes. Order classes of symmetric and dihedral groups are completely configured in [1] and [2], respectively.

Clearly, all conjugate elements have the same order. Conjugacy classes are refinement partitions to order classes. Therefore, each order class contains at least one conjugacy class. Du and Shi [3] proved that if a finite group G has conjugacy classes number one greater than its same order classes number, then G is isomorphic to one of the following groups: $A_5, L_2(7), S_5, C_3, C_4, S_4, A_4, D_{10}, \mathrm{Hol}(C_5)$, or $C_3 \rtimes C_4$.

In order to classify a finite group using its order classes, there is a new issue obtained by the size of the order classes. That is, a finite group G is said to be a perfect order subsets group (POS-group) if the cardinality of each order class divides $|G|$. Das [4] studied some of the properties of arbitrary POS-groups and constructed a couple of new families of nonabelian POS-groups. He also proved that the alternating group A_n, $n \geq 3$, is not a POS-group. Later, Jones and Toppin [5] proved that any nontrivial finite POS-group has even order.

The classification of all p-groups is not completed yet. In 1993 the classification of finite 2-generator p-groups of class 2 has been studied in [6]. Ahmad et al. [7] classified 2-generator p-groups of class 2 and defined these groups as a central extension of cyclic p-groups, that is, to obtain the exact number of conjugacy classes for these groups. In this study we will follow the same classification found in [7], to investigate the order classes of 2-generator p-groups of nilpotency class 2.

The results obtained here were found using GAP. Fortunately, using our main theorem, we have developed a practical GAP algorithm to find the order classes of 2-generator p-groups of class 2 (p odd prime).

2. Preliminaries and Definitions

Our notation is fairly standard. By $|G|$ we denote the order of a finite group G and we denote the identity element of G by e. The order of an element $a \in G$, denoted by $o(a)$, is the smallest positive integer m such that $a^m = e$. The set of all possible orders for a finite group G will be denoted by O. The class of all elements of G which have the same order of $x \in G$ is called the order class of x. Equivalently, the class of all elements of G of order $o(x) = j$ is the order class of j and is denoted by O_j. The order classes of a group G will be denoted by $\mathfrak{G}_G$, which consists of all possible pairs of the form $[j, |O_j|]$ for all $j \in O$.

The derived subgroup and the center of a group G are denoted by G' and $Z(G)$, respectively.

Let G be a group. The commutator of $x, y \in G$ is given by $[x, y] = x^{-1}y^{-1}xy$. For any subgroups A and B of a group G the commutator subgroup is $[A, B] = \langle [a, b] \mid a \in A,\ b \in B\rangle$. Note that the lower central series of a group G is

$$G = \gamma_0(G) \geq \gamma_1(G) \geq \cdots \geq \gamma_c(G) \geq \cdots, \tag{1}$$

where $\gamma_i(G) = [\gamma_{i-1}(G), G]$ for $i = 1, 2, 3, \ldots$.

Definition 1. A group G is called nilpotent if there exists c such that $\gamma_c(G) = [\gamma_{c-1}(G), G] = \{e\}$, and the smallest such c is the class of nilpotency.

All abelian groups are nilpotent of class 1. If p is prime, then the group in which every element has order a power of p is called a p-group. If G is a finite p-group, then the order of G is a power of p. Such groups are nilpotent. A group G is nilpotent group of class 2 if $\gamma_2(G) = \{e\}$; equivalently $\gamma_1(G) = [\gamma_0(G), G] = [G, G] = G' \leq Z(G)$.

In a finite p-group G of order p^n, the center $Z(G)$ is a subgroup of G. Using Lagrange's theorem, it is implied that $|Z(G)| = p^k$ for some integer $k \leq n$.

Lemma 2 (see [8]). *Let G be a group of nilpotency classes 2. Let $x, y, z \in G$ and $n \in \mathbb{N}$; then*

(1) $[x, yz] = [x, y][x, z]$,

(2) $[xy, z] = [x, z][y, z]$,

(3) $[x^n, y] = [x, y^n] = [x, y]^n$,

(4) $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$.

Lemma 3 (see [6]). *Let G be a group of nilpotency classes 2 and $g, h \in G$ with $o(g)$ and $o(h)$ being odd. Then*

(1) $o(gh) \leq \max\{o(g), o(h)\}$,

(2) $o([g, h]) \leq \min\{o(g), o(h)\}$.

The following theorem is used to describe the structure of a 2-generator p-group of nilpotency class 2 in terms of generators and relations.

Theorem 4 (see [7]). *Let p be a prime and $n > 2$ a positive integer. Every 2-generator p-group of order p^n and class 2 corresponds to an ordered 5-tuple of integers, $(\alpha, \beta, \gamma; \rho, \sigma)$, such that*

(1) $\alpha \geq \beta \geq \gamma \geq 1$,

(2) $\alpha + \beta + \gamma = n$,

(3) $0 \leq \rho \leq \gamma$ *and* $0 \leq \sigma \leq \gamma$,

where $(\alpha, \beta, \gamma; \rho, \sigma)$ corresponds to the group presented by

$$G = \left\langle a, b \mid [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = e,\ a^{p^\alpha} = [a, b]^{p^\rho},\ b^{p^\beta} = [a, b]^{p^\sigma} \right\rangle. \tag{2}$$

Moreover

(1) *if $\alpha > \beta$, then G is isomorphic to*

  (a) $(\alpha, \beta, \gamma; \rho, \sigma)$ *when* $\rho \leq \sigma$;

  (b) $(\alpha, \beta, \gamma; \rho, \sigma)$ *when* $0 \leq \sigma < \sigma + \alpha - \beta$ *or* $\sigma < \rho = \gamma$;

  (c) $(\alpha, \beta, \gamma; \rho, \sigma)$ *when* $0 \leq \sigma < \rho < \min(\gamma, \sigma + \alpha - \beta)$;

(2) *if $\alpha = \beta > \gamma$, or $\alpha = \beta = \gamma$ and $p > 2$, then G is isomorphic to $(\alpha, \beta, \gamma; \min(\rho, \sigma), \gamma)$;*

(3) *if $\alpha = \beta = \gamma$ and $p = 2$, then G is isomorphic to*

  (a) $(\alpha, \beta, \gamma; \min(\rho, \sigma), \gamma)$ *when* $0 \leq \min(\rho, \sigma) < \gamma - 1$;

  (b) $(\alpha, \beta, \gamma; \gamma - 1, \gamma - 1)$ *when* $\rho = \sigma = \gamma - 1$;

  (c) $(\alpha, \beta, \gamma; \gamma, \gamma)$ *when* $\min(\rho, \sigma) \geq \gamma - 1$ *and* $\max(\rho, \sigma) = \gamma$.

The groups listed in 1(a)–3(c) are pairwise nonisomorphic.

If p is prime and G is a 2-generator p-group of class 2, with $|G| = p^n$, $n \geq 3$, then $|Z(G)| = p^{n-2\gamma}$, where $|G'| = p^\gamma$ [7]. Let G be a 2-generator p-group of class 2. Then $C_{p^\alpha}, C_{p^\beta}$, and C_{p^γ} are the polycyclic series of G. Hence, a, b and $[a, b]$ are the polycyclic generators of G. Therefore, if $g \in G$, then g can be written uniquely as $g = a^x b^y [a, b]^z$, where $0 \leq x \leq \alpha$, $0 \leq y \leq \beta$, and $0 \leq z \leq \gamma$.

3. Order Classes of 2-Generator p-Groups of Nilpotency Class 2

The previous classification for 2-generator p-groups will be used to obtain the order classes of these groups. Let $\mathfrak{G}$ be the set of all 2-generator p-groups of nilpotency class 2 with p being an odd prime and $|G| = p^n$, $n \geq 3$, $|G'| = p^\gamma$. To find the order classes of a group G, we need to answer some important issues related to G, such as the description of the available orders j; the largest possible order $\exp(G)$, to achieve the set O; the count of elements of each order family to obtain $|O_j|$. The following lemmas will justify these issues and concepts to establish the order classes in terms of $\mathfrak{O}_G = \{[j, |O_j|] \mid j \in O\}$.

Lemma 5. *Let $G \in \mathfrak{G}$ be the group generated by a and b, with $o(a) = p^i$, $o(b) = p^j$. Then $\gamma \leq \min\{i, j\}$, where $|G'| = p^\gamma$.*

Proof. The proof follows directly using Lemma 3, since

$$p^\gamma = \left|G'\right| = |\langle [a, b]\rangle| \leq \min\left\{p^i, p^j\right\}. \tag{3}$$

□

Reasonably, for $G \in \mathfrak{G}$, the order 5-tuple of integers $(\alpha, \beta, \gamma; \rho, \sigma)$ in Theorem 4 was configured to construct the group G. But the new order pair (i, j) obtained by the generators orders is a different pair; it is clear that $i + j \leq n$ for all $G \in \mathfrak{G}$. So that (i, j) will never be used instead of (α, β), although they are occasionally similar.

Let $G \in \mathfrak{G}$. Then the order of any element in G should divides $|G| = p^n$. Therefore, if $g \in G$, then $o(g)$ should be written as a power of p. Thus, $O = \{1, p, p^2, \ldots, p^r\}$ where $r < n$ (if $r = n$, then G is cyclic group). The following lemma establishes the largest possible order p^r in terms of the generators orders (i, j).

Lemma 6. *If $G \in \mathfrak{G}$ is the group generated by a and b, such that $o(a) = p^i$, $o(b) = p^j$, letting $w = \max\{i, j\}$, then the exponent of G, denoted by $\exp(G)$, is given by*

$$\exp(G) = p^w. \tag{4}$$

Proof. Let $G \in \mathfrak{G}$ and $g \in G$. Theorem 4 gives that $g = a^x b^y [a, b]^z$, where $1 \le x \le p^i$, $1 \le y \le p^j$, and $1 \le z \le p^\gamma = |G'|$. Therefore

$$\begin{aligned} g^{p^k} &= \left(a^x b^y [a,b]^z\right)^{p^k} = \left(a^x b^y\right)^{p^k} [a,b]^{zp^k} \\ &= a^{xp^k} b^{yp^k} [a,b]^{xy\binom{p^k}{2}} [a,b]^{zp^k} \\ &= a^{xp^k} b^{yp^k} [a,b]^{(1/2)xyp^k(p^k-1)+zp^k}. \end{aligned} \tag{5}$$

Then

$$g^{p^k} = \left(a^{p^k}\right)^x \left(b^{p^k}\right)^y \left([a,b]^{p^k}\right)^{(1/2)xy(p^k-1)+z}. \tag{6}$$

Notice that $a^{p^w} = b^{p^w} = [a,b]^{p^w} = e$. Hence

$$\begin{aligned} g^{p^w} &= \left(a^{p^w}\right)^x \left(b^{p^w}\right)^y \left([a,b]^{p^w}\right)^{(1/2)xy(p^w-1)+z} \\ &= e^x e^y e^{(1/2)xy(p^w-1)+z} = e. \end{aligned} \tag{7}$$

□

Using Lemma 6, it follows that the set of all possible orders is $O = \{1, p, p^2, \ldots, p^w\}$, where $w = \max\{i, j\}$. Hence $\mathfrak{G}_G = \{[1,1], [p, Y_1], [p^2, Y_2], \ldots, [p^w, Y_w]\}$, where Y_k is the number of elements of order p^k for $k = 1, 2, \ldots, w$.

According to the previous classifications our main results will be as follows.

Theorem 7. *Let $G \in \mathfrak{G}$ be the group generated by a and b, with $|G| = p^n$, $n \ge 3$, $o(a) = p^i$, and $o(b) = p^j$ ($i + j \le n$ for all $G \in \mathfrak{G}$). Let $w = \max\{i, j\}$. Then, G has Y_k elements of order p^k, $k = 0, 1, \ldots, w$, where*

$$\begin{aligned} Y_0 &= 1, \\ Y_1 &= p^2 m_1 - 1, \\ Y_k &= p^2 m_k, \quad \text{with } k = 2, 3, \ldots, w \end{aligned} \tag{8}$$

such that

(1)

$$m_1 = \begin{cases} 1, & i + j = n \\ p, & i + j < n. \end{cases} \tag{9}$$

(2)

$$m_2 = \begin{cases} p^{n-w} - 1, & i + j = n \\ p^{n-w} - p, & i + j < n. \end{cases} \tag{10}$$

(3) *If $w \ge 3$. Then $m_w = p^{n-3}(p-1)$.*

(4) *If $w > 3$. Then $m_{(w-k)} = m_w / p^k$ for $k = 0, 1, \ldots, w-3$.*

(5) $\sum_{i=1}^{w} m_i = p^{n-2}$.

Proof. The identity element e is the only element in G of order 1; therefore $Y_0 = 1$. Without loss of generality, let $i \ge j$.

(1) Let $g \in G$; then $g = a^x b^y [a, b]^z$, where $1 \le x \le p^i$, $1 \le y \le p^j$, and $1 \le z \le p^\gamma$.

Using (6), it is implied that

$$g^p = \left(a^p\right)^x \left(b^p\right)^y \left([a,b]^p\right)^{(1/2)xy(p-1)+z}. \tag{11}$$

Since $p > 2 \ge \gamma$, hence $[a, b]^p = e$; therefore

$$g^p = \left(a^p\right)^x \left(b^p\right)^y. \tag{12}$$

Then

$$\begin{aligned} Y_1 &= \left|\{g \in G \mid g^p = e\}\right| = \left|\{(x, y) \in \mathbb{N} \right. \\ &\quad \left. \times \mathbb{N} \mid \left(a^p\right)^x \left(b^p\right)^y = e;\ 1 \le x \le p^i,\ 1 \le y \le p^j\}\right|. \end{aligned} \tag{13}$$

Case 1. If $i + j = n$, then

$$\begin{aligned} x &= c_1 p, \quad c_1 = 1, 2, \ldots, p, \\ y &= c_2 p, \quad c_2 = 1, 2, \ldots, p. \end{aligned} \tag{14}$$

Hence, there are p choices for c_1; they are originally for x and similarly there are p choices for y. Therefore there are p^2 choices for (x, y). Note that $(a^p)^x = (b^p)^y = e$ for $x = p^i$ and $y = p^j$. Then

$$\begin{aligned} &\left|\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \left(a^p\right)^x \left(b^p\right)^y = e;\ 1 \le x \le p^i,\ 1 \le y \right. \\ &\quad \left. \le p^j\}\right| = p^2 - 1. \end{aligned} \tag{15}$$

Case 2. If $i + j < n$, then

$$\begin{aligned} x &= c_1 p, \quad c_1 = 1, 2, \ldots p, \\ y &= c_2 p, \quad c_2 = 1, 2, \ldots, p^2. \end{aligned} \tag{16}$$

Therefore, there are $p \cdot p^2$ choices for (x, y). The identity element is omitted. Thus

$$\begin{aligned} &\left|\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \left(a^p\right)^x \left(b^p\right)^y = e;\ 1 \le x \le p^i,\ 1 \le y \right. \\ &\quad \left. \le p^j\}\right| = p^3 - 1. \end{aligned} \tag{17}$$

(2) Using similar arguments as Case 1, then

$$\begin{aligned} Y_2 &= \left|\{g \in G \mid g^{p^2} = e\}\right| = \left|\{(x, y) \in \mathbb{N} \right. \\ &\quad \left. \times \mathbb{N} \mid \left(a^{p^2}\right)^x \left(b^{p^2}\right)^y = e;\ 1 \le x \le p^i,\ 1 \le y \le p^j\}\right|. \end{aligned} \tag{18}$$

If $i + j < n$, then

$$\begin{aligned} x &= c_1 p, \quad c_1 = 1, 2, \ldots, p^{n-i} - p, \\ y &= c_2 p, \quad c_2 = 1, 2, \ldots, p^2. \end{aligned} \tag{19}$$

Hence, there are $p^{n-i} - p = p(p^{n-w-1} - 1)$ choices for x and p^2 choices for y. Hence, $Y_2 = p^2(p^{n-w} - p) = p^2 m_2$; else, $i + j = n$. Then

$$x = c_1 p, \quad c_1 = 1, 2, \ldots, p^{n-i} - 1, \qquad y = c_2 p, \quad c_2 = 1, 2, \ldots, p^2. \tag{20}$$

There are $p^{n-i} - 1$ choices for x and p^2 choices for y, implying that $Y_2 = p^2(p^{n-i} - 1) = p^2(p^{n-w} - 1) = p^2 m_2$.

(3) Similarly, if $w \geq 3$, then

$$Y_w = \left|\left\{g \in G \mid g^{p^w} = e\right\}\right| = \left|\left\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \left(a^{p^w}\right)^x \left(b^{p^w}\right)^y = e;\ 1 \leq x \leq p^i,\ 1 \leq y \leq p^j\right\}\right|. \tag{21}$$

Then

$$x = c_1 p, \quad c_1 = 1, 2, \ldots, p^{n-1}, \qquad y = c_2 p, \quad c_2 = 1, 2, \ldots, p - 1. \tag{22}$$

Hence, $Y_w = p^{n-1}(p-1) = p^2 p^{n-3}(p-1) = p^2(p^{n-3}(p-1)) = p^2 m_w$.

(4) If $w > 3$, for all $k = 0, 1, \ldots, w-3$, then the number of choices for x reduces in a ratio of p^k for each k. Thus $m_{w-k} = (p^{n-3}/p^k)(p-1) = m_w/p^k$.

(5) When $|G| = p^n$, then

$$\begin{aligned} p^n &= 1 + \sum_{i=1}^{w} Y_i = 1 + Y_1 + \sum_{i=2}^{w} Y_i \\ &= 1 + p^2 m_1 - 1 + \sum_{i=2}^{w} p^2 m_i = p^2 m_1 + p^2 \sum_{i=2}^{w} m_i \\ &= p^2 \left(m_1 + \sum_{i=2}^{w} m_i\right) = p^2 \left(\sum_{i=1}^{w} m_i\right). \end{aligned} \tag{23}$$

Hence, $\sum_{i=1}^{w} m_i = p^{n-2}$. □

Corollary 8. *Let $G \in \mathfrak{G}$ be the group generated by a and b, with $|G| = p^n$, $n \geq 3$. Then G is not a POS-group.*

Proof. It is enough to show that there exists $[p^r, Y_r] \in \mathfrak{O}_G$ such that $Y_r \nmid p^n = |G|$. For $[p, Y_1] \in \mathfrak{O}_G$, where

$$Y_1 = \begin{cases} p^2 - 1, & i + j = n \\ p^3 - 1, & i + j < n, \end{cases} \tag{24}$$

suppose, on the contrary, that $Y_1 \mid p^n$. Then there exists $x \in \mathbb{N}$ with $x \leq n$ and $Y_1 = p^x$. Therefore

$$p^x = p^2 - 1 \quad \text{if } i + j = n \text{ or} \qquad p^x = p^3 - 1 \quad \text{if } i + j < n. \tag{25}$$

Then

$$p^2 - p^x = 1 \quad \text{if } i + j = n \text{ or} \qquad p^3 - p^x = 1 \quad \text{if } i + j < n \tag{26}$$

so that $x < 2$. If $x = 0$, then $p^2 = p^3 = 1$, which implies that $p = 1$, a contradiction. If $x = 1$, then $p^2 - p = 1$ and $p^3 - p = 1$ have no solution for p as an integer which gives a contradiction as well. It follows that there is no integer $x \leq n$ such that $Y_1 = p^x$. Thus $Y_1 \nmid p^n$, which means that G is not a POS-group. □

4. GAP

This study includes GAP's algorithms. Algorithm 1 (see the appendix) is derived from Theorem 7 and is used to find the order classes of all 2-generator p-groups of nilpotency class 2 (as a list), by determining the values of p and n. Algorithm 2 (see the appendix) is being built using the ordinary GAP formulas and commands (already installed with GAP's packages) to give the same results as Algorithm 1.

Example 9. When both Algorithms 1 and 2 are used to find the order classes for all 2-generator p-groups of class 2, where $p = 43$ and $n = 3$, the results obtained are as follows:

Algorithm 1

```
G=(C43 x C43)  :  C43
|G| =79507 p=43 n=3 N.class 2
no of gen.=2 o(a)=43 o(b)=43 w=1
WW=[ [ 1,1 ], [ 43, 79506 ] ]

G=C1849  :  C43
|G| =79507 p=43 n=3 N.class 2
 no of gen.=2 o(a)=1849 o(b)=43 w=2
WW=[ [ 1, 1 ], [ 43, 1848 ],
  [ 1849, 77658 ] ]

time:14180
```

Algorithm 2

```
G=(C43 x C43)  :  C43
|G| =79507 p=43 n=3 N.class 2
  no of gen.=2 o(a)=43 o(b)=43 w=1
  YY=[ [ 1, 1 ], [ 43, 79506] ]

G=C1849  :  C43
|G| =79507 p=43 n=3 N.class 2
 no of gen.=2 o(a)=1849 o(b)=43 w=2
YY=[ [ 1, 1 ], [ 43, 1848 ],
  [ 1849, 77658 ] ]

time:38064
```

```
p:=○;;n:=○;;order:=p^n;; # Input the values of p and n, where the order of G is p^n
G:=AllSmallGroups(Size,order);;
D:=NumberSmallGroups(order);;
for k in [1..D] do;
    f:=G[k];;m:=Size(MinimalGeneratingSet(f));;WW:=[];;
      if NilpotencyClassOfGroup(f)=2 and m=2 then;
        Add(WW,[1,1]);
        Print(k,") G=",StructureDescription(f), " |G|=",Size(f)," p=",p,"n=",n," N.class
",NilpotencyClassOfGroup(f));
          gg:=MinimalGeneratingSet(f);;
          e:=Identity(f);;
          a:=gg[1];;b:=gg[2];;
          i:=Log(Order(a),p);;j:=Log(Order(b),p);;w:=Maximum(i,j);;
            if i+j=n then;
              m1:=1; m2:=p^n/p^w-1;Add(WW,[p,m1*p^2-1]);
              if w>=2 then;
                Add(WW,[p^2,m2*p^2]);
              fi;
            else;
              m1:=p;m2:=p^(n-w)-p;Add(WW,[p,m1*p^2-1]);
              if w>=2 then;
                Add(WW,[p^2,m2*p^2]);
              fi;
            fi;
            Yw:=p^(n-1)*(p-1);;
            for l in [0..(w-3)] do;
              Add(WW,[p^(w-l),Yw/(p^l)]);
            od;
Print("\n no. of gen.=",m,", o(a)=",p^i,", o(b)=",p^j,", w=",w,"\n OC=",WW,"\n\n");
fi;od;time;
```

Algorithm 1: Theorem 7 in GAP's algorithm.

```
p:=○;;n:=○;;order:=p^n;; # Input the values of p and n, where the order of G is p^n
G:=AllSmallGroups(Size,order);;
D:=NumberSmallGroups(order);;
for k in [1..D] do;
    f:=G[k];;x:=Elements(f);;YY:=Collected(List(x,i->[Order(i)]));;
    m:=Size(MinimalGeneratingSet(f));;
      if NilpotencyClassOfGroup(f)=2 and m=2 then;
        Print(k,") |G|=",Size(f)," p=",p," n=",n," N.class ",NilpotencyClassOfGroup(f),
          gg:=MinimalGeneratingSet(f);;
          e:=Identity(f);;
          a:=gg[1];;b:=gg[2];;c:=Comm(a,b);;
        i:=Log(Order(a),p);;j:=Log(Order(b),p);;w:=Maximum(i,j);;
 Print("\n no. of gen.=",m,", o(a)=",Order(a),", o(b)=",Order(b),",w=",w,"\n
OC=",YY,"\n\n");
fi;od;time;
```

Algorithm 2: The ordinary GAP algorithm.

Similarly, for $p = 47$ and $n = 4$

Algorithm 1

```
G=(C2209 x C47) : C47
|G| =4879681 p=47 n=4 N.class 2
 no of gen.=2 o(a)=2209 o(b)=47 w=2
 WW=[ [ 1, 1 ], [ 47, 103822 ],
[ 2209, 4775858 ] ]

G=C2209 : C2209
|G| =4879681 p=47 n=4 N.class 2
 no of gen.=2 o(a)=103823 o(b)=47 w=3
 WW=[ [ 1, 1 ], [ 47, 2208 ],
[ 2209, 101614 ], [ 103823, 4775858 ] ]

G=C103823 : C47
|G| =4879681 p=47 n=4 N.class 2
 no of gen.=2 o(a)=103823 o(b)=47 w=3
 WW=[ [ 1, 1 ], [ 47, 2208 ],
[ 2209, 101614 ], [ 103823, 4775858 ] ]

time:100355
```

Algorithm 2

```
exceeded the permitted memory
```

The time required for Algorithm 2 to find the order classes of 2-generator p-groups of class 2, when $p = 43$ and $n = 3$, is 38064 milliseconds while Algorithm 1 needs 14180 milliseconds to find the same results. Next, for $p = 47$ and $n = 4$, Algorithm 2 could not complete the process, for the group size (4879681) exceeded the permitted memory size. Conversely, Algorithm 1 takes 100355 milliseconds. Distinctly, Algorithm 1 is much better than the ordinary GAP algorithm and it can be used instead.

5. Conclusion

In this paper, the classification of 2-generator p-groups of nilpotency class 2 has been used to determine the order classes of this type of groups. This work contains an appreciable number of imperative results. We have used these results to create a GAP algorithm (Algorithm 1) to find the order classes of 2-generator p-groups of nilpotency class 2, p odd prime. When Algorithm 1 is compared to Algorithm 2, which has been used for the same purpose, we have found that Algorithm 1 does not use all of the group elements and only depends on two elements (generators) to classify the order class of this group, while Algorithm 2 uses all of the group elements to give the same results. Therefore, it works very slow and interrupts large size groups, on the contrary to Algorithm 1.

Appendix

See Algorithms 1 and 2.

Competing Interests

The authors declare that they have no competing interests.

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