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*Research Article*

# On Graceful Spider Graphs with at Most Four Legs of Lengths Greater than One

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A graceful labeling of a tree $T$ with $n$ edges is a bijection $f : V(T) \rightarrow \{0, 1, 2, \ldots, n\}$ such that $\{|f(u) - f(v)| : uv \in E(T)\}$ equal to $\{1, 2, \ldots, n\}$. A spider graph is a tree with at most one vertex of degree greater than 2. We show that all spider graphs with at most four legs of lengths greater than one admit graceful labeling.

## 1. Introduction

Labeled graphs form useful models for a wide range of disciplines and applications such as in coding theory, X-ray crystallography, radar, astronomy, circuit design, and communication network addressing [1]. A systematic presentation of diverse applications of graph labeling is presented in [2].

A graceful labeling $f$ of a tree $T$ is a bijective function from the set of vertices $V(T)$ of $T$ to the set $\{0, 1, 2, \ldots, |E(T)|\}$ such that when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A tree which admits graceful labeling is called a graceful tree. In 1964, Ringel and Rosa (see [3, 4]) gave the famous and unsolved "graceful tree conjecture" which stated that all trees are graceful.

A spider graph is a tree with at most one vertex of degree greater than 2. Gallian [1] has noted that the special case of this conjecture regarding spider graphs is still open and that very few classes of spider graphs are known to be graceful. Huang et al. [5] proved that all spider graphs with three or four legs are graceful. Bahls et al. [6] also proved that every spider graph in which the lengths of any two of its legs differ by at most one is graceful. Jampachon et al. [7] have also proven that $S_n(k, l, m)$ is graceful, if $n$ is large enough, where $S_n(k, l, m)$ is defined in Section 3.

In this paper, we prove that all spider graphs with at most four legs of lengths greater than one are graceful.

## 2. Preliminaries

Let $T$ be a tree with $n$ edges. A *graceful labeling* of $T$ is a bijection $f : V(T) \rightarrow \{0, 1, 2, \ldots, n\}$ such that when each edge $xy$ is assigned the label $|f(u) - f(v)|$, the edge label set is equal to $\{1, 2, \ldots, n\}$. A tree which admits graceful labeling is called a *graceful tree*.

To prove our results, we need some terminology and existence results which are described below. In [8], Hrnčiar and Haviar proved Lemma 1 and in [9] Jampachon and Poomsa-Ard proved Lemmas 2 and 3.

**Lemma 1.** *Let $T$ be a tree with $n$ edges and a graceful labeling $f$. Then, the function $f^* : V(T) \rightarrow \{0, 1, 2, \ldots, n\}$ defined by $f^*(v) = n - f(v)$ is also a graceful labeling of $T$.*

**Lemma 2.** *Let $P_{2n}$ be a path graph with $V(P_{2n}) = \{v_1, v_2, v_3, \ldots, v_{2n}\}$ and let $M = \{a + 1, a + 2, \ldots, a + n, m + a + 1, m + a + 2, \ldots, m + a + n\}$, $m \geq n$ and $a \geq 0$. Then, there is a bijective labeling $f' : V(P_{2n}) \rightarrow M$ such that $f'(v_1) = i$ or $f'(v_{2n}) = i$, where $i \in M$ and the edge label set is $\{m - n + 1, m - n + 2, \ldots, m + n - 1\}$.*

**Lemma 3.** *Let $P_n$, $n \geq 4$, be a path graph with $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $M = \{m, m+1, m+2, \ldots, m+n-1\}$. Then, there is a bijective labeling $f' : V(P_n) \rightarrow M$ such that $f'(v_1) = i$ or $f'(v_n) = i$, where $i \in M$ and the edge label set is $\{1, 2, 3, \ldots, n - 1\}$.*

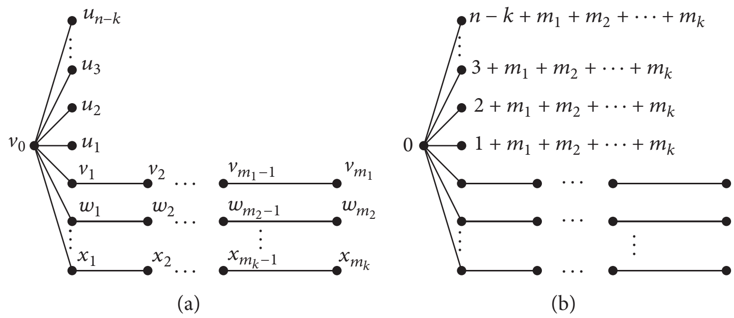


Figure 1

Let $T$ be a tree and let $u$ be a leaf of $T$. Let $T(u, u_1, u_2, \ldots, u_n)$ be the tree obtained from $T$ by adding the vertices $u_1, u_2, \ldots, u_n$ and the edges $uu_1, u_1u_2, \ldots, u_{n-1}u_n$. In [10], Sangsura and Poomsa-Ard have proved Lemma 4.

**Lemma 4.** *If a tree $T$ has a graceful labeling $f$ such that $f(u) = 0$, where $u$ is a leaf of $T$, then $T(u, u_1, u_2, \ldots, u_n)$ is graceful for each $n \geq 1$.*

## 3. Main Results

A *spider graph* or *spider* is a tree with at most one vertex of degree greater than 2 and this vertex is called the *branch vertex* and is denoted by $v_0$. A *leg* of a spider graph is a path from the branch vertex to a leaf of the tree. Let $S_n(m_1, m_2, \ldots, m_k)$, $n \geq k$, denote a spider of $n$ legs such that its legs has length one except for $k$ legs of lengths $m_1, m_2, \ldots, m_k$, where $m_i \geq 2$ for all $i = 1, 2, \ldots, k$.

**Lemma 5.** *If $S_k(m_1, m_2, \ldots, m_k)$ has a graceful labeling $f$ such that $f(v_0) = 0$, then there is a graceful labeling $f'$ of $S_n(m_1, m_2, \ldots, m_k)$ such that $f'(v_0) = 0$.*

*Proof.* Let $T = S_n(m_1, m_2, \ldots, m_k)$ be a graph such as shown in Figure 1(a). Since $f$ is a graceful labeling of a subgraph $S_k(m_1, m_2, \ldots, m_k)$ of $T$, extending $f$ to $T$ such that $f(u_i) = i + m_1 + m_2 + \cdots + m_k$, $i = 1, 2, \ldots, n-k$, then we get that $f$ is a graceful labeling of $T$ as shown in Figure 1(b). □

**Theorem 6.** *The graph $S_n(m)$ is graceful.*

*Proof.* We note that $S_1(m)$ is a path of length $m$; for example, $S_1(m) = v_0, v_1, \ldots, v_m$. Let $f$ be a labeling of $S_1(m)$ by labels alternating between the lowest and highest unused numbers in the set $\{0, 1, 2, \ldots, m\}$. We have that $f$ is a graceful labeling of $S_1(m)$ such that $f(v_0) = 0$ and by Lemma 5 we get that $S_n(m)$ is graceful. □

**Lemma 7.** *If $S_k(m_1, m_2, \ldots, m_k)$ has a graceful labeling $f$ such that $f(v_0) = 0$, then $S_n(l, m_1, m_2, \ldots, m_k)$ is also graceful.*

*Proof.* By assumption and Lemma 5, there exists a graceful labeling $f'$ of $S_n(m_1, \ldots, m_k)$ such that $f'(v_0) = 0$. From the proof of Lemma 5, we see that $f'(u_{n-k})$ is the maximum number; that is, $f'(u_{n-k}) = (n-k) + m_1 + m_2 + \cdots + m_k$. Let $f^*(v) = (n-k) + m_1 + m_2 + \cdots + m_k - f'(v)$ for all vertices of $S_n(m_1, \ldots, m_k)$. By Lemma 1, we get that the function $f^*$ is a graceful labeling of $S_n(m_1, \ldots, m_k)$ such that $f^*(u_{n-k}) = 0$. Then, $S_n(l, m_1, m_2, \ldots, m_k)$ is graceful by Lemma 4. □

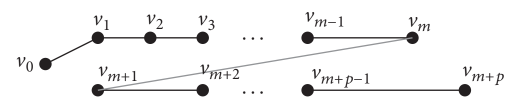


Figure 2

**Theorem 8.** *The graph $S_n(l, m)$ is graceful.*

*Proof.* From the proof of Theorem 6, we see that the labeling $f$ of $S_1(m)$ is a graceful labeling such that $f(v_0) = 0$. By Lemma 7, we therefore get that $S_n(l, m)$ is also graceful. □

**Theorem 9.** *The graph $S_n(l, m, p)$ is graceful.*

*Proof.* Without loss of generality, let $m \leq p$ and let $P$ be the path as shown in Figure 2.

Let $f$ be a labeling of $P$ by labels alternately between the lowest and highest unused numbers in the set $\{0, 1, 2, \ldots, m+p\}$. We have that $f$ is a graceful labeling of $P$ such that $f(v_0) = 0$. Note that $S_2(m, p) \neq P$ and $|f(v_m) - f(v_{m+1})| = p$. Since $f(v_{2m}) = p$ if $m = p$ and $f(v_{2m+1}) = p$ if $m < p$, then by Lemma 3 we can find a graceful labeling $f'$ of $S_2(m, p)$ such that $f'(v_0) = 0$. Hence, $S_n(l, m, p)$ is graceful by Lemma 7. □

Theorem 9 is not a new result; it follows from Jampachon and Poomsa-Ard [9], but our proof here is shorter.

Next, consider the path $P$ obtained from $S_3(m, p, q)$ by deleting the edges $v_0v_{m+1}$ and $v_0v_{m+p+1}$ and adding the edges $v_mv_{m+1}$ and $v_{m+p}v_{m+p+1}$ as shown in Figure 3.

It can be seen that $|V(P)| = m + p + q + 1$ and $|E(P)| = m + p + q$. If we now introduce a special labeling $f$ of $P$ which can be used to generate a graceful labeling, it follows that

> $f(v_i) = i/2$ if $i$ is even, and $f(v_i) = m + p + q - (i-1)/2$ if $i$ is odd.

It can be seen that with the labeling above the labels alternate between the lowest and highest unused numbers in the set $\{0, 1, 2, \ldots, m+p+q\}$. Moreover, we see that $P$ admits graceful

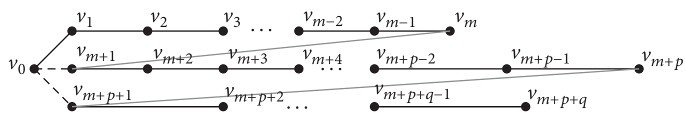


Figure 3

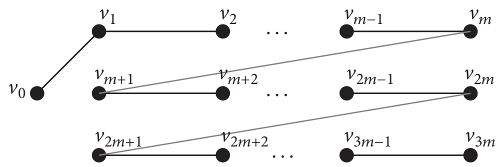


Figure 4

labeling and that the label of the branch vertex is 0; that is, $f(v_0) = 0$. For convenience, we call the labeling above Type$^*$.

*Remark 10.* Let $P$ be the path obtained from $T = S_3(m, p, q)$ as shown in Figure 3 and let $f$ be the labeling of $P$ of Type$^*$. Note that $S_3(m, p, q) \neq P$, $|f(v_m) - f(v_{m+1})| = f(v_{2m+1}) = p + q$, and $|f(v_{m+p}) - f(v_{m+p+1})| = q$. To find a graceful labeling $f'$ of $T$, we change the values of $f$ at the vertices $v_{m+1}$ and $v_{m+p+1}$ of $P$ such that $f'(v_{m+1}) = p + q$ and $f'(v_{m+p+1}) = q$.

Next we want to show that $S_n(l, m, p, q)$ are graceful and to prove this result we need the following lemmas.

**Lemma 11.** *If $m = p = q$, then there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$.*

*Proof.* Let $P$ be the path as shown in Figure 4 and let $f$ be the labeling of $P$ of Type$^*$. We see that $|f(v_m) - f(v_{m+1})| = f(v_{2m+1}) = p + q = 2m$ and $|f(v_{2m}) - f(v_{2m+1})| = f(v_{2m}) = q = m$, where $v_{2m}$ is the leaf of the path $v_{m+1}, \ldots, v_{2m}$. If we change the values of $f$ at $v_{m+1}, \ldots, v_{2m}$ by reversing their labels, then we get a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$. □

**Lemma 12.** *If $m = p < q$ and $q = 2m + 1$, then there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$.*

*Proof.* Let $P$ be the path as shown in Figure 5 and let $f$ be the labeling of $P$ of Type$^*$. We see that $|f(v_m) - f(v_{m+1})| = f(v_{2m+1}) = p + q$ and $|f(v_{2m}) - f(v_{2m+1})| = f(v_{4m+1}) = q$.

To construct a graceful labeling $f'$ of $T = S_3(m, p, q)$, we consider two cases.

*Case 1 (m is odd).* First, we switch the values of $f$ at $v_i$ and $v_{i+1}$, $i = m + 1, m + 3, m + 5, \ldots, 4m$, such that the new value of $f$ at $v_{2m}$ is $p + q$ and at $v_{4m}$ is $q$. We then change the new values of $f$ at $v_{m+1}, v_{m+2}, \ldots, v_{2m}$ by reversing their labels; then, by Lemma 3, we can find a graceful labeling $f'$ of $T$ such that $f'(v_0) = 0$.

*Case 2 (m is even).* Define $f'$ as follows: for $0 \leq i \leq m$, $f'(v_i) = f(v_i)$ such that

for $m + 1 \leq i \leq 3m/2$,

$$f'(v_i) = (m + i)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 7m/2 - (i - 3)/2 \text{ if } i \text{ is odd,}$$

for $3m/2 + 1 \leq i \leq 2m$,

$$f'(v_i) = i/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 4m - (i - 3)/2 \text{ if } i \text{ is odd,}$$

for $2m + 1 \leq i \leq 5m/2 + 1$,

$$f'(v_i) = 5m/2 - (i - 2)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 3m/2 + (i + 1)/2 \text{ if } i \text{ is odd,}$$

for $5m/2 + 2 \leq i \leq 6m/2 + 1$,

$$f'(v_i) = (m + i)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 7m/2 - (i - 3)/2 \text{ if } i \text{ is odd}$$

for $6m/2 + 2 \leq i \leq 7m/2 + 1$,

$$f'(v_i) = i/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 4m - (i - 3)/2 \text{ if } i \text{ is odd,}$$

for $7m/2 + 2 \leq i \leq 4m + 1$,

$$f'(v_i) = 5m/2 - (i - 2)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 3m/2 + (i + 1)/2 \text{ if } i \text{ is odd.}$$

In accordance with the above labeling pattern, we get a graceful labeling $f'$ of $T$ such that $f'(v_0) = 0$. □

**Lemma 13.** *If $m < p = q$, then there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$.*

*Proof.* Let $P$ be the path as shown in Figure 6 and let $f$ be the labeling of $P$ of Type$^*$. We see that $|f(v_m) - f(v_{m+1})| = f(v_{2m+1}) = p + q = 2p$ and $|f(v_{m+p}) - f(v_{m+p+1})| = f(v_{2p}) = q = p$.

If $p$ is even, then by Lemmas 2 and 3, we can find a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$. Now suppose that $p$ is odd. To construct a graceful labeling $f'$ of $T = S_3(m, p, q)$, we must consider two cases.

*Case 1 (m is even)*

*Case 1.1 ($p < 2m + 1$).* We see that $|f(v_{2m}) - f(v_{2m+1})| = |f(v_{m+p}) - f(v_{3m-p+1})|$ and $v_{3m-p+1}$ lies in the path $v_{m+1}, \ldots, v_{2m}$. Change the values of $f$ at $v_{m+1}, v_{m+2}, \ldots, v_{m+p}$ by reversing their labels; then, the new value of $f$ at $v_p$ is $2p$. Next, change the new values of $f$ at $v_{m+1}, v_{m+2}, \ldots, v_p$ by reversing their labels. Since the number of vertices of the path $v_{p+1}, v_{p+2}, \ldots, v_{p+m}$ is even, by Lemmas 2 and 3, we can find a graceful labeling $f'$ of $T$ such that $f'(v_0) = 0$.

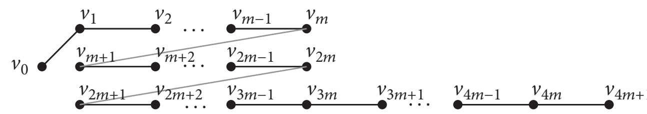


FIGURE 5

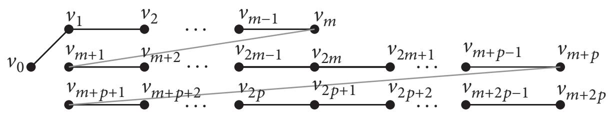


FIGURE 6

*Case 1.2 ($p = 2m + 1$).* Define $f'$ as follows: For $0 \le i \le m$, $f'(v_i) = f(v_i)$ and

for $m + 1 \le i \le 2m + 1$,

$$f'(v_i) = 3m/2 - (i-2)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 7m/2 + (i+3)/2 \text{ if } i \text{ is odd},$$

for $2m + 2 \le i \le 2m + p + 1$,

$$f'(v_i) = (m+i)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 9m/2 - (i-5)/2 \text{ if } i \text{ is odd},$$

for $2m + p + 2 \le i \le m + 2p$,

$$f'(v_i) = 7m/2 - (i-4)/2 \text{ if } i \text{ is even}$$
$$f'(v_i) = 3m/2 + (i+1)/2 \text{ if } i \text{ is odd}.$$

In accordance with the above labeling pattern, we get a graceful labeling $f'$ of $T$ such that $f'(v_0) = 0$.

*Case 1.3 ($p > 2m + 1$).* We see that $|f(v_{2m+1}) - f(v_{2m+2})| = |f(v_{m+1}) - f(v_{3m+2})|$ and $v_{3m+2}$ lies in the path $v_{2m+2}, \ldots, v_{m+p}$. Change the values of $f$ at $v_{m+1}, v_{m+2}, \ldots, v_{2m+1}$ by reversing their labels. Since the number of vertices of the path $v_{2m+2}, v_{2m+3}, \ldots, v_{m+p}$ is even, by Lemmas 2 and 3, we can find a graceful labeling $f'$ of $T$ such that $f'(v_0) = 0$.

*Case 2 (m is odd).* Switch the values of $f$ at $v_i$ and $v_{i+1}$, $i = m+1, m+3, m+5, \ldots, m+2p-1$; after that, follow a similar procedure as for Case 1. □

**Lemma 14.** *If $m < p < q$ and $p \ne 2m + 1$, then there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$.*

*Proof.* Let $P$ be the path as shown in Figure 3 and let $f$ be the labeling of $P$ of Type*. We see that $f(v_{2m+1}) = p + q$ and, since $p < q$, there must be a vertex $v_j$ laying in the path $P_q = v_{m+p+1}, \ldots, v_{m+p+q}$ such that $f(v_j) = q$. If $p$ is even, then by Lemmas 2 and 3, we can find a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$. Now suppose that $p$ is odd.

*Case 1 (m is even)*

*Case 1.1 ($p < 2m+1$).* Do similarly as in Case 1.1 of Lemma 13.

*Case 1.2 ($p > 2m + 1$).* Do similarly as in Case 1.3 of Lemma 13.

*Case 2 (m is odd).* Switch the values of $f$ at $v_i$ and $v_{i+1}$, $i = m+1, m+3, m+5, \ldots, m+2p-1$, after which do similarly as in Case 1. □

**Theorem 15.** *Let $l$, $m$, $p$, and $q$ be integers greater than one. Then, there are three of them; say $x$, $y$, and $z$, for which the spider graph $S_3(x, y, z)$ has a graceful labeling $f$ such that $f(v_0) = 0$.*

*Proof.* Without loss of generality, let $m \le p \le q$, let $P$ be the path as shown in Figure 5, and let $f$ be the labeling of $P$ of Type*.

*Case 1 ($m = p = q$).* By Lemma 11 there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$.

*Case 2 ($m = p < q$).* If $q = 2m + 1$, then by Lemma 12 there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$. Now suppose that $q \ne 2m + 1$.

If $l < m$, then by Lemma 13 there is a graceful labeling $f'$ of $S_3(l, m, p)$ such that $f'(v_0) = 0$.

If $l = m$, then by Lemma 11 there is a graceful labeling $f'$ of $S_3(l, m, p)$ such that $f'(v_0) = 0$.

If $m < l < q$ and $l = 2m + 1$, then by Lemma 12 there is a graceful labeling $f'$ of $S_3(m, p, l)$ such that $f'(v_0) = 0$.

If $m < l < q$ and $l \ne 2m + 1$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(m, l, q)$ such that $f'(v_0) = 0$.

If $l = q$, then by Lemma 13 there is a graceful labeling $f'$ of $S_3(m, l, q)$ such that $f'(v_0) = 0$.

If $l > q$ and $l = 2m + 1$, then by Lemma 12 there is a graceful labeling $f'$ of $S_3(m, p, l)$ such that $f'(v_0) = 0$.

If $l > q$ and $l \ne 2m + 1$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(m, q, l)$ such that $f'(v_0) = 0$.

*Case 3 ($m < p = q$).* By Lemma 13 there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$.

*Case 4 ($m < p < q$).* If $p \ne 2m + 1$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(m, p, q)$ such that $f'(v_0) = 0$. Now suppose that $p = 2m + 1$.

If $l < m$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(l, p, q)$ such that $f'(v_0) = 0$.

If $l = m$, then by Lemma 12 there is a graceful labeling $f'$ of $S_3(l, m, p)$ such that $f'(v_0) = 0$.

If $m < l < p$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(l, p, q)$ such that $f'(v_0) = 0$.

If $l = p$, then by Lemma 13 there is a graceful labeling $f'$ of $S_3(m, l, p)$ such that $f'(v_0) = 0$.

If $p < l < q$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(m, l, q)$ such that $f'(v_0) = 0$.

If $l = q$, then by Lemma 13 there is a graceful labeling $f'$ of $S_3(m, l, q)$ such that $f'(v_0) = 0$.

If $l > q$, then by Lemma 14 there is a graceful labeling $f'$ of $S_3(m, q, l)$ such that $f'(v_0) = 0$. □

**Theorem 16.** *The graph $S_n(l, m, p, q)$ is graceful.*

*Proof.* By Theorem 15, there is a graceful labeling $f$ of a spider graph $S_3(m, p, q)$ with three legs, such that $f(v_0) = 0$. By Lemma 7, we therefore get that $S_n(l, m, p, q)$ is graceful. □

## 4. Conclusion and Remarks

The main tools required to construct a graceful labeling of $S_n(l, m, p, q)$ are a graceful labeling $f$ with $f(v_0) = 0$, where $v_0$ is the branch vertex, and the results of Lemmas 2 and 3.

## Competing Interests

The authors declare that they have no competing interests.

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