Research Article

Bounding Regions to Plane Steepest Descent Curves of Quasiconvex Families

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Two-dimensional steepest descent curves (SDC) for a quasiconvex family are considered; the problem of their extensions (with constraints) outside of a convex body K is studied. It is shown that possible extensions are constrained to lie inside of suitable bounding regions depending on K. These regions are bounded by arcs of involutes of ∂K and satisfy many inclusions properties. The involutes of the boundary of an arbitrary plane convex body are defined and written by their support function. Extensions SDC of minimal length are constructed. Self-contracting sets (with opposite orientation) are considered: necessary and/or sufficient conditions for them to be subsets of SDC are proved.

1. Introduction

Let u be a smooth function defined in a convex body $\Omega \subset \mathbb{R}^n$. Let $Du(x) \neq 0$ in $\{x \in \Omega : u(x) > \min u\}$. A classical steepest descent curve of u is a rectifiable curve $s \to x(s)$ solution to

$$\frac{dx}{ds} = -\frac{Du}{|Du|}(x(s)). \tag{1}$$

Classical steepest descent curves are the integral curves of a unit field normal to the sublevel sets of the given smooth function u. We are interested in "generalized" steepest descent curves that are integral curves to a unit field normal to a nested family of convex sets $\{\Omega_t\}$ (see Definition 5); $\{\Omega_t\}$ will be called a quasiconvex family as in [1]. Sharp bounds about the length of the steepest descent curves for a quasi convex family have been proved in [2–4]. The geometry of these curves, equivalent definitions, related questions and generalizations have been studied in [5–9].

In the present work generalized *steepest descent curves for a quasiconvex family* (SDC for short) are defined as bounded oriented rectifiable curves $\gamma \subset \mathbb{R}^n$, with a locally Lipschitz continuous parameterization $T \ni t \to x(t)$, with ascent parameter, satisfying

$$\langle \dot{x}(t), x(\tau) - x(t) \rangle \leq 0, \quad \text{a.e. } t \in T, \ \forall \tau \leq t; \tag{2}$$

$\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. Let ordering $\preceq$ be chosen on γ, according to the orientation; let us denote

$$\gamma_x = \{y \in \gamma : y \preceq x\}. \tag{3}$$

In [8, Theorem 4.10], the SDC are characterized in an equivalent way as *self-distancing curves*, namely, oriented ($\preceq$) continuous curves with the property that the distance of x to an arbitrarily fixed previous point x_1 is not decreasing:

$$x_1 \preceq x_2 \preceq x_3 \implies$$
$$|x_2 - x_1| \leq |x_3 - x_1| \quad \forall x_1, x_2, x_3 \in \gamma. \tag{4}$$

Thus *steepest descent curves* are self-distancing curves and both denoted SDC. In [8] self-distancing curves are called self-expanding curves. With the opposite orientation these curves have been also introduced, studied, and called self-approaching curves (see [2]) or self-contracting curves (see [7]).

In our work we are interested in the behaviour and properties of a *plane* SDC γ beyond its final point x_0. One of the principal goals of the paper is to show that conditions (2) and (4) imply constraints for possible extensions of curve γ beyond x_0; these constraints are written as bounding regions for the possible extensions of γ_{x_0}.

An important property that will be used later is the property of distancing from a set A.

Definition 1. Given a set A, an absolutely continuous curve γ, $T \ni t \to x(t)$ has the distancing from A property if it satisfies

$$\langle \dot{x}(t), y - x(t) \rangle \leq 0, \quad \text{a.e. } t \in T, \ \forall y \in A. \tag{5}$$

Let us outline the content of our work. In Section 2 introductory definitions are given and covering maps for the boundary of a plane convex set, needed for later use, are introduced. In Section 3 the involutes of the boundary of a plane convex body are introduced and some of their properties are proved.

In Section 4 plane regions depending on the convex hull of γ_{x_0} have been defined; these regions fence in or fence out the possible extensions of γ_{x_0}. The boundary of these sets consists of arcs of involutes of convex bodies, constructed in Section 3. As an application, in Section 4.1 the following problem has been studied: given a convex set K, $x_0 \in \partial K$, $x_1 \notin K$, is it possible to construct SDC joining x_0 to x_1, satisfying the distancing from K property? Minimal properties of this construction have been introduced and studied. In Section 5 sets of points more general than SDC are studied. A set $\sigma \subset \mathbb{R}^2$ (not necessarily a curve) of ordered points satisfying (4) will be called *self-distancing set*; see also Definition 2; with the opposite order, σ was called in [6] self-contracting set and many properties of these sets, as only subsets of self-contracting curves, were there obtained. Another goal of the paper is the solution to the following question: given a self-distancing set $\sigma \subset \mathbb{R}^2$ does a steepest descent curve $\gamma \supset \sigma$ exist? In Section 5 examples, necessary and/or sufficient conditions are given when σ consists of a finite or countable number of points $x_i \in \mathbb{R}^2$ and/or steepest descent curves $\gamma^i \subset \mathbb{R}^2$.

In the present work the two-dimensional case is studied. Similar results for the n-dimensional case are an open problem stated at the end of the work.

2. Preliminaries and Definitions

Let

$$B(z, \rho) = \{x \in \mathbb{R}^n : |x - z| < \rho\},$$
$$S^{n-1} = \partial B(0, 1) \tag{6}$$
$$n \geq 2.$$

A nonempty, compact convex set K of $\mathbb{R}^n$ will be called a *convex body*. From now on, K will *always be a convex body not reduced to a point*. $\text{Int}(K)$ and ∂K denote the interior of K and the boundary of K, $|\partial K|$ denotes its length, $\text{cl}(K)$ is the closure of K, $\text{Aff}(K)$ will be the smallest affine space containing K,

and relint K and $\partial_{\text{rel}} K$ are the corresponding subsets in the topology of $\text{Aff}(K)$. For every set $S \subset \mathbb{R}^n$, $\text{co}(S)$ is the convex hull of S.

Let $q \in K$; the *normal cone* at q to K is the closed convex cone:

$$N_K(q) = \{x \in \mathbb{R}^n : \langle x, y - q \rangle \leq 0 \ \forall y \in K\}. \tag{7}$$

When $q \in \text{Int}(K)$, then $N_K(q)$ reduces to zero.

The *tangent cone* or support cone of K at a point $q \in \partial K$ is given by

$$T_K(q) = \text{cl} \left(\bigcup_{y \in K} \{s(y - q) : s \geq 0\} \right). \tag{8}$$

In two dimensions cones will be called sectors.

Let K be a convex body and let p be a point. A simple cap body K^p is

$$K^p = \bigcup_{0 \leq \lambda \leq 1} \{\lambda K + (1 - \lambda) p\} = \text{co}(K \cup \{p\}). \tag{9}$$

Cap bodies properties can be found in [10, 11].

2.1. Self-Distancing Sets and Steepest Descent Curves. Let us recall the following definitions.

Definition 2. Let us call *self-distancing set* a bounded subset σ of $\mathbb{R}^n$, linearly ordered (by $\preceq$), with the property

$$x_1 \preceq x_2 \preceq x_3 \implies$$
$$|x_2 - x_1| \leq |x_3 - x_1|, \quad x_1, x_2, x_3 \in \sigma. \tag{10}$$

The self-distancing sets have been introduced in [6] with the opposite order. If a self-distancing set σ is a closed connected set, not reduced to a point, then it can be proved that σ is the support of a steepest descent curve γ (see [8, Theorem 4.10, Theorem 4.8]) and it will also be called a *self-distancing curve γ*.

The short name SDC will be used both for self-distancing curves and for steepest descent curves in all the paper.

Definition 3. Let K be a convex body; $\gamma \subset \mathbb{R}^2 \setminus \text{relint } K$ will be called a *self-distancing curve from K* (denoted SDC_K) if

  (i) γ is a self-distancing curve,

 (ii) $\gamma \cap \partial_{\text{rel}} K \neq \emptyset$,

(iii) γ has the property

$$x \preceq x_1 \implies$$
$$|x - y| \leq |x_1 - y| \quad \forall y \in K, \ \forall x, x_1 \in \gamma. \tag{11}$$

When (ii) does not hold, that is $\gamma \cap \partial_{\text{rel}} K = \emptyset$, γ will be called a *deleted self-distancing curve from K*.

Remark 4. Let γ be SDC_K, since γ has an absolutely continuous parameterization ([8, Theorem 4.10, Theorem 4.8]), thus property (11) for γ is equivalent to (5).

If γ is SDC and $x \in \gamma$ then $\gamma \setminus \gamma_x$ is $\text{SDC}_{\text{co}(\gamma_x)}$. That is, the tangent vector $\dot{x}(t)$ to γ is in the normal cone at $x(t)$ to the related convex set $\Omega_t := \text{co}(\gamma_{x(t)})$. This condition is a generalization of the classicals steepest descent curves that are integral curves to a unit field normal to smooth quasiconvex families.

Nested families of convex sets have been introduced and studied by de Finetti [12] and Fenchel [1]. Let us recall some definitions.

Definition 5. Let T be a real interval. A *convex stratification* (see [12]) is a nonempty family $\mathfrak{K}$ of convex bodies $\Omega_t \subset \mathbb{R}^n$, $t \in T \subset \mathbb{R}$, linearly strictly ordered by inclusion ($\Omega_1 \subset \Omega_2$, $\Omega_1 \neq \Omega_2$), with a maximum set ($\max \mathfrak{K}$) and a minimum set ($\min \mathfrak{K}$).

Let $\mathfrak{K} = \{\Omega_t\}_{t \in T}$ be a convex stratification. If for every $s \in T \setminus \{\max T\}$ the property

$$\bigcap_{t > s} \Omega_t = \Omega_s \tag{12}$$

holds, then as in [1], $\mathfrak{K} = \{\Omega_t\}_{t \in T}$ will be called a *quasiconvex family.*

An important quasiconvex family associated with a continuous self-distancing curve from K, $\gamma: t \to x(t)$ is $\mathfrak{K} = \{\Omega_t\}_{t \in T}$, where

$$\Omega_t = \text{co}\left(\gamma_{x(t)} \cup K\right). \tag{13}$$

The couple $(\gamma, \mathfrak{K})$ is special case of *Expanding Couple*, a class introduced in [8].

Remark 6. If $\gamma \in \text{SDC}_K$, then for all $x \in \gamma$ the curve $(\gamma \setminus \gamma_x) \cup \{x\}$ is a self-distancing curve from the convex hull of the set $\gamma_x \cup K$.

This fact is a direct consequence of the following.

Proposition 7 (see [8, Lemma 4.9]). *Let* $p, q, y_i \in \mathbb{R}^n$, $i = 1, \ldots, s$. *If*

$$|p - y| \leq |q - y|, \quad \text{for } y = y_i, \ i = 1, \ldots, s \tag{14}$$

then the same holds for every $y \in \text{co}(\{y_i, \ i = 1, \ldots, s\})$.

The statement of the previous proposition holds if large inequalities are replaced by strict inequalities everywhere.

2.2. The Support Function of a Plane Convex Body. Let $K \subset \mathbb{R}^n$ be a convex body not reduced to a point.

For a convex body K, the *support function* is defined as

$$H_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n, \tag{15}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. For $n = 2$, $\vartheta \in \mathbb{R}$, let $\theta = (\cos \vartheta, \sin \vartheta) \in S^1$ and $h_K(\vartheta) := H_K(\theta)$; it will be denoted $h(\vartheta)$ if no ambiguity arises.

For every $\theta \in S^1$ there exists at least one point $x \in \partial K$ such that

$$\langle \theta, y - x \rangle \leq 0 \quad \forall y \in K; \tag{16}$$

this means that the line through x orthogonal to θ supports K. For every $x \in \partial K$ let $\widehat{N_x}$ be the set of $\theta \in S^1$ such that (16) holds. Let $F(\theta)$ be the set of all $x \in \partial K$ satisfying (16). If ∂K is strictly convex at the direction θ then $F(\theta)$ reduces to one point and it will be denoted by $x(\theta)$.

Definition 8. The set valued map $G : \partial K \to S^1, \partial K \ni x \to \widehat{N_x} \subset S^1$, is the generalized Gauss map; $x \in \partial K$ is a vertex on ∂K iff $\widehat{N_x}$ is a sector with interior points. The set valued map $F : S^1 \to \partial K, S^1 \ni \theta \to F(\theta) \subset \partial K$ is the reverse generalized Gauss map; $F(\theta)$ is a closed segment, possibly reduced to a single point, and it will be called 1-face when it has interior points.

Let P be the covering map

$$P : \mathbb{R} \longrightarrow S^1, \tag{17}$$

$$\mathbb{R} \ni \vartheta \longrightarrow \tag{18}$$

$$\theta = (\cos \vartheta, \sin \vartheta) \in S^1.$$

Let $L = |\partial K|$, and let $s \to x_l(s)$, $0 \leq s < L$ ($s \to x_r(s)$, $0 \leq s < L$) be the parametric representations of ∂K depending on the arc length counterclockwise (clockwise) with an initial point (not necessarily the same). Let us extend $x_l(\cdot)$ and $x_r(\cdot)$ by defining

$$x_l(s) := x_l(s - kL) \quad \text{if } kL \leq s < (k+1)L, \ (k \in \mathbb{Z}), \tag{19}$$

similarly for x_r.

Let us fix $x_0 \in \partial K$, $\theta_0 \in G(x_0)$, $\theta_0 = (\cos \vartheta_0, \sin \vartheta_0)$, and $\vartheta_0 \in \mathbb{R}$.

For later use, we need to have $x_0 = x_l(s_0) = x_r(s_0)$; this can be realized by choosing suitable initial points for the parameterizations x_l and x_r.

Then

$$x_l(s_0 + s) = x_r(s_0 + L - s), \quad \forall s \in \mathbb{R}. \tag{20}$$

The maps

$$x_l : \mathbb{R} \longrightarrow \partial K,$$
$$x_r : \mathbb{R} \longrightarrow \partial K \tag{21}$$

are covering maps.

The initial parameters will be

$$x_0 = x_l(s_0) = x_r(s_0) \in \partial K,$$

$$S^1 \ni \theta_0 \in F^{-1}(x_0), \tag{22}$$

$$\mathbb{R} \ni \vartheta_0 \in P^{-1}(\theta_0)$$

($F^{-1}(x_0), P^{-1}(\theta_0)$ are the back images of F, P, resp.). Let $k \in \mathbb{Z}$. Let us define, for $\vartheta_0 + 2k\pi < \vartheta < \vartheta_0 + 2(k+1)\pi$:

$$s_{l+}(\vartheta) := \sup\{s \in \mathbb{R} : kL < s \leq (k+1)L, \ x_l(s) \tag{23}$$

$$\in F(P(\vartheta))\};$$

if $\vartheta = \vartheta_0 + 2k\pi$

$$s_{l+}(\vartheta) := \sup\left\{s \in \mathbb{R} : kL \leq s < (k+1)L, \; x_l(s) \right.$$
$$\left. \in F(P(\vartheta))\right\}. \tag{24}$$

Similarly, let us define for $\vartheta_0 + 2(k-1)\pi < \vartheta < \vartheta_0 + 2k\pi$:

$$s_{r-}(\vartheta) := \inf\left\{s \in \mathbb{R} : kL \leq s < (k+1)L, \; x_r(s) \right.$$
$$\left. \in F(P(\vartheta))\right\}; \tag{25}$$

if $\vartheta = \vartheta_0 + 2k\pi$

$$s_{r-}(\vartheta) := \inf\left\{s \in \mathbb{R} : (k-1)L < s \leq kL, \; x_r(s) \right.$$
$$\left. \in F(P(\vartheta))\right\}. \tag{26}$$

The function s_{l+} is increasing in $\mathbb{R}$ and right continuous and with left limits (so called *cadlag* function). Similar properties hold for $-s_{r-}$. Let us recall that a cadlag increasing function $s(\vartheta)$, $\vartheta \in \mathbb{R}$, has a right continuous inverse defined as

$$\vartheta(s) = \inf\{\vartheta : s(\vartheta) > s\}. \tag{27}$$

Let $\vartheta_{l+}(\cdot)$ be the right continuous inverse of $s_{l+}(\cdot)$. Let $s \to \vartheta_{r-}(s)$ be the opposite of the right continuous inverse of $-s_{r-}(\cdot)$.

Let us introduce for simplicity

$$\mathbf{n}_\vartheta := (\cos\vartheta, \sin\vartheta),$$
$$\mathbf{t}_\vartheta := (-\sin\vartheta, \cos\vartheta). \tag{28}$$

Let $\vartheta \to h(\vartheta)$ be the support function of K.

It is well known ([13]) that if ∂K is C_+^2 (i.e., $\partial K \in C^2$, with positive curvature), then h is C^2 and the counterclockwise element arc ds of ∂K is given by

$$ds = \left(h + \ddot{h}\right) d\vartheta. \tag{29}$$

$h(\vartheta) + \ddot{h}(\vartheta)$ is the positive radius of curvature; moreover the reverse Gauss map $F : \theta \to x \in \partial K$ is a 1-1 map given by

$$x(\theta) := h(\vartheta)\mathbf{n}_\vartheta + \dot{h}(\vartheta)\mathbf{t}_\vartheta, \quad \vartheta \in P^{-1}(\theta). \tag{30}$$

The previous formula also holds for an arbitrary convex body, for every ϑ such that $F(\theta)$ is reduced to a point; see [10]. Let us recall that a real valued function $x \to f(x)$ is called semiconvex on $\mathbb{R}$ when there exists a positive constant C such that $f(x) + Cx^2$ is convex on $\mathbb{R}$. From (29) the function $\vartheta \to h(\vartheta) + (1/2)\vartheta^2 \max h$ is convex on $\mathbb{R}$; thus h is semiconvex. In the case that K is an arbitrary convex body, by approximation arguments with C_+^2 convex bodies (see [11]) it follows that the support function of K is also semiconvex. As consequence h is Lipschitz continuous, it has left (right) derivative $\dot{h}_-$ (resp., $\dot{h}_+$) at each point, which is left (right) continuous. Moreover at each point the right limit of $\dot{h}_-$ is $\dot{h}_+$ and the left limit of $\dot{h}_+$ is $\dot{h}_-$; see [14, pp. 228].

It is not difficult to show (from (30), with a right limit argument) that, for an arbitrary convex body, for $\vartheta \in \mathbb{R}$, the formula

$$x_l\left(s_{l+}(\vartheta)\right) = h(\vartheta)\mathbf{n}_\vartheta + \dot{h}_+(\vartheta)\mathbf{t}_\vartheta \tag{31}$$

holds. Similarly the formula

$$x_r\left(s_{r-}(\vartheta)\right) = h(\vartheta)\mathbf{n}_\vartheta + \dot{h}_-(\vartheta)\mathbf{t}_\vartheta \tag{32}$$

holds.

If ∂K is not strictly convex at the direction $\theta = (\cos\vartheta, \sin\vartheta)$ then h is not differentiable at ϑ and

$$\dot{h}_+(\vartheta) - \dot{h}_-(\vartheta) = \left|x_l\left(s_{l+}(\vartheta)\right) - x_r\left(s_{r-}(\vartheta)\right)\right| = |F(\theta)|. \tag{33}$$

If $x_1, x_2 \in \partial K$ let us define $\mathrm{arc}^+(x_1, x_2)$ as the set of points of ∂K between x_1 and x_2 according to the counterclockwise orientation of ∂K and $\mathrm{arc}^-(x_1, x_2)$ as the set of points between x_1 and x_2, according to the clockwise orientation; $|\mathrm{arc}^\pm(x_1, x_2)|$ denote their length.

Remark 9. It is well known that a sequence of convex body $K^{(n)}$ converges uniformly to K if and only if the corresponding sequence of support functions converges in the uniform norm; see [11, pp. 66]. Moreover as the two sequences of the end points of a closed counterclockwise oriented arc of $\partial K^{(n)}$ converge, then the sequence of the corresponding arcs converges to a connected arc of ∂K and the sequence of the corresponding lengths converges too.

Proposition 10. *Let K be a convex body and h its support function; then*

$$s_{l+}(\vartheta) - s_{l+}(\vartheta_0) = \int_{\vartheta_0}^\vartheta h(\tau)\,d\tau + \left(\dot{h}_+(\vartheta) - \dot{h}_+(\vartheta_0)\right), \tag{34}$$
$$\forall \vartheta \geq \vartheta_0;$$

$$s_{r-}(\vartheta_0) - s_{r-}(\vartheta) = \int_{\vartheta_0}^\vartheta h(\tau)\,d\tau + \left(\dot{h}_-(\vartheta) - \dot{h}_-(\vartheta_0)\right), \tag{35}$$
$$\forall \vartheta \leq \vartheta_0.$$

Proof. For every convex body K not reduced to a point the function $\vartheta \to s_{l+}(\vartheta)$ is defined everywhere and satisfies the weak form of (29); namely,

$$-\int_{\mathbb{R}} s_{l+}(\eta)\dot{\phi}(\eta)\,d\eta = \int_{\mathbb{R}} \left(\phi + \ddot{\phi}\right)(\eta)h(\eta)\,d\eta, \tag{36}$$
$$\forall \phi \in C_0^\infty(\mathbb{R}).$$

Using the fact that $\vartheta \to h(\vartheta)$ is Lipschitz continuous, integrating by parts (36), the formula

$$-\int_{\mathbb{R}} s_{l+}(\eta)\dot{\phi}(\eta)\,d\eta$$
$$= -\int_{\mathbb{R}} \dot{\phi}(\eta)\left(\int_0^\eta h(\tau)\,d\tau + \dot{h}(\eta)\right) d\eta, \tag{37}$$
$$\forall \phi \in C_0^\infty(\mathbb{R})$$

holds. Thus

$$s_{l+}(\eta) = c + \int_0^\eta h(\tau)\,d\tau + \dot{h}(\eta), \quad \text{a.e.} \qquad (38)$$

with c constant. Passing to the right limit, the equality

$$s_{l+}(\eta) = c + \int_0^\eta h(\tau)\,d\tau + \dot{h}_+(\eta), \quad \forall \eta \in \mathbb{R} \qquad (39)$$

holds. Formula (34) follows, by computing $s_{l+}(\vartheta) - s_{l+}(\vartheta_0)$, using the previous equality. Similarly (35) is proved. $\quad\square$

3. Involutes of a Closed Convex Curve

Definition 11. Let I be an interval. A plane curve $I \ni t \to x(t)$ is convex if at every point x it has right tangent vector $T^+(x)$ and $\arg T^+(x(t))$ is not decreasing function.

Let $s \to x(s)$ be the arc length parameterization of a smooth curve; the classical definition of involute starting at a point $x_0 = x(s_0)$ of the curve $x(\cdot)$ is

$$i(s) = x(s) - (s - s_0)x'(s) \quad s \geq s_0. \qquad (40)$$

Let us notice that s is the arc length of the curve, not of the involute; if $s_0 = 0$, then the starting point of the involute coincides with the starting point of the curve. It is easy to construct an involute of a convex polygonal line (even if classical definition (40) does not work) by using arcs of circle centered at its corner points; moreover the involute depends on the orientation of the curve.

In this section, involutes for the boundary of an arbitrary plane convex body K, not reduced to a point, will be defined. The assumption that K is an arbitrary convex body is needed to work with the involutes of the convex sets, not smooth, studied in Section 4.

Let $K \in C_+^2$; let x_0 be a fixed point of ∂K; $s \to x(s)$ can be the clockwise parameterization of ∂K or the counterclockwise parameterization. Since there exist two orientations, then two different involutes have to be considered. As noted previously one can assume that the parameterizations of ∂K have been chosen so that $x_0 = x_l(s_0) = x_r(s_0)$.

Definition 12. Let one denote by i_{l,x_0} the left involute of ∂K starting at x_0 corresponding to the counterclockwise parameterization of ∂K and by i_{r,x_0} the right involute corresponding to the clockwise parameterization. When one needs to emphasize the dependence on K of involutes, they will be written as i_{l,x_0}^K, i_{r,x_0}^K.

Remark 13. Let us notice that if ρ is a plane reflection with respect to a fixed axis then

$$i_{r,x_0}^K = \rho\left(i_{l,\rho(x_0)}^{\rho(K)}\right). \qquad (41)$$

This relation allows us to prove our results for the left involutes only and to state without proof the analogous results for the right involutes.

Theorem 14. *Let one fix the initial parameters x_0, s_0, θ_0, and ϑ_0. The left and the right involutes of a plane convex curve starting at $x_0 \in \partial K$, boundary of a C_+^2 plane convex body K with support function h, are parameterized by the value ϑ related to the outer normal $\mathbf{n}_\vartheta$ to K, as follows:*

$$i_{l,x_0}(\vartheta) = h(\vartheta)\,\mathbf{n}_\vartheta - \left(\int_{\vartheta_0}^\vartheta h(\tau)\,d\tau - \dot{h}(\vartheta_0)\right)\mathbf{t}_\vartheta, \qquad (42)$$

$$\text{for } \vartheta \geq \vartheta_0,$$

$$i_{r,x_0}(\vartheta) = h(\vartheta)\,\mathbf{n}_\vartheta - \left(\int_{\vartheta_0}^\vartheta h(\tau)\,d\tau - \dot{h}(\vartheta_0)\right)\mathbf{t}_\vartheta, \qquad (43)$$

$$\text{for } \vartheta \leq \vartheta_0.$$

Proof. In the present case there is a 1-1 mapping between ϑ and s; from (29), it follows that

$$s - s_0 = \int_{\vartheta_0}^\vartheta h(\tau)\,d\tau + \dot{h}(\vartheta) - \dot{h}(\vartheta_0); \qquad (44)$$

then, changing the variable s with ϑ in (40), with elementary computation, (42) is obtained (since $x'(s) = \mathbf{t}_\vartheta$ and (30) holds). Formula (43) follows from (32) and (35). $\quad\square$

For an arbitrary convex body K in place of (30), formulas (31) and (32) have to be used.

Definition 15. Let K be a plane convex body; let

$$x_0 = x(s_0) \in \partial K,$$
$$\vartheta_0^+ := \vartheta_{l+}(s_0), \qquad (45)$$
$$s_0^+ := s_{l+}(\vartheta_0^+).$$

The left involute of ∂K starting at x_0 will be defined as

$$i_{l,x_0}(\vartheta) = x_l(s_{l+}(\vartheta)) - (s_{l+}(\vartheta) - s_0)\mathbf{t}_\vartheta \quad \text{for } \vartheta \geq \vartheta_0^+; \quad (46)$$

similarly if $\vartheta_0^- := \vartheta_{r-}(s_0)$, $s_0^- := s_{r-}(\vartheta_0^-)$, the right involute starting at x_0 will be defined as

$$i_{r,x_0}(\vartheta) = x_r(s_{r-}(\vartheta)) + (s_{r-}(\vartheta) - s_0)\mathbf{t}_\vartheta \quad \text{for } \vartheta \leq \vartheta_0^-. \quad (47)$$

From (46) and (34) it follows that

$$i_{l,x_0}(\vartheta) = h(\vartheta)\,\mathbf{n}_\vartheta - \left(\int_{\vartheta_0^+}^\vartheta h(\tau)\,d\tau - \dot{h}_+(\vartheta_0^+)\right)\mathbf{t}_\vartheta \qquad (48)$$

$$- \left|x_0 - x_l(s_0^+)\right|\mathbf{t}_\vartheta, \quad \vartheta \geq \vartheta_0^+;$$

similarly from (47), (35) it follows that

$$i_{r,x_0}(\vartheta) = h(\vartheta)\,\mathbf{n}_\vartheta - \left(\int_{\vartheta_0^-}^\vartheta h(\tau)\,d\tau - \dot{h}_-(\vartheta_0^-)\right)\mathbf{t}_\vartheta \qquad (49)$$

$$+ \left|x_0 - x_r(s_0^-)\right|\mathbf{t}_\vartheta, \quad \vartheta \leq \vartheta_0^-.$$

Let us notice that in (48) and (49) the same parameter ϑ is used, but with different range; it turns out that i_l is counterclockwise oriented; instead i_r is clockwise oriented; $x_0 = i_{l,x_0}(\vartheta_0^+) = i_{r,x_0}(\vartheta_0^-)$.

Remark 16. The following facts can be derived from the above equations:

(i) since h is Lipschitz continuous for every convex body K, then the involute i_{l,x_0} is a rectifiable curve;

(ii) $i_{l,x_0}(\vartheta_0^+) = x_0$ and

$$\left| i_{l,x_0}(\vartheta) - x_l\left(s_{l+}(\vartheta)\right) \right| = s_{l+}(\vartheta) - s_0; \tag{50}$$

(iii) if x is a vertex of ∂K then $i_{l,x_0}(\vartheta)$, for $(\cos\vartheta, \sin\vartheta) \in N_K(x)$, lies on an arc of circle centered at x with radius $s_{l+}(\vartheta) - s_0$;

(iv) the involute (42) satisfies

$$i_{l,x_0}(\vartheta + 2\pi) = i_{l,x_0}(\vartheta) - L\mathbf{t}_\vartheta, \quad \forall\vartheta \geq \vartheta_0^+. \tag{51}$$

Lemma 17. *Parameterization (48) of the involute i_{l,x_0} is 1-1 in the interval $[\vartheta_0^+, \vartheta_0^+ + 2\pi)$; moreover, except for at most a finite or countable set $\mathfrak{F}$ of values ϑ_i, $i = 1, 2, \ldots$ (corresponding to the 1-face F_{θ_i} of ∂K), i_{l,x_0} is differentiable and*

$$\frac{d}{d\vartheta}i_{l,x_0}(\vartheta) = \left(s_{l+}(\vartheta) - s_0\right)\mathbf{n}_\vartheta \quad \text{for } \vartheta > \vartheta_0^+, \ \vartheta \notin \mathfrak{F}; \tag{52}$$

furthermore i_{l,x_0} has left and right derivative with common direction $\mathbf{n}_\vartheta$ at $\vartheta = \vartheta_i \in \mathfrak{F}$.

Proof. By differentiating (48) and using (34), equality (52) is proved. Similar argument, at $\vartheta = \vartheta_i \in \mathfrak{F}$, proves that $\mathbf{n}_\vartheta$ is the common direction of the left and right derivatives. □

Remark 18. Let $\vartheta \to i_{l,x_1}(\vartheta), \vartheta \to i_{l,x_2}(\vartheta)$, $x_i = x(s_i)$, $i = 1, 2$, be left involutes of K. Since

$$i_{l,x_2}(\vartheta) - i_{l,x_1}(\vartheta) = \left(s_2 - s_1\right)\mathbf{t}_\vartheta,$$
$$\text{for } \vartheta > \max\left\{\vartheta_l^+(s_2), \vartheta_l^+(s_1)\right\}, \tag{53}$$

then they will be called *parallel* curves. Moreover, by (51), $i_{l,x_0}(\vartheta)$ and $i_{l,x_0}(\vartheta + 2\pi)$ will also be called parallel.

Theorem 19. *If $d\varrho$ is the arc element of the involute i_{l,x_0} then $\vartheta \to \varrho(\vartheta)$ is continuous and invertible in $\vartheta \geq \vartheta_0^+$ with continuous inverse $[0, +\infty) \ni \varrho \to \vartheta(\varrho)$. Moreover*

$$d\varrho = \left(s_{l+}(\vartheta) - s_0\right)d\vartheta \quad \text{for } \vartheta \geq \vartheta_0^+, \ \vartheta \notin \mathfrak{F}; \tag{54}$$

the involute is a convex curve with positive curvature a.e.:

$$\frac{d\vartheta}{d\varrho} = \frac{1}{\left(s_{l+}(\vartheta) - s_0\right)} \quad \text{for } \vartheta > \vartheta_0^+, \ \vartheta \notin \mathfrak{F}, \tag{55}$$

$\varrho \to i_{l,x_0}(\vartheta(\varrho))$ is C^1 everywhere, and

$$\frac{d}{d\varrho}i_{l,x_0} = \mathbf{n}_{\vartheta(\varrho)}. \tag{56}$$

Moreover the following properties hold.

(i) *For every $\varrho > 0$ the right derivative*

$$\left(\frac{d\vartheta}{d\varrho}\right)^+ = \frac{1}{s_{l+}(\vartheta(\varrho)) - s_0} \tag{57}$$

exists everywhere and it is a decreasing cadlag function.

(ii) *$(d/d\varrho)i_{l,x_0}$ has everywhere right derivative given by*

$$\left(\frac{d^2}{d\varrho^2}i_{l,x_0}\right)^+ = -\frac{1}{s_{l+}(\vartheta(\varrho)) - s_0}\mathbf{t}_{\vartheta(\varrho)}. \tag{58}$$

Theorem 20. *Let $K^{(n)}$ be a sequence of plane convex bodies which converges uniformly to K, $x^{(n)} \in \partial K^{(n)}$, $x^{(n)} \to x_0$; then the corresponding sequences of left involutes $i_{l,x^{(n)}}^{K^{(n)}}$ converge uniformly to i_{l,x_0} in compact subsets of $[\vartheta_0^+, +\infty]$; moreover the corresponding sequence of their derivatives (with respect to the arc length) converges uniformly to $(d/d\varrho)i_{l,x_0}$.*

Proof. By Remark 9 the sequence of functions s_{l+}^n converges to s_{l+}. From (54) the arclengths of the left involutes $i_{l,x^{(n)}}^{K^{(n)}}$

$$\varrho^{(n)}(\vartheta) = \int_{\vartheta_0}^\vartheta \left(s_{l+}^{(n)}(\vartheta) - s_0^{(n)}\right)d\vartheta \tag{59}$$

converges uniformly in compact subsets of $[\vartheta_0^+, +\infty)$ to the arc length $\varrho(\vartheta)$ of i_{l,x_0}; from (56) the same fact holds for their derivatives. □

Let us consider the arc of the involute

$$\eta := \left\{i_{l,x_0}(\vartheta) : \vartheta_0^+ \leq \vartheta \leq \vartheta_0^+ + 3\pi/2\right\} \tag{60}$$

and the set valued map F (Definition 8). Let

$$Q$$
$$= \bigcup_{\vartheta_0^+ \leq \vartheta \leq \vartheta_0^+ + 3\pi/2} \left\{\lambda F(\theta) + (1 - \lambda)i_{l,x_0}(\vartheta), \ 0 \leq \lambda \leq 1\right\}, \tag{61}$$
$$\theta = (\cos\vartheta, \sin\vartheta),$$

the union of segments joining the points of η with the corresponding points on ∂K.

Definition 21. If the tangent sector $T(x_0)$ to K has an opening less than or equal to $\pi/2$ as in Figure 1, then $Q \cup K$ is convex; let one define

$$\vartheta_l^* = \vartheta_0^+ + 3\pi/2. \tag{62}$$

If $Q \cup K$ is not convex then let us consider $\text{co}(Q \cup K)$. Let us notice that $\partial\text{co}(Q \cup K) \setminus \partial(Q \cup K)$ is an open segment with end points A, B, with $A \in \eta$, $B \in \partial K$. Let us define ϑ_l^*, with $\vartheta_0^+ + 3\pi/2 \leq \vartheta_l^* < \vartheta_0^+ + 2\pi$ such that (see Figure 2) $\theta_l^* = (\cos\vartheta_l^*, \sin\vartheta_l^*)$ is orthogonal to AB, $B \in F(\theta_l^*)$. Let $\vartheta_{1,l}$ be the smallest $\theta > \theta_0^+$ satisfying $A = i_{l,x_0}(\vartheta_{1,l})$. Clearly $\vartheta_l^* = \vartheta_{1,l} + (3/2)\pi$.
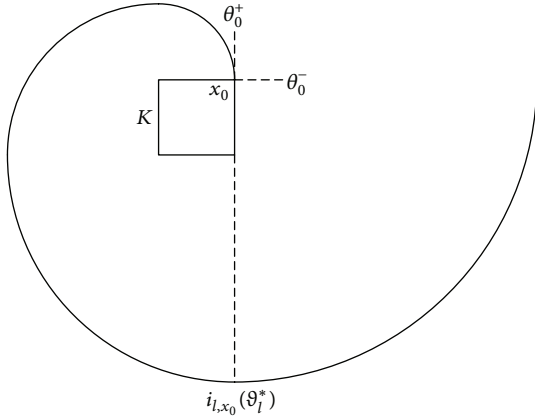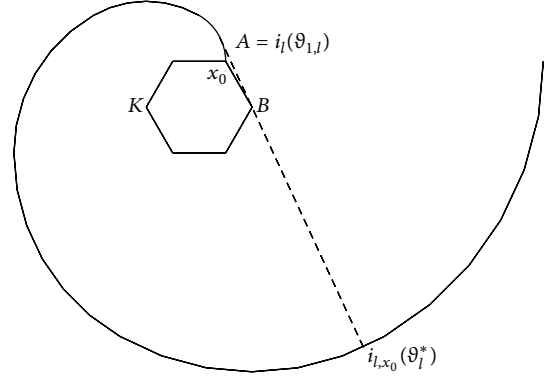
FIGURE 1: Left involute of a square.

For the right involutes a value ϑ_r^* is defined similarly, with $\vartheta_0^- - 2\pi < \vartheta_r^* \leq \vartheta_0^- - 3\pi/2$, such that the line orthogonal to θ_r^* supporting K at $F(\theta_r^*)$ is tangent to the right involute at $i_{r,x_0}(\vartheta_{1,r})$ (see Figure 3) where $F(\theta_r^*)$ is the point $x(s_{r-}(\vartheta_r^*))$, written as $x(\vartheta_r^*)$ for short.

Theorem 22. *Let $i_l := i_{l,x_0}$ be the left involute starting at x_0 on the boundary of a plane convex body K; then*

(i) *the left involute $\vartheta \to i_l(\vartheta)$ has the distancing from K property for $\vartheta \geq \vartheta_0^+$ but is not SDC for $\vartheta \geq \vartheta_l^*$;*

(ii) *the curve $\vartheta \in [\vartheta_0^+, \vartheta_l^*] \to i(\vartheta)$ is SDC;*

(iii) *for $y \in \text{Int}(K)$ the distance function $J_y(\vartheta) = |i_l(\vartheta) - y|$ is strictly increasing for $\vartheta \geq \vartheta_0^+$;*

(iv) *if $y \in \partial K$, then $J_y(\vartheta)$ is not decreasing for $\vartheta \geq \vartheta_0^+$ and $(d/d\vartheta)J > 0$ for $(\cos \vartheta, \sin \vartheta) \notin N_K(y)$.*

Proof. As i_l is rectifiable, then the function $J_y^2(\vartheta) = |i_l(\vartheta) - y|^2$ is an absolutely continuous function for $\vartheta \geq \vartheta_0^+$, and from (52) for $\vartheta \notin \mathfrak{F}$

$$\frac{1}{2}\frac{d}{d\vartheta}J_y^2 = \left\langle \frac{d}{d\vartheta}i_l, i_l(\vartheta) - y \right\rangle$$
$$= \left\langle (s_l(\vartheta) - s_0)\mathbf{n}_\vartheta, x_l(s_{l+}(\vartheta)) + (s_{l+}(\vartheta) - s_0)\mathbf{t}_\vartheta - y \right\rangle \quad (63)$$
$$= (s_{l+}(\vartheta) - s_0)\langle \mathbf{n}_\vartheta, x_l(s_{l+}(\vartheta)) - y \rangle \geq 0;$$

the last inequality holds since $\mathbf{n}_\vartheta$ is the outer normal to ∂K at $x_l(s_{l+}(\vartheta))$. Moreover the previous inequality is strict for all ϑ if $y \in \text{Int}(K)$, and it is also a strict inequality for $y \in \partial K$ and $y \notin F(\theta)$. This proves (iii) and (iv). Then (i) follows from (iii) and Definition 1 of distancing from K property for a curve. To prove (ii) let us recall that SDC satisfies (2); then one has to prove that the angle at $i_l(\vartheta)$ between the vector $i_l(\vartheta) - i_l(\tau)$, $\vartheta_0^+ < \tau < \vartheta \leq \vartheta_l^*$, and $\mathbf{n}_\vartheta$, the tangent vector at $i_l(\vartheta)$, is greater than or equal to $\pi/2$; this is equivalent to show that the half line r_ϑ through $i_l(\vartheta)$ and $x(s_{l+}(\vartheta))$ orthogonal to $\mathbf{n}_\vartheta$ supports at $i_l(\vartheta)$ the arc of i_l from x_0 to $i_l(\vartheta)$. By Definition 21 this is the case for all ϑ between ϑ_0^+ and ϑ_l^*. □
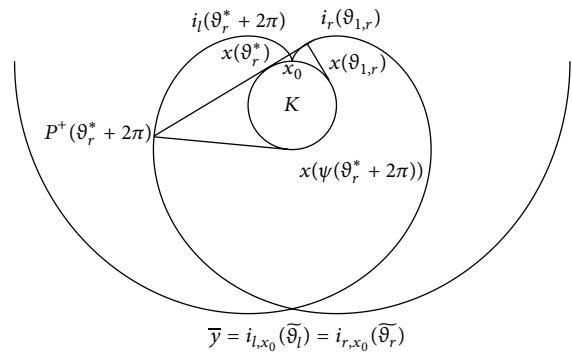

FIGURE 2: Left involute of an hexagon.

FIGURE 3: Involutes of a circumference.

Corollary 23. *The left involute $\vartheta \to i_{l,x_0}(\vartheta)$ of the boundary of a plane convex body K is a self-distancing curve from K for $\vartheta \in [\vartheta_0^+, \vartheta_l^*]$; similarly right involute (49) is a self-contracting curve from K for $\vartheta \in [\vartheta_r^*, \vartheta_0^-]$.*

Proof. From (i) of Theorem 22 the left involute is a curve such that the distance of its points from all $y \in K$ is not decreasing; (ii) of the same theorem proves that it is a SDC. Let us recall that a self-contracting curve is a self-distancing curve with opposite orientation. □

Theorem 24. *Let K be a plane convex body not reduced to a single point and let $x_0, s_0, \theta_0, \vartheta_0$ be the initial parameters. Let $[\vartheta_0^+, \vartheta_0^+ + 2\pi] \ni \vartheta \to i_l(\vartheta)$ be an arc of the left involute starting at x_0, and let $[\vartheta_0^- - 2\pi, \vartheta_0^-] \ni \vartheta \to i_r(\vartheta)$ be an arc of the right involute ending at x_0; then there exists only one point $\overline{y} \neq x_0$ which belongs to both arcs and*

$$\overline{y} = i_l(\widetilde{\vartheta}_l) = i_r(\widetilde{\vartheta}_r), \quad (64)$$

with

$$\vartheta_0^- \leq \vartheta_r^* + 2\pi < \widetilde{\vartheta}_l < \vartheta_0^+ + \frac{3\pi}{2} \leq \vartheta_l^*,$$
$$\vartheta_r^* \leq \vartheta_0^- - \frac{3\pi}{2} < \widetilde{\vartheta}_r < \vartheta_l^* - 2\pi \leq \vartheta_0^+. \quad (65)$$

Proof. For simplicity, first let us prove the existence of $\overline{y}$ assuming that $K \in C_+^2$. With the assumed conditions,

$\mathbb{R} \ni \vartheta \to x(\vartheta) := x(\theta)$ defined by (30) is a parameterization of ∂K.

Let $\vartheta \in [\vartheta_0, \vartheta_0 + 2\pi]$ and let $P^+(\vartheta)$ be the first common point of the half line $\{x(\vartheta) + \lambda \mathbf{t}_\vartheta, \lambda > 0\}$ and of i_l. Moreover, let $[\vartheta_0, \vartheta_l^*] \ni \vartheta \to \psi(\vartheta)$ be the function satisfying

$$P^+(\vartheta) = i_l(\psi(\vartheta)). \tag{66}$$

Let

$$\phi(\vartheta) := |P^+(\vartheta) - i_l(\vartheta)|. \tag{67}$$

First the following sentence will be proved.

Claim 1. $P^+(\overline{\vartheta_l})$ belongs to i_r iff the equality

$$\phi(\overline{\vartheta_l}) = L \tag{68}$$

holds for some $\overline{\vartheta_l} \in [\vartheta_0, \vartheta_0 + 2\pi]$, $L = |\partial K|$.

Proof of Claim 1. If (68) holds, then

$$
\begin{aligned}
\left| P^+(\overline{\vartheta_l}) - x(\overline{\vartheta_l}) \right| &= \left| P^+(\overline{\vartheta_l}) - i_l(\overline{\vartheta_l}) \right| \\
&\quad - \left| i_l(\overline{\vartheta_l}) - x(\overline{\vartheta_l}) \right| \\
&= L - \left| \operatorname{arc}^+(x_0, x(\overline{\vartheta_l})) \right| \\
&= \left| \operatorname{arc}^+(x(\overline{\vartheta_l}), x_0) \right| \\
&= \left| \operatorname{arc}^-(x_0, x(\overline{\vartheta_l} - 2\pi)) \right|.
\end{aligned} \tag{69}
$$

Thus

$$
\begin{aligned}
P^+(\overline{\vartheta_l}) &= x(\overline{\vartheta_l}) + \left| \operatorname{arc}^+(x(\overline{\vartheta_l}), x_0) \right| \mathbf{t}_{\overline{\vartheta_l}} \\
&= x(\overline{\vartheta_l} - 2\pi) + \left| \operatorname{arc}^-(x_0, x(\overline{\vartheta_l} - 2\pi)) \right| \mathbf{t}_{\overline{\vartheta_l} - 2\pi} \\
&= i_r(\overline{\vartheta_l} - 2\pi).
\end{aligned} \tag{70}
$$

Thus $P^+(\overline{\vartheta_l})$ is on both arcs of involutes and the other way around.

Our aim is to prove that there exists $\overline{\vartheta_l} \in [\vartheta_0, \vartheta_0 + 3\pi/2]$ such that (68) holds. For this goal we prove next Claims 2 and 3.

Claim 2. The following facts hold in $[\vartheta_0, \vartheta_l^*]$:

(i) ψ is continuously differentiable and $\psi' > 0$;

(ii) $\phi' > 0$.

Proof of Claim 2. Let us prove that $\mathbf{n}_\vartheta$ and $\mathbf{n}_{\psi(\vartheta)}$ satisfy

$$\langle \mathbf{n}_\vartheta, \mathbf{n}_{\psi(\vartheta)} \rangle < 0. \tag{71}$$

Let us consider the triangle with vertices $x(\vartheta), i_l(\psi(\vartheta))$, $x(\psi(\vartheta))$. As

$$
\begin{aligned}
\left| i_l(\psi(\vartheta)) - x(\psi(\vartheta)) \right| &= \left| \operatorname{arc}^+(x_0, x(\psi(\vartheta))) \right| \\
&\geq \left| \operatorname{arc}^+(x(\vartheta), x(\psi(\vartheta))) \right| \geq \left| x(\psi(\vartheta)) - x(\vartheta) \right|,
\end{aligned} \tag{72}
$$

the angle between $x(\psi(\vartheta)) - P^+(\vartheta)$ and $x(\vartheta) - P^+(\vartheta)$ is acute and the angle between $\mathbf{n}_\vartheta$ and $\mathbf{n}_{\psi(\vartheta)}$ is obtuse. Thus (71) follows. By definition, $\psi(\vartheta)$ solves (66); thus $\psi(\vartheta)$ is the implicit solution to

$$\langle i_l(\psi(\vartheta)) - x(\vartheta), \mathbf{n}_\vartheta \rangle = 0. \tag{73}$$

As

$$\left\langle \frac{d}{d\psi} i_l(\psi), \mathbf{n}_\vartheta \right\rangle = (s(\psi) - s_0) \langle \mathbf{n}_\psi, \mathbf{n}_\vartheta \rangle \tag{74}$$

is negative by (71), then by Dini's Theorem equation (73) has a solution $\psi(\theta)$ satisfying

$$
\begin{aligned}
(s(\psi) - s_0) \langle \mathbf{n}_\psi, \mathbf{n}_\vartheta \rangle \psi'(\vartheta) + \langle i_l(\psi(\vartheta)) - x(\vartheta), \mathbf{t}_\vartheta \rangle \\
= 0.
\end{aligned} \tag{75}
$$

As $i_l(\psi(\vartheta)) - x(\vartheta) = \lambda \mathbf{t}_\vartheta$ ($\lambda > 0$) and (71) holds, then $\psi' > 0$, and ψ is strictly increasing and continuously differentiable.

Let us prove (ii).

The formula

$$
\begin{aligned}
\frac{d}{d\vartheta} &\left| i_l(\vartheta) - i_l(\psi(\vartheta)) \right|^2 \\
&= 2 \left\langle i_l(\vartheta) - i_l(\psi(\vartheta)), \frac{d}{d\vartheta} i_l(\vartheta) - \frac{d}{d\vartheta} i_l(\psi(\vartheta)) \right\rangle
\end{aligned} \tag{76}
$$

holds. Let us notice that $i_l(\vartheta) - i_l(\psi(\vartheta))$ is parallel to $\mathbf{t}_\vartheta$; thus by (52)

$$\left\langle i_l(\vartheta) - i_l(\psi(\vartheta)), \frac{d}{d\vartheta} i_l(\vartheta) \right\rangle = 0. \tag{77}$$

On the other hand

$$
\begin{aligned}
- &\left\langle i_l(\vartheta) - i_l(\psi(\vartheta)), \frac{d}{d\vartheta} i_l(\psi(\vartheta)) \right\rangle \\
&= - \left\langle -s(\vartheta) \mathbf{t}_\vartheta - \lambda \mathbf{t}_\vartheta, (s(\psi(\vartheta)) - s_0) \mathbf{n}_{\psi(\vartheta)} \right\rangle \psi' \\
&= (s(\vartheta) + \lambda)(s(\psi(\vartheta)) - s_0) \langle \mathbf{t}_\vartheta, \mathbf{n}_{\psi(\vartheta)} \rangle \psi'.
\end{aligned} \tag{78}
$$

As the angle between $\mathbf{t}_\vartheta$ and $\mathbf{n}_{\psi(\vartheta)}$ is acute, then last term in the above equalities is positive; thus the derivative in the left hand side of (76) is positive and (ii) of Claim 2 follows.

Claim 3. In the interval $[\vartheta_0, \vartheta_l^*]$ the function ϕ has values smaller than L and greater than L.

Proof of Claim 3. The angles ϑ_r^* and $\vartheta_{1,r}$ have been introduced in Definition 21. For simplicity $x(s_{r-}(\vartheta_{1,r}))$ will be denoted with $x(\vartheta_{1,r})$. Let us consider the convex set bounded by $\operatorname{arc}^+(x(\psi(\vartheta_r^* + 2\pi)), x(\vartheta_{1,r}))$ and by the polygonal line with vertices $x(\vartheta_{1,r}), i_r(\vartheta_{1,r}), P^+(\vartheta_r^* + 2\pi), x(\psi(\vartheta_r^* + 2\pi))$; see Figure 3.

Clearly the inequalities

$$
\begin{aligned}
\left|i_r\left(\vartheta_{1,r}\right) - P^+\left(\vartheta_r^* + 2\pi\right)\right| \\
< \left|i_r\left(\vartheta_{1r}\right) - x\left(\vartheta_{1,r}\right)\right| \\
+ \left|\mathrm{arc}^-\left(x\left(\vartheta_{1,r}\right), x\left(\psi\left(\vartheta_r^* + 2\pi\right)\right)\right)\right| \\
+ \left|x\left(\psi\left(\vartheta_r^* + 2\pi\right)\right) - P^+\left(\vartheta_r^* + 2\pi\right)\right| \\
= \left|\mathrm{arc}^-\left(x_0, x\left(\vartheta_{1,r}\right)\right)\right| \\
+ \left|\mathrm{arc}^-\left(x\left(\vartheta_{1,r}\right), x\left(\psi\left(\vartheta_r^* + 2\pi\right)\right)\right)\right| \\
+ \left|\mathrm{arc}^-\left(x\left(\psi\left(\vartheta_r^* + 2\pi\right)\right), x_0\right)\right| = L
\end{aligned}
\tag{79}
$$

hold. As

$$
\begin{aligned}
\phi\left(\vartheta_r^* + 2\pi\right) &= \left|i_l\left(\vartheta_r^* + 2\pi\right) - P^+\left(\vartheta_r^* + 2\pi\right)\right| \\
&< \left|i_r\left(\vartheta_{1,r}\right) - P^+\left(\vartheta_r^* + 2\pi\right)\right|,
\end{aligned}
\tag{80}
$$

using the previous inequalities, one obtains

$$
\phi\left(\vartheta_r^* + 2\pi\right) < L. \tag{81}
$$

Let us show now that

$$
\phi\left(\vartheta_0 + \frac{3\pi}{2}\right) > L \tag{82}
$$

holds.

Let ρ be the half line with origin x_0 and direction $-\mathbf{t}_{\vartheta_0}$; $\rho - \{x_0\}$ crosses the arc i_r in a first point $y_1 = i_r(\alpha_1)$, with $\alpha_1 < \vartheta_0 - \pi/2$. Then

$$
\begin{aligned}
r := \left|x_0 - y_1\right| &< \left|y_1 - x\left(\alpha_1\right)\right| + \left|\mathrm{arc}^-\left(x\left(\alpha_1\right), x_0\right)\right| \\
&= L.
\end{aligned}
\tag{83}
$$

The half line ρ meets the arc i_l in a point y_2 and $|y_2 - x_0| = L$.

Property (iii) of Theorem 22 implies that the arc D of the left involute after y_2 lies outside of the circle centered in x_0 and with radius L. Similar property for the right involute implies that the arc C of the right involute joining x_0 to y_1 lies in the circle with center x_0 and radius r; thus the straight line tangent to K at $x(\vartheta_0 + 3\pi/2)$ meets the arc C in $i_r(\vartheta_0 - \pi/2)$ and D in $P^+(\vartheta_0 + 3\pi/2)$. Therefore

$$
\begin{aligned}
\phi\left(\vartheta_0 + \frac{3}{2\pi}\right) &= \left|i_l\left(\vartheta_0 + \frac{3\pi}{2}\right) - P^+\left(\vartheta_0 + \frac{3\pi}{2}\right)\right| \\
&= \left|i_l\left(\vartheta_0 + \frac{3\pi}{2}\right) - x\left(\vartheta_0 + \frac{3\pi}{2}\right)\right| \\
&\quad + \left|x\left(\vartheta_0 + \frac{3\pi}{2}\right) - P^+\left(\vartheta_0 + \frac{3\pi}{2}\right)\right| \\
&> \left|i_l\left(\vartheta_0 + \frac{3\pi}{2}\right) - x\left(\vartheta_0 + \frac{3\pi}{2}\right)\right| \\
&\quad + \left|x\left(\vartheta_0 + \frac{3\pi}{2}\right) - i_r\left(\vartheta_0 - \frac{\pi}{2}\right)\right| \\
&= \left|\mathrm{arc}^+\left(x_0, x\left(\vartheta_0 + \frac{3\pi}{2}\right)\right)\right| \\
&\quad + \left|\mathrm{arc}^-\left(x_0, x\left(\vartheta_0 + \frac{3\pi}{2}\right)\right)\right| = L.
\end{aligned}
\tag{84}
$$

Inequality (82) is proved.

The intermediate values theorem implies that there exists $\overline{\vartheta_l} \in [2\pi + \vartheta_r^*, \vartheta_0 + 3\pi/2]$ such that (68) holds. Claim 1 implies that

$$
P^+\left(\overline{\vartheta_l}\right) = i_l\left(\psi\left(\overline{\vartheta_l}\right)\right) = i_r\left(\overline{\vartheta_l} - 2\pi\right), \tag{85}
$$

so the right involute and the left involute meet each other in one point and (64) is proved with $\overline{\vartheta_l} = \psi(\overline{\vartheta_l})$, $\overline{\vartheta_r} = \overline{\vartheta_l} - 2\pi$.

By approximation argument the same result holds for an arbitrary convex body K.

Let us prove now that the point $\overline{y}$ is unique. Let us argue by contradiction. Let P, Q be two distinct points on $i_l \cap i_r$, with $P \prec Q$ on i_l and i_r; then since i_l is a distancing curve from x_0,

$$
\left|P - x_0\right| \leq \left|Q - x_0\right|, \tag{86}
$$

and since i_r is a contracting curve to x_0,

$$
\left|P - x_0\right| \geq \left|Q - x_0\right|, \tag{87}
$$

therefore all the points on the arc of i_l and of i_r between P and Q have the same distance from x_0; thus, between P and Q, i_l and i_r (arc of involutes of a same convex body K) coincide with the same arc of circle centered at x_0; this implies that K reduce to the point x_0, which is not possible for the assumption. □

Definition 25. Let $z \notin K$. Let $z_l(z_r) \in \partial K$ on the contact set on the "left" (right) support line to K through z. If the contact set is a 1-face on these support lines, then z_l and z_r are identified as the closest ones to z. The triangle zz_lz_r is counterclockwise oriented.

Theorem 26. *For every $\xi \in \partial K$ let one consider the left involutes $i_{l,\xi}$ and the right involutes $i_{r,\xi}$ parameterized by their arc length ϱ. The maps*

$$
\begin{aligned}
\partial K \times (0, +\infty) \ni (\xi, \varrho) &\longrightarrow \\
i_{l,\xi}\left(\theta\left(\varrho\right)\right) &\in \mathbb{R}^2 \setminus K, \\
\partial K \times (0, +\infty) \ni (\xi, \varrho) &\longrightarrow \\
i_{r,\xi}\left(\theta\left(\varrho\right)\right) &\in \mathbb{R}^2 \setminus K
\end{aligned}
\tag{88}
$$

are 1-1 maps.

Proof. Assume, in the proof, that $x_0 \in \partial K, \theta_0 \in G(x_0), \vartheta_0, s_0$ are fixed. Let $z \notin K$. The tangent sector to the cap body K^z with vertex z has two maximal segments zz_l and zz_r on the sides that do not meet K (except at the end points z_l and z_r). Let ϑ_l be such that $z_l = x_l(s_{l+}(\vartheta_l))$, and let $\overline{s}$ be such that

$$
\left|z - z_l\right| = s_{l+}\left(\vartheta_l\right) - \overline{s}. \tag{89}
$$

Let $\xi_l = x_l(\overline{s})$; let $\overline{\vartheta} = \vartheta_l^+(\overline{s})$. From (50) and from the definition of left involute (46) (with ξ_l in place of x_0, $\overline{\vartheta}$ in place of ϑ_0^+, and $\overline{s}$ in place s_0)

$$
z = i_{l,\xi_l}\left(\vartheta_l\right) \tag{90}
$$

holds; thus the map $(\xi, \varrho) \rightarrow i_{l,\xi}(\varrho)$ is surjective. Moreover the map is also injective, since the left involutes do not cross each other since they are parallel (see Remark 18). Similar proof holds for the right involutes. □

Let $\xi_l = x_l(\bar{s})$ be the starting point of the left involute i_{l,ξ_l} through z, defined in the previous theorem; similarly let ξ_r be the starting point of the right involute i_{r,ξ_r} through z. Let us notice that i_{l,ξ_l} and i_{r,ξ_r} meet each other in a countable ordered set of points.

3.1. $\mathfrak{F}$-Fence and $\mathfrak{G}$-Fence

Definition 27. Let K be a convex body in $\mathbb{R}^2$, $|\partial K| > 0$, $x_0 \in \partial K$, $\theta_0 \in G(x_0)$, $\theta_0 = (\cos \vartheta_0, \sin \vartheta_0)$, $s_0 \in \mathbb{R}$. Let $i_l := i_{l,x_0}$, and let $i_r := i_{r,x_0}$. Let

$$\overline{y} = i_l\left(\widetilde{\vartheta}_l\right) = i_r\left(\widetilde{\vartheta}_r\right) \in \mathbb{R}^2 \setminus K \tag{91}$$

be the first point where the two involutes cross each other (see Theorem 24). Let one define

$$\mathfrak{F}_l(K, x_0) := \left\{ y \in \mathbb{R}^2 : y = tx_0 + (1-t) i_l(\vartheta), \ 0 \le t \right.$$
$$\left. \le 1, \ \vartheta_0^+ \le \vartheta \le \widetilde{\vartheta}_l \right\},$$

$$\mathfrak{F}_r(K, x_0) := \left\{ y \in \mathbb{R}^2 : y = tx_0 + (1-t) i_r(\vartheta), \ 0 \le t \tag{92} \right.$$
$$\left. \le 1, \ \widetilde{\vartheta}_r \le \vartheta \le \vartheta_0^- \right\},$$

$$\mathfrak{F}(K, x_0) := \left(\mathfrak{F}_l(K, x_0) \cup \mathfrak{F}_r(K, x_0)\right) \setminus \mathrm{Int}(K).$$

$\mathfrak{F}(K, x_0)$ will be called the $\mathfrak{F}$-fence of K at x_0.

Let us notice that $\mathfrak{F}_l(K, x_0)$ and $\mathfrak{F}_r(K, x_0)$ are two convex bodies in common with the segment $x_0\overline{y}$ only.

From Theorem 26 the starting point ξ_l (ξ_r) of a left (right) involute is uniquely determined from any point $z \notin K$ of the involute. The arc of the points on the left (right) involute between the starting point and z will be denoted by i_{l,ξ_l}^z (i_{r,ξ_r}^z) or i_l^z (i_r^z) for short. For $y \preceq w$ let us denote with $i_l^{y,w}$ ($i_r^{y,w}$) the oriented arc of the left (right) involute between y and w.

Let us introduce now other regions which are bounded by left and right involutes.

Let us fix the initial parameters $x_0, s_0, \theta_0, \vartheta_0$.

Definition 28. Given $z \in \mathbb{R}^2 \setminus K$, let $i_l = i_{l,\xi_l}$ ($i_r = i_{r,\xi_r}$) be the left (right) involute through z with starting point ξ_l (ξ_r) and let z_l (z_r) $\in \partial K$ be as in Definition 25. Let $\vartheta_{\xi_l}^+$ satisfying $x_l(s_{l+}(\vartheta_{\xi_l}^+)) = \xi_l$. Let $\vartheta_l > \vartheta_{\xi_l}^+$ be the smallest angle for which $x_l(s_{l+}(\vartheta_l)) = z_l$. Let one consider the parameterization (46); let one define

$$\mathfrak{G}_l(K, z) := \left\{ tx_l\left(s_{l+}(\vartheta)\right) + (1-t) i_l(\vartheta), \ 0 < t \right. \tag{93}$$
$$\left. < 1, \ \vartheta_{\xi_l}^+ < \vartheta < \vartheta_l \right\}.$$

If i_l^z does not cross the open segment zz_l, the region $\mathfrak{G}_l(K, z)$ is an open set bounded by the convex arc of left involute i_l^z, the

segment zz_l, and the convex arc of ∂K: $\mathrm{arc}^+(\xi_l, z_l)$; otherwise let w be the nearest point to z where i_l^z crosses the open segment zz_l; the region $\mathfrak{G}_l(K, z)$ is an open set bounded by the arc $i_l^{w,z}$, the segment wz, and ∂K. Similarly let us define $\mathfrak{G}_r(K, z)$.

$\mathfrak{G}_l(K, z), \mathfrak{G}_r(K, z)$ are open and bounded sets. Let us define

$$\mathfrak{G}(K, z) := \mathrm{Int}\left(\mathrm{cl}\left(\mathfrak{G}_l(K, z) \cup \mathfrak{G}_r(K, z)\right)\right). \tag{94}$$

$\mathfrak{G}(K, z)$ is an open, bounded, connected set. $\mathfrak{G}(K, z)$ will be called the $\mathfrak{G}$-fence of K at z.

Remark 29. If z is the first crossing point of i_l and i_r and $\xi_l = \xi_r$, then $\mathfrak{G}(K, z) = \mathrm{Int}(\mathfrak{F}(K, \xi_l))$.

Let us conclude this section with the following result, which follows from Theorem 20.

Theorem 30. *Let K be limit of a sequence of convex bodies $K^{(n)}$, $x_0 = \lim x_0^{(n)}$, and $x_0^{(n)} \in \partial K^{(n)}$. Then*

$$\mathfrak{F}\left(K^{(n)}, x_0^{(n)}\right) \longrightarrow \mathfrak{F}(K, x_0). \tag{95}$$

Moreover if $z \notin K$, $z = \lim z^{(n)}$, $z^{(n)} \notin K^{(n)}$, then

$$\mathrm{cl}\left(\mathfrak{G}\left(K^{(n)}, z^{(n)}\right)\right) \longrightarrow \mathrm{cl}\left(\mathfrak{G}(K, z)\right). \tag{96}$$

4. Bounding Regions for SDC in the Plane

Let us assume that x_0 is the end point of one of the following sets:

   (a) a steepest descent curve γ;

   (b) γ^K: a self-distancing curve from a convex body K; see Definition 3.

The following questions arise: can one extend γ, γ^K beyond x_0? Which regions delimit that extension? Which regions are allowed and which are forbidden?

Lemma 31. *Let $z \in \mathbb{R}^2 \setminus K$. If $u \in \mathfrak{G}_l(K, z)$ then the arc i_l^u of the left involute to K ending at u is contained in $\mathfrak{G}_l(K, z)$. Similarly if $u \in \mathfrak{G}_r(K, z)$, then $i_r^u \subset \mathfrak{G}_r(K, z)$.*

Proof. Since $u \in \mathfrak{G}_l(K, z)$, by (93) there exist $\overline{\vartheta}_l \in (\vartheta_{\xi_l}^+, \vartheta_l)$, $\tau \in (0, 1)$ such that

$$u = \tau x_l\left(s_{l+}\left(\overline{\vartheta}_l\right)\right) + (1-\tau) i_l\left(\overline{\vartheta}_l\right). \tag{97}$$

Then the arc i_l^u is parallel to an arc of the left involute i_l (through z) for $\vartheta \in (\vartheta_{\xi_l}^+, \overline{\vartheta}_l)$. Then any left tangent segment to K from a point of i_l^u is contained in the left tangent segment from the corresponding point of i_l^z. □

Lemma 32. *Let $z \in \mathbb{R}^2 \setminus K$ and let $u \in \mathfrak{G}_l(K, z)$. There are two possible cases:*

(i) *if the right involute ending at u does not cross the tangent segment $z_l z$ or it crosses $z_l z$ at a point $q \in \mathfrak{G}_l(K, z)$, then in both cases $i_r^u \subset \mathfrak{G}_l(K, z)$;*

(ii) *if the right involute ending at u crosses the tangent segment $z_l z$ at a point $q \in z_l z \cap \partial\mathfrak{G}_l(K, z)$, then $i_r^{q,u} \setminus \{q\} \subset \mathfrak{G}_l(K, z)$.*

Proof. Since the starting point $\xi_r(u)$ of the right involute ending at u is on ∂K, the distance from $\xi_r(u)$ to a point of the left involute i_l^z is not decreasing; see (iv) of Theorem 22; similarly the distance from $\xi_r(u)$ to a point of i_r^u is not decreasing. In case (i) the arc i_r^u has its end points in $\mathfrak{G}_l(K, z)$ and by the above distance property it can not cross two times the left involute; then it can not cross the boundary of $\mathfrak{G}_l(K, z)$; therefore $i_r^u \subset \mathfrak{G}_l(K, z)$; similarly in case (ii) the arc $i_r^{q,u}$ can not cross the boundary of $\mathfrak{G}_l(K, z)$ at most in q; therefore all the points of this arc, except to that q, belong to $\mathfrak{G}_l(K, z)$. □

From the previous lemma the following follows.

Theorem 33. *Let $z \notin K$. The following inclusions hold:*

(a) *if $u \in \mathfrak{G}_l(K, z)$, then*

$$\mathrm{cl}\left(\mathfrak{G}_l(K, u)\right) \setminus \partial K \subset \mathfrak{G}_l(K, z); \tag{98}$$

(b) *if $u \in \mathfrak{G}_r(K, z)$, then*

$$\mathrm{cl}\left(\mathfrak{G}_r(K, u)\right) \setminus \partial K \subset \mathfrak{G}_r(K, z); \tag{99}$$

(c) *if $u \in \mathfrak{G}(K, z)$, then*

$$\mathrm{cl}\left(\mathfrak{G}(K, u)\right) \setminus \partial K \subset \mathfrak{G}(K, z). \tag{100}$$

Proof. By Lemma 31 the left involute that bounds $\mathfrak{G}_l(K, u)$ is inside $\mathfrak{G}_l(K, z)$; then (98) is proved. Inclusion (99) is proved similarly. Let $u \in \mathfrak{G}(K, z) = \mathrm{Int}(\mathrm{cl}(\mathfrak{G}_l(K, z) \cup \mathfrak{G}_r(K, z)))$ and let us consider $u \in \mathfrak{G}_l(K, z)$; then in case (i) of Lemma 32 also the open arc of the right involute i_r^u is inside $\mathfrak{G}_l(K, z) \subset \mathfrak{G}(K, z)$. Besides $i_l^u \subset \partial\mathfrak{G}_l(K, u)$; then (100) is trivial. In case (ii) of Lemma 32 the open arc $i_r^{q,u}$ is inside $\mathfrak{G}_l(K, z)$. On the other hand q is inside $\mathfrak{G}_r(K, z)$ and by (99) the arc $i_r^q \subset i_r^u$ is in $\mathfrak{G}_r(K, z) \subset \mathfrak{G}(K, z)$. Similar arguments hold if $u \in \mathfrak{G}_r(K, z)$. Then (100) holds in this case too. □

Lemma 34. *Let $w \notin K$. Let η be polygonal deleted SDC_K with end point $y \in \mathfrak{G}(K, w)$. Then*

$$\eta \subset \mathfrak{G}(K, w), \tag{101}$$

$$\eta \subset \mathrm{cl}\left(\mathfrak{G}(K, y)\right). \tag{102}$$

Proof. To prove (101), let us assume, by contradiction, that η has a point $z \notin \mathfrak{G}(K, w)$. With no loss of generality it can be assumed that $z \in \partial\mathfrak{G}(K, w)$ and

$$\eta \setminus \eta_z \subset \mathfrak{G}(K, w). \tag{103}$$

Then, z is the end point of a segment $z w_i$, where $w_i \in \mathfrak{G}(K, w) \cap \eta$ and $z \prec w_i$ on η. As $z \in \partial\mathfrak{G}(K, w)$, then there exists an involute through z which is a piece of the boundary of $\mathfrak{G}(K, w)$ (to fix the ideas it is assumed that it is the left involute i_l). Let us consider $z_l \in \partial K$ so that the tangent vector $\mathbf{t}_z$ to i_l at z satisfies

$$\langle \mathbf{t}_z, z - z_l \rangle = 0. \tag{104}$$

As w_i is inside the orthogonal angle centered in z with sides $\mathbf{t}_z$ and $z_l - z$, then

$$\langle w_i - z, z - z_l \rangle < 0. \tag{105}$$

Then as for $\varepsilon > 0$ sufficiently small, $z_\varepsilon := z + \varepsilon(w_i - z) \in \eta_{w_i}$ and at z_ε the curve η has tangent vector $w_i - z$ that satisfies

$$\langle w_i - z, z_\varepsilon - z_l \rangle < 0, \tag{106}$$

contradicting the fact that η_w has the distancing from K property (5). This proves (101).

If $w_n \to y$, with $y \in \mathfrak{G}(K, w_n)$, also the inclusions

$$\eta \subset \mathrm{cl}\left(\mathfrak{G}(K, w_n)\right) \tag{107}$$

hold. Then (102) is obtained by approximation Theorem 30. □

Theorem 35. *Let K be a convex body and let γ^K be SDC_K, $w \in \gamma, w \notin K$. Then*

$$\gamma_w^K \subset \mathrm{cl}\left(\mathfrak{G}(K, w)\right). \tag{108}$$

Proof. Let us choose a sequence $\{w_n\}, w_n \in \gamma^K, w_n \preceq w, w_n \to w$. Let us fix the arc $\gamma_{w_n}^K$. By [8, Theorem 6.16], $\gamma_{w_n}^K$ is limit of polygonal SDC_K with end point w_n. From Lemma 34, these polygonal SDC_K are enclosed in $\mathrm{cl}(\mathfrak{G}(K, w_n))$; then

$$\gamma_{w_n}^K \subset \mathrm{cl}\left(\mathfrak{G}(K, w_n)\right) \tag{109}$$

holds too. Inclusion (108) is now obtained from the limit of the previous inclusions and by the approximation Theorem 30. □

Theorem 36. *Let K be a convex body not reduced to a point. If γ^K is a self-distancing curve from K with starting point $x_0 \in \partial K$, then*

$$\gamma^K \subset \mathrm{cl}\left(\mathbb{R}^2 \setminus \left(\mathfrak{F}(K, x_0) \cup K\right)\right). \tag{110}$$

Proof. Let z be the first crossing point of the left and right involutes of K starting at x_0. Then

$$\mathrm{Int}\left(\mathfrak{F}(K, x_0)\right) = \mathfrak{G}(K, z). \tag{111}$$

By contradiction, if γ^K has a point $w \in \mathfrak{G}(K, z)$, then, by Theorem 35, the following inclusion holds:

$$\gamma_w^K \subset \mathrm{cl}\left(\mathfrak{G}(K, w)\right); \tag{112}$$

since, by the distancing from K property, γ^K has in common with K only the starting point x_0 then the inclusion

$$\gamma_w^K \setminus \{x_0\} \subset \mathrm{cl}\left(\mathfrak{G}(K,w)\right) \setminus \partial K \tag{113}$$

holds too. Moreover by (100) the set $\mathrm{cl}(\mathfrak{G}(K,w)) \setminus \partial K$ has positive distance from the $\mathbb{R}^2 \setminus \mathfrak{G}(K,z)$; then $\gamma_w^K \setminus \{x_0\}$ has a positive distance from $\mathbb{R}^2 \setminus \mathfrak{G}(K,z) = \mathbb{R}^2 \setminus \mathrm{Int}(\mathfrak{F}(K,x_0))$. This is in contradiction with $x_0 \in \partial \mathfrak{F}(K,x_0)$. $\qquad\square$

Corollary 37. *Let γ be SDC and let $z_1 \in \gamma$; then*

$$\gamma \setminus \gamma_{z_1} \subset \mathrm{cl}\left(\mathbb{R}^2 \setminus \mathfrak{F}\left(\mathrm{co}\left(\gamma_{z_1}\right), z_1\right)\right). \tag{114}$$

Proof. Since $\gamma \setminus \gamma_{z_1}$ is a self-distancing curve from $\mathrm{co}(\gamma_{z_1})$ and $z_1 \in \partial \mathrm{co}(\gamma_{z_1})$ (see [8, (i) of Lemma 4.6]), then Theorem 36 applies to $\gamma^K = \gamma \setminus \gamma_{z_1}$ with $K = \mathrm{co}(\gamma_{z_1})$. $\qquad\square$

Definition 38. Let γ be SCD. If $z_1, z \in \gamma$, with $z_1 \preceq z$ let

$$\gamma_{z_1,z} := \gamma_z \setminus \gamma_{z_1}. \tag{115}$$

For $z \notin K$, let K^z be the cap body, introduced in (9). Next theorem shows the principal result on bounding regions for arcs of SDC γ.

Theorem 39. *Let K be a convex body and let γ be SDC_K. If $z_1, z \in \gamma$, with $z_1 \preceq z$ then*

$$\gamma_{z_1,z} \subset \mathrm{cl}\left(\mathfrak{G}(K,z) \setminus \mathfrak{F}\left(K^{z_1}, z_1\right)\right). \tag{116}$$

Proof. First let us notice that $\gamma_{z_1,z}$ has the distancing from K and from the set point $\{z_1\}$ property; thus by Proposition 7 it has the distancing from K^{z_1} property. Then inclusion (116) follows from Theorems 35 and 36. $\qquad\square$

Let us conclude the section with the following inclusion result for $\mathfrak{F}$-fences.

Theorem 40. *Let K, H be two convex bodies not reduced to a point, $K \subset H$. Let $x_0 \in \partial K \cap \partial H$. Then*

$$\mathfrak{F}(K, x_0) \subset \mathfrak{F}(H, x_0). \tag{117}$$

Proof. The boundary of $\mathfrak{F}(H, x_0)$ consists of two arcs of the left and right involutes of H starting at x_0. By Corollary 23 they are SDC_H, and then they are SDC_K; therefore by Theorem 36 they cannot intersect the boundary of $\mathfrak{F}(K, x_0)$. $\qquad\square$

4.1. Minimally Connecting Plane Steepest Descent Curves. Given a point $x_1 \notin K$, the segment joining it with its projection x_0 on ∂K is SDC_K which *minimally* connects the two points.

This subsection is devoted to consider when it would be possible to connect a given point x_0 on the boundary of a plane convex body K, with an arbitrarily given point $x_1 \notin K$, by using a steepest descent curve $\gamma \in SDC_K$. Let us denote with Γ_{x_0,x_1}^K the class of the curves $\gamma \in SDC_K$ starting at x_0 and ending at x_1.
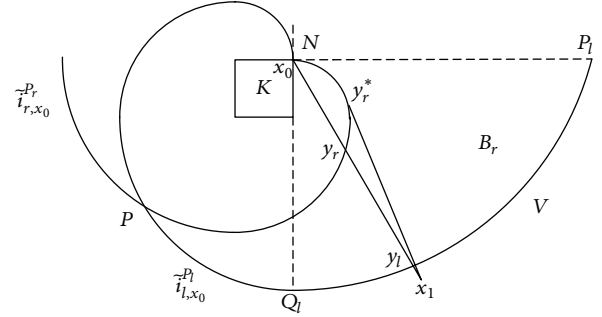

Figure 4: The regions N, B_r, and V when K is a square.

Definition 41. Let γ be SDC with end point y and let η be SDC with starting point y; let us denote by $\gamma * \eta$ the curve joining γ with η in the natural order, if it is SDC curve.

Theorem 42. *Let $x_0 \in \partial K$, $x_1 \notin K$. Then $\Gamma_{x_0,x_1}^K \neq \emptyset$ iff*

$$x_1 \in \mathrm{cl}\left(\mathbb{R}^2 \setminus \left(\mathfrak{F}(K, x_0)\right) \cup K\right). \tag{118}$$

If (118) holds, there exist at most two $\eta_i \in \Gamma_{x_0,x_1}^K$, $i = 1, 2$ such that the following properties are true:

$$\forall \gamma \in \Gamma_{x_0,x_1}^K \implies$$
$$\mathrm{co}\left(\eta_1\right) \subset \mathrm{co}\left(\gamma\right) \tag{119}$$
$$or \ \mathrm{co}\left(\eta_2\right) \subset \mathrm{co}\left(\gamma\right), \ (or \ both),$$

$$\forall \gamma \in \Gamma_{x_0,x_1}^K \implies$$
$$|\gamma| \geq \min_{i=1,2}\left\{|\eta_i|\right\}. \tag{120}$$

Proof. Let $\gamma \in \Gamma_{x_0,x_1}^K$. From (110) of Theorem 36, since $x_1 \in \gamma$, then (118) follows.

Let us prove now that (118) is sufficient. Let us notice that $\mathbb{R}^2 \setminus (\mathfrak{F}(K, x_0) \cup K)$ can be divided into four regions N, B_l, B_r, and V (see Figure 4) defined as follows:

(i) the closed normal sector $N := x_0 + N_K(x_0)$ is the angle bounded by the two half lines t_l, t_r tangent at $x_0 \in \partial K$ to the left and right involute $i_l := i_{l,x_0}$, $i_r := i_{r,x_0}$, respectively; this angle can be reduced to an half line, starting at x_0;

(ii) let P be the first crossing point between i_l and i_r; see Theorem 24; i_l is SDC to $i_l(\vartheta_l^*)$, which will be a point Q_l following P; after Q_l the involute i_l is no more SDC; see (i) of Theorem 22.

Let us change i_l after Q_l with j_{l,Q_l}, the left involute of $\mathrm{co}(K \cup i_l^{Q_l})$ at Q_l.

Let us define P_l as the first intersection point of j_{l,Q_l} with ∂N, and let

$$\vec{i}_{l,x_0}^{P_l} := i_{l,x_0}^{Q_l} * j_{l,Q_l}^{P_l}. \tag{121}$$

It is not difficult to see that $\widetilde{i}_{l,x_0}^{P_l} \in \Gamma_{x_0,P_l}^K$. Changing the left with the right, $\widetilde{i}_{r,x_0}$ and the point P_r can be constructed. Let B_r be the union of the arc $i_{r,x_0}^P \setminus \{P\}$ with the plane open region bounded by the segment $x_0 P_l$, the arc i_{r,x_0}^P, and the arc $\widetilde{i}_l^{P,P_l}$; let B_l be the union of the arc $i_{l,x_0}^P \setminus \{P\}$ with the plane open region bounded by the segment $x_0 P_r$, the arc i_{l,x_0}^P, and the arc $\widetilde{i}_r^{P,P_r}$;

(iii) let V be the remaining region; that is, $V = \mathbb{R}^2 \setminus (K \cup \mathfrak{F}(K,x_0) \cup N \cup B_l \cup B_r)$.

Let $x_1 \in B_r \cup V$. On the oriented curve $\widetilde{i}_{r,x_0}^{P_r}$ there are two points so that their tangent lines contain x_1. Let y_r^* be the first tangency point.

Let y_l, y_r be the intersection points of the half line m starting at x_0 and containing x_1, with $\widetilde{i}_{l,x_0}^{P_l}$ and with $\widetilde{i}_{r,x_0}^{P_r}$, respectively; see Figure 4.

Under assumption (118), x_1 belongs to one of the four regions N, B_l, B_r, V; let us prove now (119), (120) in the four corresponding cases.

(1) If $x_1 \in N$, then let $\eta_1 = \eta_2 \in \Gamma_{x_0,x_1}^K$ be the segment $x_0 x_1$. Then (119) and (120) are trivial.

(2) Let $x_1 \in V$. Let

$$\eta_r := \widetilde{i}_{r,x_0}^{y_r^*} * y_r^* x_1. \tag{122}$$

The curve η_r is SDC_K, since the normal lines at all the points on the segment $y_r^* x_1$ have the same directions and support $\widetilde{i}_{r,x_0}^{y_r^*}$ up to y_r^*; then η_r is SDC_K and joins x_0 with x_1. Similarly SDC_K is defined:

$$\eta_l = \widetilde{i}_{l,x_0}^{y_l^*} * y_l^* x_1. \tag{123}$$

Thus Γ_{x_0,x_1}^K is nonempty and contains at least the two elements η_l, η_r.

Let us consider the connected closed curve

$$c_{x_1} := \widetilde{i}_{r,x_0}^{y_r} \cup y_r y_l \cup \widetilde{i}_{l,x_0}^{y_l}. \tag{124}$$

Let $\gamma \in \Gamma_{x_0,x_1}^K$, and let $T \ni t \to x(t) \in \gamma$ be a continuous parameterization of γ. Let us project from x_0 the curve γ on c_{x_1} and let D be this projection. That is, for $t \in T$, let $\lambda_t := \{x_0 + \lambda x(t), 0 \leq \lambda\}$ and let

$$D = \bigcup_{t \in T} (c_{x_1} \cap \lambda_t). \tag{125}$$

Clearly D is a closed connected subset of c_{x_1} containing x_0 and the segment $y_r y_l$. Thus D contains at least one of the two connected components of c_{x_1} joining x_0 with y_r, y_l. Therefore the inclusions

$$(\widetilde{i}_{r,x_0}^{y_r} \cup y_r y_l) \subset D \tag{126}$$

or

$$(\widetilde{i}_{l,x_0}^{y_l} \cup y_r y_l) \subset D \tag{127}$$

(or both) hold.

Assume that (126) holds and let η_r be defined as in (122). Since, by construction of D, the set $\text{co}(D \cup \{x_1\})$ is contained in $\text{co}(\gamma)$, then

$$\text{co}(\eta_r) \subset \text{co}(\gamma). \tag{128}$$

Similarly if (127) holds, then

$$\text{co}(\eta_l) \subset \text{co}(\gamma), \tag{129}$$

with η_l defined by (123). Then (119) is proved. It is not difficult to see that the region bounded by $\gamma \cup x_0 x_1$ contains the convex region bounded by $i_{r,x_0}^{y_r^*} \cup y_r^* x_1 \cup x_0 x_1$. Thus the bound

$$|\gamma| \geq \left| i_{r,x_0}^{y_r^*} \right| + |y_r^* x_1| \tag{130}$$

holds; similar procedure can be used for left case. This proves (120).

(3) Let $x_1 \in B_r$; the same argument as in case (2) can be carried on up to the inclusions (126) and (127). As in step (2), when case (126) holds, the curve η_r can be constructed and η_r is SDC_K.

Let us show that if $x_1 \in B_r$ then (127) cannot occur, so the curve η_l can not to be constructed.

Let us argue by contradiction. If (127) occurs, then let $z_1 \neq x_0$ be the first point where γ crosses the half line $x_0 P$. The point z_1 exists, since, under the assumption (127), $P \in D$. Then (Theorem 36) z_1 does not belong to the open segment $x_0 P$. Then, from (127) (see Definition 27)

$$\text{co}(\gamma_{z_1}) \supset \mathfrak{F}_l(K,x_0). \tag{131}$$

Moreover $\gamma \setminus \gamma_{z_1} \cup \{z_1\}$ is $\text{SDC}_{\text{co}(K \cup \gamma_{z_1})}$; see Remark 6. Then by (131) it is $\text{SDC}_{\text{co}(K \cup \mathfrak{F}_l(K,x_0))}$. Let us consider the convex body $H = \text{co}(K \cup i_{l,x_0}^P)$. Since

$$\text{co}(K \cup \gamma_{z_1}) \supset \text{co}(K \cup \mathfrak{F}_l(K,x_0)) \supset H, \tag{132}$$

then $\gamma \setminus \gamma_{z_1}$ is a deleted SDC_H. From Theorem 39, with H in place of K, x_1 in place of z, it follows that

$$\gamma_{z_1,x_1} \subset \text{cl}\left(\mathbb{R}^2 \setminus \mathfrak{F}(H^{z_1}, z_1)\right). \tag{133}$$

Let P_1 be where the right tangent from z_1 to H crosses the arc $\widetilde{i}_{l,P}^{P_l}$. Let us notice that $\widetilde{i}_{l,P}^{P_l}$ is also an arc of the left involute of H at P. Moreover $\widetilde{i}_l^{P_1,P_l}$ is an arc of the left involute of H^{z_1} starting at P_1; then it is parallel to the left involute of H^{z_1} at z_1 until it crosses the sector N; it turns out that

$$B_r \subset \mathfrak{F}(H^{z_1}, z_1). \tag{134}$$

From the two previous inclusions a contradiction comes out since

$$x_1 \in \gamma_{z_1, x_1} \cap B_r = \emptyset. \tag{135}$$

Then (127) cannot occur.

(4) The case $x_1 \in B_l$ is similar to the previous one.

The proof is complete. $\qquad \square$

Definition 43. Under the assumptions of Theorem 42, let one define E_{x_0, x_1}^K the set of $\eta_i, i = 1, 2$ (possibly coinciding) as they are constructed in the proof of Theorem 42, which satisfy (119) and (120). These curves will be called *minimally connecting* steepest descent curves for the class Γ_{x_0, x_1}^K.

Definition 44. Let $\gamma : T \ni t \to x(t)$ be an absolutely continuous curve and let $x(t)$ be a point of γ, with tangent vector $\dot{x}(t)$. Let

$$\mathscr{H}_{x(t)} := \left\{ y \in \mathbb{R}^2 : \langle \dot{x}(t), y - x(t) \rangle \leq 0 \right\}. \tag{136}$$

$\mathscr{H}_{x(t)}$ is half plane bounded by the normal line to γ at $x(t)$ and it is defined almost everywhere in T. For the curve $\gamma \setminus \gamma_{x_1}$ (consisting of the points of γ following x_1) let us define the region:

$$\mathfrak{H}(\gamma, x_1) := \bigcap_{x_1 \leq x(t), x(t) \in \gamma} \mathscr{H}_{x(t)}. \tag{137}$$

If $\mathfrak{H}(\gamma, x_1) \neq \emptyset$, then it is a convex set.

If γ is SDC (γ is SDC_K), then condition (2) (resp., (5)) implies that

$$\gamma_{x_1} \subset \mathfrak{H}(\gamma, x_1) \quad \left(\text{resp.,} \ \gamma_{x_1} \cup K \subset \mathfrak{H}(\gamma, x_1) \right). \tag{138}$$

Theorem 45. *Let $x_0 \in \partial K, x_1 \notin K$. Let γ_1 be SDC with first point x_1. Necessary and sufficient conditions for the existence of a curve γ, self-distancing curve from K, starting at x_0 and satisfying $(\gamma \setminus \gamma_{x_1}) \cup \{x_1\} = \gamma_1$, are as follows:*

(a) $x_1 \in cl(\mathbb{R}^2 \setminus \mathfrak{F}(K, x_0))$;

(b) *there exists $\eta \in E_{x_0, x_1}^K$ such that $(K \cup \eta) \subset \mathfrak{H}(\gamma_1, x_1)$;*

moreover if (a) and (b) are satisfied, then $\gamma_1 \in SDC_{co(K \cup \eta)}$.

Proof. (a) is necessary by Theorem 36. (b) is necessary by Theorem 42 and by (138) since $\mathfrak{H}(\gamma_1, x_1) = \mathfrak{H}(\gamma, x_1)$. Conversely if (a) and (b) hold, let us define $\gamma := \eta * \gamma_1$; then (by definition of E_{x_0, x_1}^K) η is SDC_K; thus, γ is SDC_K too and γ_1 is $SDC_{co(K \cup \eta)}$ (see Remark 6). $\qquad \square$

5. Self-Distancing Sets and Steepest Descent Curves

A self-distancing set σ will be called SDC-extendible if there exists a steepest descent curve γ such that $\sigma \subset \gamma$.

This section is devoted to investigate the following question:

Can a self-distancing set σ be extended to a steepest descent curve γ?

Let us call Γ_σ the family of SDC γ which extends σ. The following example shows that Γ_σ can be empty.

Example 46. Let us consider in a coordinate system xy the points

$$
\begin{aligned}
x_1 &= (0, 0), \\
x_2 &= (0, 2), \\
x_3 &= \left(1, \sqrt{8}\right), \\
x_4 &= \left(-1, \sqrt{8}\right).
\end{aligned}
\tag{139}
$$

The set $\tilde{\sigma} = \{x_i, \ i = 1, \ldots, 4\}$ is a self-distancing set not SDC-extendible.

Proof. By contradiction let $\gamma \in \Gamma_{\tilde{\sigma}}$; then any point x on the arc γ_{x_3, x_4} satisfies the inequalities

$$3 = |x_3 - x_1| \leq |x - x_1| \leq |x_4 - x_1| = 3, \tag{140}$$
$$|x_3 - x_2| \leq |x - x_2|.$$

That is, $x \in \partial B(x_1, 3)$ and $x \in \mathbb{R}^2 \setminus B(x_2, |x_3 - x_2|)$. Since $\partial B(x_1, 3) \cap (\mathbb{R}^2 \setminus B(x_2, |x_3 - x_2|)) = \{x_3, x_4\}$, the arc γ_{x_3, x_4} consists of two points only, which is impossible. $\qquad \square$

Next theorem gives a necessary condition (141) in order to extend a finite self-distancing set σ to SDC; this condition is based on the bounding sets introduced in Section 3.1.

Let us define σ_x as the subset of σ consisting of the point x and of the previous ones on σ (consistent with (3)).

Theorem 47. *Let σ be a self-expanding SDC-extendible set; then for all $x_0 \in \sigma$ such that $\sigma_{x_0} \neq \{x_0\}$, the inclusion*

$$\left(\sigma \setminus \sigma_{x_0}\right) \subset cl\left(\mathbb{R}^2 \setminus \mathfrak{F}\left(co\left(\sigma_{x_0}\right), x_0\right)\right) \tag{141}$$

holds.

Proof. Let $\gamma \in \Gamma_\sigma$; then $\sigma \subset \gamma$ and γ is SDC. Then $co(\gamma_{x_0}) \supset co(\sigma_{x_0})$ and $x_0 \in \partial co(\gamma_{x_0}) \cap \partial co(\sigma_{x_0})$ (see [8, (i) of Lemma 4.6]); from Theorem 40,

$$\mathfrak{F}\left(co\left(\gamma_{x_0}\right), x_0\right) \supset \mathfrak{F}\left(co\left(\sigma_{x_0}\right), x_0\right); \tag{142}$$

moreover $\sigma \setminus \sigma_{x_0} \subset \gamma \setminus \gamma_{x_0}$ and from Corollary 37,

$$\gamma \setminus \gamma_{x_0} \subset cl\left(\mathbb{R}^2 \setminus \mathfrak{F}\left(co\left(\gamma_{x_0}\right), x_0\right)\right). \tag{143}$$

The previous inclusions prove (141). $\qquad \square$

Remark 48. In Example 46 it has been proved, in a simple way, that $\tilde{\sigma}$ is not SDC-extendible. Another way to prove this

fact is to check that condition (141) does not hold for the point x_4; let us notice that $\partial \mathfrak{F}(\operatorname{co}(\widetilde{\sigma}_{x_3}), x_3) \cap \{x \leq 0, y \geq 0\}$ consists of a circular arc centered at x_1 with radius $2 + \sqrt{13 - 4\sqrt{8}}$; then it is easy to see that x_4 is in the interior of $\mathfrak{F}(\operatorname{co}(\widetilde{\sigma}_{x_3}), x_3)$ and (141) is not satisfied.

Let us show in the following example that (141) is not sufficient for a self-distancing set σ to be SDC-extendible.

Example 49. Let us consider in a coordinate system xy the points

$$\xi_1 = (0, 0),$$
$$\xi_2 = (0, 2),$$
$$\xi_3 = (2, 0),$$
$$\xi_4 = (\rho, 2).$$
(144)

For $\sqrt{8} < \rho < \pi$, the set $\sigma := \{\xi_i, \ i = 1, \ldots, 4\}$ is a self-distancing set satisfying condition (141) not SDC-extendible.

Proof. It easy to see that σ is a self-distancing set. Moreover the initial piece of the left involute of $\operatorname{co}(\{\xi_1, \xi_2, \xi_3\})$ starting at ξ_3 consists of a circular arc centered at ξ_2 of ray $\sqrt{8}$ and amplitude $(3/4)\pi$. Then $\xi_4 \notin \mathfrak{F}(\operatorname{co}(\{\xi_1, \xi_2, \xi_3\}), \xi_3)$ and (141) is verified with $x_0 = \xi_3$, $\sigma \setminus \sigma_{x_0} = \{\xi_4\}$. Trivially (141) is verified also at $x_0 = \xi_2$. Let us prove now that Γ_σ is empty. By contradiction let $\gamma \in \Gamma_\sigma$. Let us consider γ_{ξ_3}. Since ξ_2, ξ_3 have the same distance from ξ_1, arguing as in Example 46, γ_{ξ_3} is a circular arc C centered at ξ_1 from ξ_2 to ξ_3. Since the arc γ_{ξ_3} has the distancing property from the segment $\xi_1\xi_2$, it is necessarily the arc of amplitude $\pi/4$ and not the complementary arc. Let $\eta = \xi_1\xi_2 * C$. Since $C \subset \gamma$ and $\xi_2, \xi_3 \in \gamma$, then $\operatorname{co}(\eta) \subset \operatorname{co}(\gamma_{\xi_3})$. Thus by Theorem 40

$$\mathfrak{F}\left(\operatorname{co}\left(\gamma_{\xi_3}\right), \xi_3\right) \supset \mathfrak{F}\left(\operatorname{co}(\eta), \xi_3\right).$$
(145)

Since the segment $\xi_2\xi_4$ is tangent to η at ξ_2 and it has length ρ, less than π, the length of the arc η_{ξ_2,ξ_3}, then

$$\xi_4 \in \operatorname{Int}\left(\mathfrak{F}\left(\operatorname{co}\left(\gamma_{\xi_3}\right), \xi_3\right)\right).$$
(146)

This is in contradiction with Corollary 37 at the point ξ_3. □

Let us introduce definitions and preliminary facts needed to obtain necessary and sufficient conditions for the extendibility of a self-distancing set σ structured as follows.

Definition 50. Let one denote with $\widetilde{\cup}_i \sigma_i$ a self-distancing set with a finite (or countable) family of closed connected components $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$, ordered as the points of σ; that is, if $i < j$, $x \in \sigma_i$, $y \in \sigma_j \Rightarrow x \preceq y$. Let x_i^- be the first point and let x_i^+ be the last point of σ_i; if they are distinct (i.e. σ_i does not reduce to a point) as noticed in the introduction ([6, Theorem 3.3] and [8, Theorem 4.10]), σ_i is SDC and it will be denoted by γ_i.

Lemma 51. *Let $\sigma = \widetilde{\cup}_i \sigma_i$ be a self-distancing set. A necessary condition for $\Gamma_\sigma \neq \emptyset$ is that for all components σ_i, which are curves γ_i, the fact*

$$\bigcup_{j=1}^{i} \sigma_j \subset \mathfrak{H}\left(\gamma_{i+1}, x_{i+1}^-\right)$$
(147)

holds.

Proof. Let $\gamma \in \Gamma_\sigma$. Then $\cup_{j=1}^{i} \sigma_j \subset \gamma_{x_{i+1}^-}$. Then (147) follows from (138). □

Definition 52. Let $\sigma = \widetilde{\cup}_i \sigma_i$ be a self-distancing set. A subfamily $E_\sigma \subset \Gamma_\sigma$ is called *essential* for Γ_σ if the facts

(a) $\forall \gamma \in \Gamma_\sigma \exists \eta \in E_\sigma : \operatorname{co}(\eta) \subset \operatorname{co}(\gamma),$ (148)

(b) $\gamma \in \Gamma_\sigma \Longrightarrow$
$$|\gamma| \geq \min\{|\eta|, \ \eta \in E_\sigma\},$$
(149)

(c) $\gamma \in \Gamma_\sigma, \ |\gamma| = \min\{|\eta|, \ \eta \in E_\sigma\} \Longrightarrow$
$$\gamma \in E_\sigma$$
(150)

hold.

If $\Gamma_\sigma = \emptyset$, let us define $E_\sigma = \emptyset$ essential for Γ_σ.

Let us start to study a self-distancing set with two closed connected components.

Lemma 53. *Let $\sigma^* = \widetilde{\cup}_{i=1,2} \sigma_i$ be a self-distancing set and let ρ be the segment joining x_1^+, x_2^-. There are five possibilities:*

(p1) *Let $\sigma_1 = \{x_1\}, \sigma_2 = \{x_2\}$; then $E_{\sigma^*} = \{\rho\} \neq \emptyset$ is essential for Γ_σ.*

(p2) *Let $\sigma_1 = \{x_1\}, \sigma_2 = \gamma_2$ (γ_2 is SDC); then a necessary and sufficient condition for the extensibility of σ^* is*

$$\sigma_1 \subset \mathfrak{H}\left(\gamma_2, x_2^-\right);$$
(151)

moreover $E_{\sigma^*} = \{\rho * \gamma_2\}$ is essential for Γ_{σ^*}.

(p3) *Let $\sigma_1 = \gamma_1$ be a SDC, $\sigma_2 = \{x_2\}$; then a necessary and sufficient condition for the extensibility of σ^* is*

$$\sigma_2 \subset \operatorname{cl}\left(\mathbb{R}^2 \setminus \left(\mathfrak{F}\left(\operatorname{co}(\gamma_1), x_1^+\right)\right) \cup \operatorname{co}(\gamma_1)\right);$$
(152)

moreover $E_{\sigma^*} = \{\gamma_1 * \eta : \eta \in E_{x_1^+, x_2}^{\operatorname{co}(\gamma_1)}\}$ (see Definition 43).

(p4) *Let $\sigma_1 = \gamma_1, \sigma_2 = \gamma_2$; then a necessary and sufficient condition for the extensibility of σ^* is that there exists SDC η such that*

$$\eta \in E_{x_1^+, x_2^-}^{\operatorname{co}(\gamma_1)},$$
$$\sigma_1 \cup \eta \subset \mathfrak{H}\left(\gamma_2, x_2^-\right);$$
(153)

the related essential family is $E_{\sigma^*} = \{\gamma_1 * \eta * \gamma_2 : \eta \text{ satisfies} (153)\}$.

(p5) *If in cases (p2), (p3), and (p4) the corresponding necessary and sufficient conditions do not hold, then*

$$E_{\sigma^*} = \Gamma_{\sigma^*} = \emptyset. \tag{154}$$

Proof. Case (p1) is trivial; in case (p2) the inclusion (151) follows from Lemma 51 with $i = 2$ in (147). It is also trivial that it is sufficient. Case (p3) follows from Theorem 42 with $K = \mathrm{co}(\gamma_1)$. Case (p4) follows from Theorem 45 with γ_2 in place of γ_1, $K = \mathrm{co}(\gamma_1)$, x_1^+ in place of x_0 and x_2^- in place of x_1. □

An easy sufficient condition to check if $\sigma = \widetilde{\cup}_i \sigma_i$ is extendible is the following.

Theorem 54. *Let $\sigma = \widetilde{\cup}_i \sigma_i$ be a self-distancing set. Let $\sigma^{(i)} = \widetilde{\cup}_{j=1}^{i} \sigma_j$. If (147) and*

$$x_{i+1}^- \in N_{\mathrm{co}(\sigma^{(i)})}\left(x_i^+\right), \quad \forall i \geq 1 \tag{155}$$

hold, then $\Gamma_\sigma \neq \emptyset$ and $\overline{\gamma}$, which linearly and orderly connects σ_i, σ_{i+1} with the segments $x_i^+ x_{i+1}^-$, is SDC and it has minimal length in Γ_σ.

Proof. Let us argue by induction on the self-distancing set $\sigma^{(i)}$. The case $i = 1$ is contained in Lemma 53, since the assumptions (147) and (155) are enough to get the corresponding assumptions in cases (p1), (p2), (p3), and (p4). Moreover in the case (p4) the curve $\eta = x_1^+ x_2^-$ is such that $\gamma^{(2)} = \gamma_1 * \eta * \gamma_2$ is the SDC of minimal length extending $\sigma^{(2)}$.

Let $\gamma^{(i)}$ be the curve of minimal length extending $\sigma^{(i)}$. Since the normal sector to $\mathrm{co}(\sigma^{(i)})$ at x_i^+ coincides with the sector N in the proof of Theorem 42, with x_i^+ in place of x_0, x_{i+1}^- in place of x_1, then assumption (155) implies that case (1) of the proof of Theorem 42 occurs. It follows that

$$\gamma^{(i+1)} = \gamma^{(i)} * x_i^+ x_{i+1}^- * \sigma_{i+1} \tag{156}$$

is of minimal length in $\Gamma_{\sigma^{(i+1)}}$. Then, $\overline{\gamma} = \cup_i \gamma^{(i+1)}$ is of minimal length in Γ_σ. □

Remark 55. Since, as noticed in [4, II, Section 2], $\forall u, w \in \sigma_{x_i^+}$ the angle $u x_i^+ w$ has opening less than $\pi/2$, thus the related normal sector in (155) has opening greater than or equal to $\pi/2$; then checking that x_{i+1}^- satisfies (155) is easier than checking that x_{i+1}^- is outside of the $\mathfrak{F}$-fence as in (152).

Lemma 51 and Theorem 54 give only necessary and only sufficient conditions, respectively, for the extensibility of self-expanding sets. Let us give definitions in order to get necessary and sufficient conditions.

Definition 56. Let $\sigma = \widetilde{\cup}_j \sigma_j$ be a self-distancing set. Let E_i, $i = 2, \ldots, n, \ldots$ be defined by induction as follows.

E_2 is the essential family related to $\widetilde{\cup}_{j=1}^{2} \sigma_j$, as given by Lemma 53; if $i \geq 2$, the E_{i+1} related to $\widetilde{\cup}_{j=1}^{i+1} \sigma_j$ is defined as follows:

(i) if $E_i = \emptyset$ then $E_{i+1} = \emptyset$;

(ii) if $E_i \neq \emptyset$, let one consider for all $\eta \in E_i$ the essential family $E(\eta)$ (see Lemma 53) related to $\eta \widetilde{\cup} \sigma_{i+i}$ (see Definition 50). Let $E_{i+1} = \cup_{\eta \in E_i} E(\eta)$.

Let us notice that $\{E_i\}$ is ordered by inclusion and E_{i+1}, if it is nonempty, consists of 2^i curves at most.

Theorem 57. *Let $\sigma = \widetilde{\cup} \sigma_j$ be a self-expanding set and let $E_2, E_3, \ldots, E_i, \ldots$ be the sequence (finite or countable) of the essential families associated with σ. Then $\Gamma_\sigma \neq \emptyset$ iff $\forall i \geq 2$ the essential family E_i is nonempty.*

Proof. If there exists $\gamma \in \Gamma_\sigma$ then, for all $i \geq 1$, $\gamma_{x_{i+1}^+} \in \Gamma_{\widetilde{\cup}_{j=1}^{i+1} \sigma_j}$; thus $E_{i+1} \neq \emptyset$ by Theorem 42. Conversely if at each step $i \geq 1$ the essential family $E_{i+1} \neq \emptyset$, then, by definition, there exists a sequence $\{\eta^{i+1}\}$ of SDC such that $\eta^{i+1} \in E_{i+1}$ and such that $\eta^s \subset \eta^{s+1}, s \geq 1$ (i.e., at each step, η^{s+1} is an extension of a previous one η^s) and $\eta^{s+1} \in E_{\widetilde{\cup}_{j=1}^{i+1} \sigma_j}$; see Definition 56. Then $\gamma = \cup_{i=1}^{\infty} \eta^{i+1}$ is well defined; obviously $\gamma \in \Gamma_\sigma$. □

Open Problem. In the present work only two-dimensional problems are studied. In three (or more) dimensions the construction of boundary regions to SDC and to SDC_K is open. The boundary regions should probably be constructed by using the space involutes of the geodesics curves on ∂K.

Competing Interests

The authors declare that they have no competing interests.

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