Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2016, Article ID 4146323, 6 pages
http://dx.doi.org/10.1155/2016/4146323

Research Article

The Variational Homotopy Perturbation Method for Solving $((n \times n)+1)$ Dimensional Burgers' Equations

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Received 2 February 2016; Accepted 30 March 2016

Academic Editor: Jisheng Kou

The variational homotopy perturbation method VHPM is used for solving n-dimensional Burgers' system. Some examples are examined to validate that the method reduced the calculation size, treating the difficulty of nonlinear term and the accuracy.

1. Introduction

The variational iteration method VIM and the homotopy perturbation method HPM were proposed by He in [1–6]. Many researchers used these methods in a variety of scientific fields of partial differential equations PDEs including Burgers' equation which arises in many of physically important phenomena [7–9]. It was shown that the methods are stronger than other techniques such as the Adomian decomposition method [10–18]. In our work n-dimensional Burgers' equation is solved by the variational homotopy perturbation method VHPM which is combination of VIM and HPM. The VHPM was proposed in [19–21]. Vector Burgers' system is given by [22]

$$U_t + (U \cdot \nabla) U = \mu \Delta U, \tag{1}$$

where $u_1, u_2, \dots, u_n$ are the velocity components and μ is the kinematic viscosity. t is time and Δ and ∇ are

$$\begin{aligned} \Delta &= \frac{\partial^2}{x_1^2} + \frac{\partial^2}{x_2^2} + \cdots + \frac{\partial^2}{x_n^2}, \\ \nabla &= \frac{\partial}{x_1} + \frac{\partial}{x_2} + \cdots + \frac{\partial}{x_n}. \end{aligned} \tag{2}$$

Equation (1) can be written as

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \mu \Delta u_i, \quad i = 1, 2, \dots, n. \tag{3}$$

2. Variational Iteration Method

According to the variational iteration method [2, 3, 10–14] we can write the correction functional for (3) as

$$u_{n+1} = u_n + \int_0^t \lambda_i(\xi) \left[\frac{\partial u_i}{\partial \xi} + \sum_{j=1}^{n} \widetilde{u_j} \frac{\partial \widetilde{u_i}}{\partial x_j} - \mu \Delta \widetilde{u_i} \right] d\xi, \tag{4}$$

where $i = 1, 2, \dots, n$, $u = u(x_i, \xi)$, λ is a general Lagrangian multiplier which can be found via variational theory, and $\widetilde{u_i}$ are restricted variation which means $\delta \widetilde{u_i} = 0$. The solution is given by

$$u_i(x_j, t) = \lim_{n \to \infty} u_{(i,n)}(x_j, t), \quad j = 1, 2, \dots, n. \tag{5}$$

3. Homotopy Perturbation Method

Applying HPM according to [4–6, 15–17] for (3), we construct the following homotopy:

$$\begin{aligned} &(1-p)\left[\frac{\partial u_{(i,k)}}{\partial t} - \frac{\partial u_{(i,0)}}{\partial t}\right] \\ &\quad + p\left[\sum_{j=1}^{n} u_{(j,k)} \frac{\partial u_{(i,k)}}{\partial x_j} - \mu \Delta u_{(i,k)}\right] = 0 \end{aligned} \tag{6}$$

or

$$\frac{\partial u_{(i,k)}}{\partial t} - \frac{\partial u_{(i,0)}}{\partial t} + p\left[\frac{\partial u_{(i,0)}}{\partial t} + \sum_{j=1}^{n} u_{(j,k)} \frac{\partial u_{(i,k)}}{\partial x_j} - \mu \Delta u_{(i,k)}\right] = 0, \tag{7}$$

where $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p \in [0, 1]$ is an embedding parameter, while $u_{(1,0)} = f_1(x_j, 0)$, $u_{(2,0)} = f_2(x_j, 0)$, $\ldots, u_{(n,0)} = f_n(x_j, 0)$ are initial approximations of (3). Assume the solution of (3) has the form

$$u_i = \sum_{\ell=0}^{\infty} p^{\ell} u_{(i,\ell)}\left(x_j, t\right), \quad i, j = 1, 2, \ldots, n. \tag{8}$$

Now, substituting u_i from (8) in (7) and comparing coefficients of terms with identical powers of p we get

$$\begin{aligned} p^0 &: \frac{\partial u_{(i,0)}}{\partial t} - \frac{\partial u_{(i,0)}}{\partial t} = 0, \\ p^1 &: \frac{\partial u_{(i,0)}}{\partial t} + u_{(j,0)} \frac{\partial u_{(i,0)}}{\partial x_j} - \mu \Delta u_{(i,0)}, \\ p^2 &: u_{(j,1)} \frac{\partial u_{(i,1)}}{\partial x_j} - \mu \Delta u_{(i,1)}, \\ p^3 &: u_{(j,2)} \frac{\partial u_{(i,2)}}{\partial x_j} - \mu \Delta u_{(i,2)}, \\ &\vdots \end{aligned} \tag{9}$$

The solution of (7) is

$$u_i\left(x_j, t\right) = u_{(i,0)} + u_{(i,1)} + u_{(i,2)} + \cdots. \tag{10}$$

4. Variational Homotopy Perturbation Method

Consider (3) according to [19–21]. In HPM, assume that the solution of (3) has the form

$$\begin{aligned} u_i &= \sum_{\ell=0}^{\infty} p^{\ell} u_{(i,\ell)}\left(x_j, t\right) = v_i, \quad i, j = 1, 2, \ldots, n, \\ u_j &= \sum_{\ell=0}^{\infty} p^{\ell} u_{(j,\ell)}\left(x_j, t\right) = v_j, \quad i, j = 1, 2, \ldots, n. \end{aligned} \tag{11}$$

From (11), (3) can be written as

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} = \mu \Delta v_i, \quad i = 1, 2, \ldots, n. \tag{12}$$

In VIM, from the correction functional for (12) we can write

$$v_{i+1} = v_0 + p \int_0^t \lambda_i(\xi) \left[-\sum_{j=1}^{n} \widetilde{v}_j \frac{\partial \widetilde{v}_i}{\partial x_j} + \mu \Delta \widetilde{v}_i\right] d\xi, \tag{13}$$

where $i = 1, 2, \ldots, n$, $v = v(x_i, \xi)$; from (11) in (13) and by comparing the coefficients of like powers of p, we get

$$\begin{aligned} p^0 &: u_{(i,0)}\left(x_j, 0\right) = f_i\left(x_j, 0\right), \\ p^1 &: u_{(i,1)}\left(x_j, t\right) = \int_0^t \lambda_i(\xi) \cdot \left[-u_{(j,0)}\left(x_j, \xi\right) \frac{\partial u_{(i,0)}\left(x_j, \xi\right)}{\partial x_j} + \mu \Delta u_{(i,0)}\left(x_j, \xi\right)\right] d\xi, \\ p^2 &: u_{(i,2)}\left(x_j, t\right) = \int_0^t \lambda_i(\xi) \cdot \left[-u_{(j,1)}\left(x_j, \xi\right) \frac{\partial u_{(i,1)}\left(x_j, \xi\right)}{\partial x_j} + \mu \Delta u_{(i,1)}\left(x_j, \xi\right)\right] d\xi, \\ p^3 &: u_{(i,3)}\left(x_j, t\right) = \int_0^t \lambda_i(\xi) \cdot \left[-u_{(j,2)}\left(x_j, \xi\right) \frac{\partial u_{(i,2)}\left(x_j, \xi\right)}{\partial x_j} + \mu \Delta u_{(i,2)}\left(x_j, \xi\right)\right] d\xi, \\ &\vdots \end{aligned} \tag{14}$$

The approximations solution is given by

$$u_i\left(x_j, t\right) = u_{(i,0)} + u_{(i,1)} + u_{(i,2)} + \cdots. \tag{15}$$

To demonstrate the efficiency of the methods we have solved some examples by VHPM as (1 + 1)-dimensional, (1 + 2)-dimensional, (1 + 3)-dimensional, and 2-dimensional. Then we can generalize it for $(n+1)$-dimensional or n-dimensional.

5. Application

Example 1. Consider (1 + 1)-dimensional Burgers' equation [17]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{16}$$

with the initial condition

$$u(x, 0) = u_0(x) = 2x. \tag{17}$$

The correction functional for (16) is

$$U_{n+1} = U_n + p\int_0^t \lambda(\xi)\left[\frac{\partial U}{\partial t} + \widetilde{U}_n(x,\xi)\frac{\partial \widetilde{U}_n(x,\xi)}{\partial x} - \frac{\partial^2 \widetilde{U}_n(x,\xi)}{\partial x^2}\right]d\xi. \tag{18}$$

The general Lagrangian multiplier λ can be found as follows:

$$\lambda' = 0, \qquad 1 + \lambda(\xi)\big|_{\xi=t} = 0. \tag{19}$$

Then, $\lambda = -1$.

Equation (11) can be written as

$$U = \sum_{\ell=0}^{\infty} p^{\ell} u_{\ell}(x,t). \tag{20}$$

Applying VHPM, we have

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \cdots = 2x - p\int_0^t \Big[\big(u_0 + pu_1 + p^2u_2 + \cdots\big)\frac{\partial}{\partial x}\big(u_0 + pu_1 + p^2u_2 + \cdots\big)\Big]d\xi \\ - p\int_0^t \left[-\frac{\partial^2}{\partial x^2}\big(u_0 + pu_1 + p^2u_2 + \cdots\big)\right]d\xi. \end{aligned} \tag{21}$$

Comparing the coefficient of like powers of p, we get

$$\begin{aligned} p^0 &: u_0 = 2x, \\ p^1 &: u_1 = -\int_0^t \left[u_0\frac{\partial u_0}{\partial x} - \frac{\partial^2}{\partial x^2}u_0\right]d\xi = -4xt, \\ p^2 &: u_2 = -\int_0^t \left[u_1\frac{\partial u_1}{\partial x} - \frac{\partial^2}{\partial x^2}u_1\right]d\xi = 8xt^2, \\ p^3 &: u_3 = -\int_0^t \left[u_2\frac{\partial u_2}{\partial x} - \frac{\partial^2}{\partial x^2}u_2\right]d\xi = -16xt^2, \\ &\vdots \end{aligned} \tag{22}$$

The approximations solution is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots. \tag{23}$$

Exact solution is ($u^* = 2x/(1+2t)$).

The results are in Table 1.

Example 2. Consider (1 + 2)-dimensional Burgers' equation [17]

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{24}$$

with the initial condition

$$u(x,y,0) = u_0(x,y) = x + y. \tag{25}$$

Table 1: Comparison of VHPM solutions with exact solution at $t = 0.01$ (Example 1).

x	$u^*(x,y,t)$	$u(x,y,t)$	$\lvert u^* - u\rvert$
0.1	0.1960784314	0.1960784314	0
0.2	0.3921568628	0.3921568627	1×10^{-10}
0.3	0.5882352942	0.5882352941	1×10^{-10}
0.4	0.7843137254	0.7843137254	0
0.5	0.9803921568	0.9803921568	0
0.6	1.176470588	1.176470588	0
0.7	1.372549020	1.372549020	0
0.8	1.568627451	1.568627451	0
0.9	1.764705882	1.764705882	0

As above, we have

$$\begin{aligned} U_{n+1} &= u_0 - p\int_0^t \left[-U_n(x,y,\xi)\frac{\partial U_n(x,y,\xi)}{\partial x} - \frac{\partial^2 U_n(x,y,\xi)}{\partial x^2} - \frac{\partial^2 U_n(x,y,\xi)}{\partial y^2}\right]d\xi, \\ u_0 + pu_1 + p^2u_2 + \cdots &= x + y - p\int_0^t \Big[-\big(u_0 + pu_1 + p^2u_2 + \cdots\big)\frac{\partial}{\partial x}\big(u_0 + pu_1 + p^2u_2 + \cdots\big)\Big]d\xi \\ &\quad - p\int_0^t \Big[-\frac{\partial^2}{\partial x^2}\big(u_0 + pu_1 + p^2u_2 + \cdots\big) - \frac{\partial^2}{\partial y^2}\big(u_0 + pu_1 + p^2u_2 + \cdots\big)\Big]d\xi. \end{aligned} \tag{26}$$

Comparing the coefficient of like powers of p, we get

$$\begin{aligned} p^0 &: u_0 = x + y, \\ p^1 &: u_1 = (x+y)t, \\ p^2 &: u_2 = (x+y)t^2, \\ p^3 &: u_3 = (x+y)t^3, \\ &\vdots \end{aligned} \tag{27}$$

The approximations solution is given by

$$u(x,y,t) = u_0 + u_1 + u_2 + u_3 + \cdots. \tag{28}$$

Exact solution is ($u^* = (x+y)/(1-t)$).

The results are in Table 2.

Example 3. Consider (1 + 3)-dimensional Burgers' equation [17]

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{29}$$

Table 2: Comparison of VHPM solutions with exact solution at $x = 0.1$ and $y = 0.1$ (Example 2).

t	$u^*(x,y,t)$	$u(x,y,t)$	$\lvert u^* - u\rvert$
0.01	0.2020202020	0.2020202020	0
0.02	0.2040816327	0.2040816326	1×10^{-10}
0.03	0.2061855670	0.2061855669	1×10^{-10}
0.04	0.2083333333	0.2083333325	8×10^{-10}
0.05	0.2105263158	0.2105263125	3.3×10^{-9}
0.06	0.2127659574	0.2127659475	9.9×10^{-9}
0.07	0.2150537634	0.2150537381	2.53×10^{-8}
0.08	0.2173913043	0.2173912474	5.69×10^{-8}
0.09	0.2197802198	0.2197801030	1.168×10^{-7}
0.10	0.2222222222	0.2222220000	2.222×10^{-7}

Table 3: Comparison of VHPM solutions with exact solution at $x = 0.1$, $y = 0.1$, and $z = 0.1$ (Example 3).

t	$u^*(x,y,t)$	$u(x,y,t)$	$\lvert u^* - u\rvert$
0.01	0.3030303030	0.3030303030	0
0.02	0.3061224490	0.3061224490	0
0.03	0.3092783505	0.3092783503	2×10^{-10}
0.04	0.3125000000	0.3124999987	1.3×10^{-9}
0.05	0.3157894737	0.3157894688	4.9×10^{-9}
0.06	0.3191489362	0.3191489213	1.49×10^{-8}
0.07	0.3225806452	0.3225806072	3.80×10^{-8}
0.08	0.3260869565	0.3260868710	8.55×10^{-8}
0.09	0.3296703297	0.3296701545	1.752×10^{-7}
0.10	0.3333333333	0.3333330000	3.333×10^{-7}

with the initial condition

$$u(x,y,z,0) = u_0(x,y,z) = x + y + z. \tag{30}$$

We have

$$\begin{aligned}
U_{n+1} &= u_0 - p\int_0^t \left[-U_n(x,y,\xi)\frac{\partial U_n(x,y,\xi)}{\partial x} \right. \\
&\quad - \frac{\partial^2 U_n(x,y,\xi)}{\partial x^2} - \frac{\partial^2 U_n(x,y,\xi)}{\partial y^2} \\
&\quad \left. - \frac{\partial^2 U_n(x,y,\xi)}{\partial z^2} \right] d\xi, \\
u_0 + pu_1 + p^2u_2 + \cdots &= x + y + z - p\int_0^t \Big[-\big(u_0 \\
&\quad + pu_1 + p^2u_2 \\
&\quad + \cdots\big)\frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\Big] d\xi \\
&\quad - p\int_0^t \left[-\frac{\partial^2}{\partial x^2}\left(u_0 + pu_1 + p^2u_2 + \cdots\right) - \frac{\partial^2}{\partial y^2}\big(u_0 \right. \\
&\quad + pu_1 + p^2u_2 + \cdots\big) - \frac{\partial^2}{\partial z^2}\big(u_0 + pu_1 + p^2u_2 \\
&\quad \left. + \cdots\big) \right] d\xi.
\end{aligned} \tag{31}$$

Comparing the coefficient of like powers of p, we get

$$\begin{aligned}
p^0 &: u_0 = x + y + z, \\
p^1 &: u_1 = (x + y + z)t, \\
p^2 &: u_2 = (x + y + z)t^2, \\
p^3 &: u_3 = (x + y + z)t^3, \\
&\vdots
\end{aligned} \tag{32}$$

The approximations solution is given by

$$u(x,y,z,t) = u_0 + u_1 + u_2 + u_3 + \cdots. \tag{33}$$

Exact solution is $(u^* = (x + y + z)/(1 - t))$.

The results are in Table 3.

Example 4. Consider two-dimensional Burgers' equations [23]

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)
\end{aligned} \tag{34}$$

with the initial conditions:

$$\begin{aligned}
u_0 = u(x,y,0) &= \frac{3}{4} - \frac{1}{4\left[1 + e^{(y-x)R/8}\right]}, \\
v_0 = v(x,y,0) &= \frac{3}{4} + \frac{1}{4\left[1 + e^{(y-x)R/8}\right]}.
\end{aligned} \tag{35}$$

The correction functional for (34) is

$$\begin{aligned}
U_{n+1} &= U_n + p\int_0^t \lambda_1(\xi)\left[\frac{\partial U}{\partial t} \right. \\
&\quad + \widetilde{U}_n(x,y,\xi)\frac{\partial \widetilde{U}_n(x,y,\xi)}{\partial x} \\
&\quad \left. + \widetilde{V}_n(x,y,\xi)\frac{\partial \widetilde{U}_n(x,y,\xi)}{\partial x}\right] d\xi + p\int_0^t \lambda_1(\xi) \\
&\quad \cdot \left[-\frac{1}{R}\left(\frac{\partial^2 \widetilde{U}_n(x,y,\xi)}{\partial x^2} + \frac{\partial^2 \widetilde{U}_n(x,y,\xi)}{\partial y^2}\right)\right] d\xi,
\end{aligned}$$

Table 4: Comparison of VHPM solutions with exact solution at $x = 1$, $y = 0$, and $R = 100$ (Example 4).

t	$u^*(x,y,t)$	$u(x,y,t)$	$\lvert u^* - u\rvert$	$v^*(x,y,t)$	$v(x,y,t)$	$\lvert v^* - v\rvert$
0.01	0.5000009030	0.5000009571	5.41×10^{-8}	0.9999990970	0.9999990321	6.49×10^{-8}
0.02	0.5000008752	0.5000009862	1.110×10^{-7}	0.9999991248	0.9999990031	1.217×10^{-7}
0.03	0.5000008483	0.5000010153	1.670×10^{-7}	0.9999991517	0.9999989741	1.776×10^{-7}
0.04	0.5000008222	0.5000010445	2.223×10^{-7}	0.9999991778	0.9999989451	2.327×10^{-7}
0.05	0.5000007969	0.5000010736	2.767×10^{-7}	0.9999992031	0.9999989161	2.870×10^{-7}
0.06	0.5000007724	0.5000011027	3.303×10^{-7}	0.9999992276	0.9999988871	3.405×10^{-7}
0.07	0.5000007486	0.5000011318	3.832×10^{-7}	0.9999992514	0.9999988581	3.933×10^{-7}
0.08	0.5000007256	0.5000011609	4.353×10^{-7}	0.9999992744	0.9999988281	4.463×10^{-7}
0.09	0.5000007032	0.5000011900	4.868×10^{-7}	0.9999992968	0.9999987991	4.977×10^{-7}
0.10	0.5000006816	0.5000012191	5.375×10^{-7}	0.9999993184	0.9999987701	5.483×10^{-7}

$$
\begin{aligned}
V_{n+1} = {} & V_n + p\int_0^t \lambda_2(\xi)\left[\frac{\partial V}{\partial t}\right. \\
& + \widetilde{V}_n(x,y,\xi)\frac{\partial \widetilde{V}_n(x,y,\xi)}{\partial x} \\
& \left. + \widetilde{U}_n(x,y,\xi)\frac{\partial \widetilde{V}_n(x,y,\xi)}{\partial x}\right] d\xi + p\int_0^t \lambda_2(\xi) \\
& \cdot\left[-\frac{1}{R}\left(\frac{\partial^2 \widetilde{V}_n(x,y,\xi)}{\partial x^2} + \frac{\partial^2 \widetilde{V}_n(x,y,\xi)}{\partial y^2}\right)\right] d\xi.
\end{aligned} \tag{36}
$$

The general Lagrangian multipliers are $\lambda_1 = -1 = \lambda_2$.

Equation (11) can be written as

$$
\begin{aligned}
U &= \sum_{\ell=0}^{\infty} p^{\ell} u_{\ell}(x,y,t), \\
V &= \sum_{\ell=0}^{\infty} p^{\ell} v_{\ell}(x,y,t).
\end{aligned} \tag{37}
$$

By VHPM, we have

$$
\begin{aligned}
u_0 + pu_1 + p^2u_2 + \cdots = {} & u_0 - p\int_0^t \left[-\left(u_0 + pu_1\right.\right. \\
& \left.\left. + p^2u_2 + \cdots\right)\frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right] d\xi \\
& - p\int_0^t \left[\left(v_0 + pv_1 + p^2v_2\right.\right. \\
& \left.\left. + \cdots\right)\frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right] d\xi \\
& - p\int_0^t \left[\frac{1}{R}\left(\frac{\partial^2}{\partial x^2}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right.\right. \\
& \left.\left. + \frac{\partial^2}{\partial y^2}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right)\right] d\xi,
\end{aligned}
$$

$$
\begin{aligned}
v_0 + pv_1 + p^2v_2 + \cdots = {} & v_0 - p\int_0^t \left[-\left(u_0 + pu_1\right.\right. \\
& \left.\left. + p^2u_2 + \cdots\right)\frac{\partial}{\partial x}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\right] d\xi \\
& - p\int_0^t \left[\left(v_0 + pv_1 + p^2v_2\right.\right. \\
& \left.\left. + \cdots\right)\frac{\partial}{\partial x}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\right] d\xi \\
& - p\int_0^t \left[\frac{1}{R}\left(\frac{\partial^2}{\partial x^2}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\right.\right. \\
& \left.\left. + \frac{\partial^2}{\partial y^2}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\right)\right] d\xi.
\end{aligned} \tag{38}
$$

Comparing the coefficient of like powers of p, we get

$$
\begin{aligned}
& p^0 : \begin{cases} u_0 = \dfrac{3}{4} - \dfrac{1}{4\left[1 + e^{(y-x)R/8}\right]} \\ v_0 = \dfrac{3}{4} + \dfrac{1}{4\left[1 + e^{(y-x)R/8}\right]}, \end{cases} \\
& p^1 : \begin{cases} u_1 = \dfrac{1}{128}\dfrac{Re^{-(1/8)(-y+x)R}t}{\left[1 + e^{-(1/8)(-y+x)R}\right]^2} \\ v_1 = \dfrac{-1}{128}\dfrac{Re^{-(1/8)(-y+x)R}t}{\left[1 + e^{-(1/8)(-y+x)R}\right]^2}, \end{cases} \\
& p^2 : \begin{cases} u_2 = \dfrac{-1}{2048}\dfrac{R^2e^{-(1/4)(-y+x)R}t^2}{\left[1 + e^{-(1/8)(-y+x)R}\right]^4} \\ v_2 = \dfrac{1}{2048}\dfrac{R^2e^{-(1/4)(-y+x)R}t^2}{\left[1 + e^{-(1/8)(-y+x)R}\right]^4}, \end{cases} \\
& p^3 : \begin{cases} u_3 = \dfrac{1}{32768}\dfrac{R^3e^{-(3/8)(-y+x)R}t^3}{\left[1 + e^{-(1/8)(-y+x)R}\right]^6} \\ v_3 = \dfrac{-1}{32768}\dfrac{R^3e^{-(3/8)(-y+x)R}t^3}{\left[1 + e^{-(1/8)(-y+x)R}\right]^6}, \end{cases} \\
& \qquad \vdots
\end{aligned} \tag{39}
$$

The approximations solution is given by

$$u(x,y,t) = u_0 + u_1 + u_2 + u_3 + \cdots,$$
$$v(x,y,t) = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{40}$$

Exact solution $u^* = 3/4 - 1/4(1 + e^{(4y-4x-t)R/32})$; $v^* = 3/4 + 1/4(1 + e^{(4y-4x-t)R/32})$.

The results are in Table 4.

6. Conclusion

In this work, the approximate solutions of n-dimensional Burgers' equations are obtained by combination of two powerful methods VIM and HPM in VHPM. The examples have shown the efficiency and accuracy of the VHPM; it reduces the size of computation without the restrictive assumption to handle nonlinear terms and it gives the solutions rapidly.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

This paper was funded by King Abdulaziz City for Science and Technology (KACST) in Saudi Arabia. The authors therefore thank them for their full collaboration.

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