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Research Article

Limit Cycles for the Class of *D*-Dimensional Polynomial Differential Systems

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We perturb the differential system $\dot{x}_1 = -x_2(1 + x_1)$, $\dot{x}_2 = x_1(1 + x_1)$, and $\dot{x}_k = 0$ for $k = 3, \dots, d$ inside the class of all polynomial differential systems of degree n in $\mathbb{R}^d$, and we prove that at most n^{d-1} limit cycles can be obtained for the perturbed system using the first-order averaging theory.

1. Introduction

One of the main problems in the theory of differential systems is the study of their periodic orbits, their existence, their number, and their stability. As usual, a limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the differential system.

In [1], the authors studied the differential system

$$\begin{aligned}\dot{x} &= -y + \varepsilon\left(ax + P(x, y)\right),\\ \dot{y} &= x + \varepsilon\left(ay + Q(x, y)\right),\end{aligned} \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are arbitrary polynomials of degree n starting with terms of degree 2, a is a real parameter, and ε is small parameter. They proved that for $\varepsilon \neq 0$ sufficiently small, the maximum number of limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$, obtained using the averaging theory of first order, is $(n - 1)/2$ if n is odd and $(n - 2)/2$ if n is even. In the same paper, the authors studied the limit cycles of the differential system

$$\begin{aligned}\dot{x} &= -y + \varepsilon\left(ax + P(x, y, z)\right),\\ \dot{y} &= x + \varepsilon\left(ay + Q(x, y, z)\right),\\ \dot{z} &= \varepsilon\left(bz + R(x, y, z)\right),\end{aligned} \tag{2}$$

where $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are arbitrary polynomials of degree n starting with terms of degree 2 and $a, b \in \mathbb{R}$. Then, there exists $\varepsilon_0 > 0$ sufficiently small such that for $|\varepsilon| < \varepsilon_0$ there are systems (2) having at least $n(n - 1)/2$ limit cycles bifurcating from the periodic orbits of the system $\dot{x} = -y$, $\dot{y} = x$, and $\dot{z} = 0$.

In [2], the authors studied the number of limit cycles of the differential system

$$\begin{aligned}\dot{x} &= -y(1 + x) + \varepsilon\left(ax + P(x, y, z)\right),\\ \dot{y} &= x(1 + x) + \varepsilon\left(ay + Q(x, y, z)\right),\\ \dot{z} &= \varepsilon\left(bz + R(x, y, z)\right),\end{aligned} \tag{3}$$

where $F(x, y, z)$, $G(x, y, z)$, and $R(x, y, z)$ are polynomials of degree n starting from terms of degree 2. Then, there exists $\varepsilon_0 > 0$ sufficiently small such that for $|\varepsilon| < \varepsilon_0$ there are systems (3) having at least n^2 limit cycles bifurcating from the periodic orbits of the system $\dot{x} = -y(1 + x)$, $\dot{y} = x(1 + x)$, and $\dot{z} = 0$.

In general, to obtain analytically periodic solutions of a differential system is a very difficult work, usually impossible.

Here, using the averaging theory of first order, we will study the number of limit cycles of the differential system

$$\begin{aligned}\dot{x}_1 &= -x_2(1+x_1) + \varepsilon(ax_1 + P_1(x_1,\dots,x_d)),\\ \dot{x}_2 &= x_1(1+x_1) + \varepsilon(ax_2 + P_2(x_1,\dots,x_d)),\\ \dot{x}_k &= \varepsilon(b_k x_k + P_k(x_1,\dots,x_d)), \quad k=3,\dots,d,\end{aligned} \tag{4}$$

in $\mathbb{R}^d$, where $P_k(x_1,\dots,x_d)$ for $k = 1,\dots,d$ is a polynomial of degree n starting with terms of degree 2, $a, b_k \in \mathbb{R}$, and ε is a small parameter.

The problem of studying the limit cycles of system (4) is reduced using the averaging theory of first order to find the zeros of a nonlinear system of $d-2$ equations with $d-2$ unknowns. It is known that in general the averaging theory for finding periodic solutions does not provide all the periodic solutions of the system; this is due to two main reasons. First, the averaging theory for studying the periodic solutions of a differential system is based on the so-called displacement function, whose zeros provide periodic solutions of the differential system. This displacement function in general is not global and consequently it cannot control all the periodic solutions of the differential system, only the ones which are in its domain of definition and are hyperbolic. Second, the displacement function is expanded in power series of a small parameter ε, and the averaging theory only controls the zeros of the dominant term of this displacement function. When the dominant term is ε^k, we talk about the averaging theory of order k. For more details, see, for instance, [3] and the references quoted there. The averaging theory of first order necessary for the results of this paper is summarized in Section 2.

Our main result on the limit cycles of the differential system (4) is as follows.

Theorem 1. *By applying the first-order averaging theory to the polynomial differential system (4), for $\varepsilon \neq 0$ sufficiently small at most n^{d-1} limit cycles bifurcate from the periodic orbits of the differential system $\dot{x}_1 = -x_2(1+x_1)$, $\dot{x}_2 = x_1(1+x_1)$, $\dot{x}_3 = 0$, and $\dot{x}_4 = 0,\dots,\dot{x}_d = 0$.*

Theorem 1 is proved in Section 3.

2. Limit Cycles via Averaging Theory

Roughly speaking, we can say that the averaging theory gives a quantitative relation between periodic solutions of a nonautonomous periodic differential system and the solutions of its averaged differential system, which is autonomous. The next result provides a first-order approximation in ε for the limit cycles of a periodic differential system; for a proof, see Theorem 2.6.1 of [4] and Theorem 11.5 of [5].

Theorem 2. *One considers the following two initial-value problems:*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \quad x(0) = x_0, \tag{5}$$

$$\dot{y} = \varepsilon f_0(y), \quad y(0) = x_0, \tag{6}$$

where x, y, $x_0 \in D$, and D is an open subset of $\mathbb{R}^n$, $t \in [0,\infty)$, $|\varepsilon| \le \varepsilon_0$, $t \in [0,\infty)$, $|\varepsilon| \le \varepsilon_0$, f and g are periodic of period T in the variable t, and $f_0(y)$ is the averaged function of $f(t,x)$ with respect to t; that is,

$$f_0(y) = \frac{1}{T}\int_0^T f(t,y)\,dt. \tag{7}$$

Assume that

(i) *f, its Jacobian $\partial f/\partial x$, its Hessian $\partial^2 f/\partial x^2$, g, and its Jacobian $\partial g/\partial x$ are defined, continuous, and bounded by a constant independent of ε in $[0,\infty)\times D$ and $|\varepsilon| \le \varepsilon_0$;*

(ii) *T is a constant independent of $|\varepsilon|$;*

(iii) *$y(t)$ belongs to D on the time interval $[0, 1/|\varepsilon|]$.*

Then, the following statements hold:

(a) *On the time scale $1/|\varepsilon|$, we have that $x(t) - y(t) = O(\varepsilon)$ as $\varepsilon \to 0$.*

(b) *If p is a singular point of the averaged system (6) such that the determinant of the Jacobian matrix $\partial f_0/\partial y|_{y=p}$ is not zero, then there exists a limit cycle $\varphi(t,\varepsilon)$ of period T for system (5) such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(c) *The stability or instability of the limit cycle $\varphi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system (6) when p is a hyperbolic singular point.*

To prove Theorem 1, we need the following three lemmas which are proved in [6].

Before doing the proof of Theorem 1, we recall the Bézout theorem which will be used later on; for a proof of this result, see [7].

Theorem 3 (Bézout theorem). *Let q_j be polynomials in the variables $(x_1,\dots,x_d)$ of degree d_j for $j = 1,\dots,d$. Consider the following polynomial system: $q_1(x_1,\dots,x_d) = 0,\dots,q_d(x_1,\dots,x_d) = 0$, where $(x_1,\dots,x_d) \in \mathbb{R}^d$. If the number of solutions of this system is finite, then it is bounded by $d_1\cdots d_d$.*

Lemma 4. *For $i, j \in \mathbb{N}$, one defines*

$$I_{i,j} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{1+\cos\theta}\,d\theta. \tag{8}$$

Then, $I_{i,j} = 0$ if and only if j is even. For $i, j \in \mathbb{N}$ with j even, one has

$$I_{i,j} = \sum_{\substack{s=0\\ s\ \text{even}}}^{j} (-1)^{s/2} \begin{pmatrix} \frac{j}{2} \\ \frac{s}{2} \end{pmatrix} I_{i+s,0}. \tag{9}$$

Lemma 5. *The following equalities hold. For $k \in \mathbb{N}$, one has*

$$E_k = \frac{1}{2\pi}\int_0^{2\pi} \cos^k\theta \, d\theta = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ C_k^{k/2} 2^{-k} & \text{if } k \text{ is even}; \end{cases} \tag{10}$$
$$I_{0,0} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+r\cos\theta} d\theta = \frac{1}{\sqrt{1-r^2}}.$$

Lemma 6. *For $i \in \mathbb{N}$, one has*

$$I_{i,0} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^i\theta}{1+r\cos\theta} d\theta = \frac{(-1)^i}{r^i\sqrt{1-r^2}} + \sum_{\substack{l=1 \\ l\equiv i (\mathrm{mod}\, 2)}}^{i} (-1)^{l-1} 2^{l-i} \binom{i-l}{\frac{(i-l)}{2}} r^{-l}. \tag{11}$$

3. Proof of Theorem 1

Doing the change to polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, system (4) becomes

$$\begin{aligned} \dot{r} &= \varepsilon\left(ar + \sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} \left(a^{i,1}_{i_1,i_2,\ldots,i_d} \cos^{i_1+1}\theta \sin^{i_2}\theta + a^{i,2}_{i_1,i_2,\ldots,i_d} \cos^{i_1}\theta \sin^{i_2+1}\theta\right)\right) \\ \dot{\theta} &= 1 + r\cos\theta + \frac{\varepsilon}{r}\left(\sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} \left(a^{i,2}_{i_1,i_2,\ldots,i_d} \cos^{i_1+1}\theta \sin^{i_2}\theta - a^{i,1}_{i_1,i_2,\ldots,i_d} \cos^{i_1}\theta \sin^{i_2+1}\theta\right)\right) \\ \dot{x}_k &= \varepsilon\left(b_k x_k + \sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} a^{i,k}_{i_1,i_2,\ldots,i_d} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} \cos^{i_1}\theta \cdot \sin^{i_2}\theta\right), \end{aligned} \tag{12}$$

where $k = 3, \ldots, d$. Taking θ as the new independent variable instead of t, this differential system can be written as

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon F(\theta, r, x_3, \ldots, x_d) + O(\varepsilon^2), \\ \frac{dx_k}{d\theta} &= \varepsilon (G_k(\theta, r, x_3, \ldots, x_d)) + O(\varepsilon^2), \end{aligned} \tag{13}$$

for $k = 3, \ldots, d$, where

$$\begin{aligned} F(\theta, r, x_3, \ldots, x_d) &= arD_{0,0} + \sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} \left(a^{i,1}_{i_1,i_2,\ldots,i_d} D_{i_1+1,i_2} + a^{i,2}_{i_1,i_2,\ldots,i_d} D_{i_1,i_2+1}\right), \\ G_k(\theta, r, x_3, \ldots, x_d) &= b_k x_d D_{0,0} + \sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} a^{i,k}_{i_1,i_2,\ldots,i_d} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} D_{i_1,i_2}, \end{aligned} \tag{14}$$

with

$$D_{i_1,i_2} = \frac{\cos^{i_1}\theta \sin^{i_2}\theta}{1+r\cos\theta}. \tag{15}$$

Now, using the notation introduced in Lemma 4 and applying the first-order averaging method, we must find the zeros of the system

$$\begin{aligned} f(r, x_3, \ldots, x_d) &= 0, \\ g_k(r, x_3, \ldots, x_d) &= 0, \quad \text{for } k = 3, \ldots, d, \end{aligned} \tag{16}$$

where

$$\begin{aligned} f(r, x_3, \ldots, x_d) &= \frac{1}{2\pi}\int_0^{2\pi} F(\theta, r, x_3, \ldots, x_d) \, d\theta \\ &= arI_{0,0} + \sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} \left(a^{i,1}_{i_1,i_2,\ldots,i_d} I_{i_1+1,i_2} + a^{i,2}_{i_1,i_2,\ldots,i_d} I_{i_1,i_2+1}\right), \\ g_k(r, x_3, \ldots, x_d) &= \frac{1}{2\pi}\int_0^{2\pi} G_k(\theta, r, x_3, \ldots, x_d) \, d\theta \\ &= b_k x_k I_{0,0} + \sum_{i=2}^{n} \sum_{i_1+\cdots+i_d=i} a^{i,k}_{i_1,i_2,\ldots,i_d} r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} I_{i_1,i_2}, \end{aligned} \tag{17}$$

for $k = 3, \ldots, d$, and

$$I_{i,j} = \frac{1}{2\pi}\int_0^{2\pi} D_{i_1,i_2} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^{i_1}\theta \sin^{i_2}\theta}{1+\cos\theta} d\theta. \tag{18}$$

Theorem 7. *Let $t = \sqrt{1-r^2}$ and $k = 3, \ldots, d$. The function $tg_k(t, x_3, \ldots, x_d)$ is a polynomial of degree n in the variables t and x_k, while $rtf(t, x_3, \ldots, x_d)$ is a polynomial of degree $n+1$. Moreover, $rtf(t, x_3, \ldots, x_d) = (t-1)Q(t, x_3, \ldots, x_d)$, where Q is a polynomial in the variables t and $x_3, x_4, \ldots, x_d$ of the degree at most n.*

Proof. The function g_k is a linear combination of $x_k I_{0,0}$ and $r^{i_1+i_2} x_3^{i_3} \cdots x_d^{i_d} I_{i_1,i_2}$, where $2 \le i_1 + i_2 + \cdots + i_d \le n$.

Lemma 4 claims that

$$r^m I_{m,0} = (-1)^m \left(\sqrt{1-r^2}\right)^{-1} + X_m(r), \tag{19}$$

where X_m is an even polynomial of the degree $m-1$ if m is odd and of degree $m-2$ otherwise. Using the variable $t = \sqrt{1-r^2}$, we conclude that

$$r^m I_{m,0} = \frac{(-1)^m + \widehat{X}_m(t)}{t}, \tag{20}$$

where $\widehat{X}_m(t)$ is an odd polynomial of degree m or $m-1$. Since $r^m I_{m,0}$ vanishes at $r = 0$, the functions $r^k I_{k,0}$ for $k = 2, \ldots, m$ span the space of functions of the form $[A + \widehat{X}(t)]/t$ vanishing at $t = 1$ with $\deg \widehat{X}(t) = m$ or $m-1$, respectively. Lemma 4 implies that any function $r^{i+j} I_{i,j}$ is of the form

$$\phi_{i,j} = \frac{Y_{i,j}(t) + \widehat{X}_{i+j}(t)}{t}, \tag{21}$$

where $Y_{i,j}(t)$ is an even polynomial in t of the degree j (j is necessarily even by Lemma 4) and $\widehat{X}_{i+j}(t)$ is a polynomial in t of the degree $i+j$ or $i+j-1$. We conclude that the functions $r^{i_1+i_2}I_{i_1,i_2}$, where $2-(i_3+\cdots+i_d) \le i_1+i_2 \le n-(i_3+\cdots+i_d)$, generate the space of functions $Z(t)/t$, where $\deg Z \le n-(i_3+\cdots+i_d)$ (and, in addition, $Z(1)=0$). Therefore, $\{P_k(t,x_3,\ldots,x_d)/t,\ \deg P_k \le n\}$, $k=3,\ldots,d$.

In a similar way, f is a linear combination of r and terms $r^{i_1+i_2}x_3^{i_3}\cdots x_d^{i_d}I_{i_1+1,i_2}$ and $r^{i_1+i_2}x_3^{i_3}\cdots x_d^{i_d}I_{i_1,i_2+1}$, where $2 \le i_1+i_2+\cdots+i_d \le n$. We conclude that the functions $r^{i_1+i_2}I_{i_1+1,i_2}$ and $r^{i_1+i_2}I_{i_1,i_2+1}$, where $2-(i_3+\cdots+i_d) \le i_1+i_2 \le n-(i_3+\cdots+i_d)$, generate the space of functions $Z(t)/rt$, where $\deg Z \le n+1-(i_3+\cdots+i_d)$. We have $f(0,x_3,\ldots,x_d)=0$ which implies $Z(1)=0$. Therefore, $\{(t-1)Q(t,x_3,\ldots,x_d)/rt,\ \deg Q \le n\}$. So the polynomials $Q(t,x_3,\ldots,x_d)$ and $tg_k(t,x_3,\ldots,x_d)$ in the variables $t,x_3,\ldots,x_d$ have at most degree n. Hence, by the Bézout theorem, the maximum number of solutions of $tg_k(t,x_3,\ldots,x_d)=0$ for $k=3,\ldots,d$ and $Q(t,x_3,\ldots,x_d)=0$ is at most n^{d-1} for $0<t<1$. □

Thus, from Theorems 2 and 3, it follows that the maximum number of limit cycles bifurcating from the differential system (4) is n^{d-1} obtained using the averaging theory of first order. This completes the proof of Theorem 1.

4. An Application of Theorem 1

In system (4), we consider the case n even and

$$
\begin{aligned}
P_1(x_1,\ldots,x_d) &= \sum_{i=2}^{n} a_{0,0,\ldots,i}^{i,1} x_d^i + a_{1,0,\ldots,0,1}^{2,1} x_1 x_d,\\
P_2(x_1,\ldots,x_d) &= 0,\\
P_k(x_1,\ldots,x_d) &= \sum_{i=2,i \text{ even}}^{n} \left(a_{i,0,\ldots,0}^{i,k} x_1^i + a_{0,i,\ldots,0}^{i,k} x_2^i \right) \quad \text{for } k=3,\ldots,d.
\end{aligned}
\tag{22}
$$

Computing the averaged functions and taking $t=\sqrt{1-r^2}$, we have

$$
\begin{aligned}
r\sqrt{1-r^2} f(r,x_3,x_4,\ldots,x_d) &= ar^2 + \left(a_{1,0,\ldots,0,1}^{2,1} x_d - \sum_{i=2}^{n} a_{0,0,\ldots,i}^{i,1} x_d^i \right)\left(1-\sqrt{1-r^2}\right)\\
&= (1-t)\left(a(1+t) + a_{1,0,\ldots,0,1}^{2,1} x_d - \sum_{i=2}^{n} a_{0,0,\ldots,i}^{i,1} x_d^i \right)\\
&= (1-t)\left(a(1+t) - \overline{Q}(x_d) \right),
\end{aligned}
\tag{23}
$$

where $\overline{Q}(x_d)$ is an arbitrary polynomial in x_d of degree n such that $\overline{Q}(0)=0$. At the same time, the averaged function corresponding to $P_k(x_1,\ldots,x_d)$ satisfies

$$
\begin{aligned}
&\sqrt{1-r^2}\, g_k(r,x_3,x_4,\ldots,x_d)\\
&\quad = b_k x_k + \sqrt{1-r^2} \sum_{\substack{i=2\\ i \text{ even}}}^{n} r^i \left(a_{i,0,0,\ldots,0}^{i,k} I_{i,0} + a_{0,i,0,\ldots,0}^{i,k} I_{0,i} \right),
\end{aligned}
\tag{24}
$$

for $k=3,\ldots,d$. It is easy to obtain the following relations:

$$
\begin{aligned}
\frac{r^k \cos^k\theta}{1+r\cos\theta} &= (-1)^k \frac{1}{1+r\cos\theta} + \sum_{\nu=1}^{k} (-1)^k \cos^{k-\nu}\theta\, r^{k-\nu},\\
\frac{r^k \sin^k\theta}{1+r\cos\theta} &= \sum_{s=0,s \text{ even}}^{k} (-1)^{s/2} \begin{pmatrix} \frac{k}{2} \\ \frac{s}{2} \end{pmatrix} \cdot \left[(-1)^k \frac{r^{k-s}}{1+r\cos\theta} + \sum_{\nu=1}^{s} (-1)^{\nu-1} \cos^{s-\nu}\theta\, r^{k-\nu} \right].
\end{aligned}
\tag{25}
$$

Looking at the second term of the first relation and at the first term of the second relation, we obtain that $r^i I_{i,0}$ and $r^i I_{0,i}$ for even $2 \le i \le n$ are independent. In particular, using Lemmas 4, 5, and 6, we obtain

$$
\begin{aligned}
&\sqrt{1-r^2}\, g_k(r,x_3,x_4,\ldots,x_d)\\
&\quad = b_k x_k + g_{1,k}(r) + g_{2,k}(r), \quad \text{for } k=3,\ldots,d,
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
g_{1,k} &= \sum_{\substack{i=2\\ i \text{ even}}}^{n} a_{i,0,0,\ldots,0}^{i,k} - \sqrt{1-r^2} \sum_{\substack{m=0\\ m \text{ even}}}^{n-1} r^m \sum_{\substack{i=m+1\\ i \text{ even}}}^{n} a_{i,0,\ldots,0}^{i,k} 2^{-m} \begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix},\\
g_{2,k} &= \sum_{\substack{m=0\\ m \text{ even}}}^{n} A_{m,k} r^m + \sqrt{1-r^2} \sum_{\substack{m=0\\ m \text{ even}}}^{n-1} B_{m,k} r^m,
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
A_{m,k} &= \sum_{\substack{i=0\\ i \text{ even}}}^{n} a_{0,i,0,\ldots,0}^{i,k} d_{i-m,i},\\
B_{m,k} &= \sum_{\substack{i=0\\ i \text{ even}}}^{n} a_{0,i,0,\ldots,0}^{i,k} d_{i-m,i} \sum_{\substack{l>0\\ l \text{ even}}}^{n} e_{i-m,l},
\end{aligned}
$$

$$d_{i-m,i} = (-1)^{(i-m)/2} \begin{pmatrix} \dfrac{i}{2} \\ \dfrac{(i-m)}{2} \end{pmatrix},$$

$$e_{i-m,l} = (-1)^{l-1} 2^{l-(i-m)} \begin{pmatrix} i-m-l \\ \dfrac{(i-m-l)}{2} \end{pmatrix}. \tag{28}$$

Writing $t = \sqrt{1-r^2}$, the polynomials $g_{s,k}(r) = P_{s,k}(t)$ satisfy the conditions $g_{s,k}(0) = P_{s,k}(1) = 0$ for $s = 1, 2$ and $k = 3, \ldots, d$. Then, we can define a polynomial of degree n in t:

$$\overline{P}_k(t) = P_{1,k}(t) + P_{2,k}(t) = (t-1)\widetilde{P}_k(t). \tag{29}$$

Due to the independence of $r^i I_{i,0}$ and $r^i I_{0,i}$ and the arbitrariness of the coefficients $a^{i,k}_{i,0,0,\ldots,0}$ and $a^{i,k}_{0,i,0,\ldots,0}$, the polynomial $\overline{P}_k(t)$ is an arbitrary polynomial such that $\overline{P}_k(1) = 0$. In fact, it is obvious that $g_{1,k}$ and $g_{2,k}$ have $n/2$ parameters, respectively, where $n/2$ coefficients $a^{i,k}_{0,i,0,\ldots,0}$ allow choosing the first term of $g_{2,k}$ arbitrarily except for the term with $m = 0$, implying that the even terms of $\overline{P_k}(t)$ are arbitrary except for the constant term, while the other $n/2$ coefficients $a^{i,k}_{i,0,0,\ldots,0}$ allow choosing the second term in $g_{1,k}$ arbitrarily, implying that the odd terms of $\overline{P_k}(t)$ are arbitrary. Therefore, the polynomial $\overline{P_k}(t)$ of the degree n satisfies $\overline{P_k}(1) = 0$ and has n arbitrary coefficients. The number of solutions of $f(r, x_3, x_4, \ldots, x_d) = 0$, $g_k(r, x_3, x_4, \ldots, x_d) = 0$, for $k = 3, \ldots, d$, is equal to the number of the intersection points of the curves

$$\begin{aligned} &l_1 : a(1+t) - \overline{Q}(x_d) = 0. \\ &l_{k-1} : b_k x_k + \overline{P}_k(t) = 0, \quad \text{for } k = 3, \ldots, d. \end{aligned} \tag{30}$$

Hence, by the Bézout theorem, the maximum number of the common solutions of system (30) is at most n^{d-1} for $0 < t < 1$. We can find n^{d-1} intersection points on $f(r, x_3, \ldots, x_d) = 0$ with $g_k(r, x_3, \ldots, x_d) = 0$ for $k = 3, \ldots, d$, $r \in (r_0, 1)$, $0 < r_0 \ll 1$, which (using the averaging theory; see Theorem 2) give rise to n^{d-1} limit cycles bifurcating from periodic orbits of the system $\dot{x}_1 = -x_2(1+x_1)$, $\dot{x}_2 = x_1(1+x_1)$, $\dot{x}_3 = 0$, and $\dot{x}_4 = 0, \ldots, \dot{x}_d = 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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