Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 945049, 7 pages
http://dx.doi.org/10.1155/2015/945049

Research Article

On the Riesz Basisness of Systems Composed of Root Functions of Periodic Boundary Value Problems

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Received 30 June 2014; Accepted 19 December 2014

Academic Editor: Feliz Minhós

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We consider the nonself-adjoint Sturm-Liouville operator with $q \in L_1[0,1]$ and either periodic or antiperiodic boundary conditions. We obtain necessary and sufficient conditions for systems of root functions of these operators to be a Riesz basis in $L_2[0,1]$ in terms of the Fourier coefficients of q.

1. Introduction

Let L be Sturm-Liouville operator generated in $L_2[0,1]$ by the expression

$$y'' + (\lambda - q)\, y = 0, \tag{1}$$

either with the periodic boundary conditions

$$y(1) = y(0), \qquad y'(1) = y'(0), \tag{2}$$

or with the antiperiodic boundary conditions

$$y(1) = -y(0), \qquad y'(1) = -y'(0), \tag{3}$$

where q is a complex-valued summable function on $[0,1]$. We will consider only the periodic problem. The antiperiodic problem is completely similar. The operator L is regular, but not strongly regular. It is well known [1, 2] that the system of root functions of an ordinary differential operator with strongly regular boundary conditions forms a Riesz basis in $L_2[0,1]$. Generally, the normalized eigenfunctions and associated functions, that is, the root functions of the operator with only regular boundary conditions, do not form a Riesz basis. Nevertheless, Shkalikov [3, 4] showed that the system of root functions of an ordinary differential operator with regular boundary conditions forms a basis with parentheses. In [5], they proved that under the conditions

$$q(1) \neq q(0), \quad q \in C^{(4)}[0,1] \tag{4}$$

the system of root functions of L forms a Riesz basis in $L_2[0,1]$. A new approach in terms of the Fourier coefficients of q is due to Dernek and Veliev [6]. They proved that if the following conditions

$$q_{2m} \sim q_{-2m}, \qquad \lim_{m\to\infty} \frac{\ln|m|}{m q_{2m}} = 0, \tag{5}$$

hold, then the root functions of L form a Riesz basis in $L_2[0,1]$, where

$$q_m =: \left(q, e^{i2m\pi x}\right) =: \int_0^1 q(x)\, e^{-i2m\pi x} dx \tag{6}$$

is the Fourier coefficient of q and without loss of generality we always suppose that $q_0 = 0$ and the notation $a_m \sim b_m$ means that there exist constants c_1 and c_2 such that $0 < c_1 < c_2$ and $c_1 < |a_m/b_m| < c_2$ for all large m. Makin [7] extended this result as follows.

Let the first condition in (5) hold. But the second condition in (5) is replaced by a less restrictive one: $q \in W_1^s[0,1]$,

$$q^{(k)}(1) = q^{(k)}(0), \quad \forall k = 0, 1, \ldots, s-1 \tag{7}$$

holds and $|q_{2m}| > cm^{-s-1}$ with some $c > 0$ for large m, where s is a nonnegative integer. Then the root functions of the operator L form a Riesz basis in $L_2[0,1]$.

In addition, some conditions which imply that the system of root functions does not form a Riesz basis of $L_2[0,1]$ were established in [7] (see also [8–10]). In [11], we proved that the Riesz basis property is valid if the first condition in (4) holds, but the second is replaced by $q \in W_1^1[0,1]$. The results of Veliev and Shkalikov [12] are more general and inclusive. The assertions in various forms concerning the Riesz basis property were proved. One of the basic results in [12] is the following statement.

Let $p \geq 0$ be an arbitrary integer, $q \in W_1^p[0,1]$ and (7) holds with some $s \leq p$, and let one of the following conditions hold:

$$\left|q_{2m}\right| > \varepsilon m^{-s-1} \quad \text{or} \quad \left|q_{-2m}\right| > \varepsilon m^{-s-1} \quad \forall \text{large } m \tag{8}$$

with some $\varepsilon > 0$. Then a normal system of root functions of the operator L forms a Riesz basis if and only if $q_{2m} \sim q_{-2m}$.

Here, for large m, by $\Psi_{m,j}(x)$ for $j = 1,2$ denote the normalized eigenfunctions corresponding to the simple eigenvalues $\lambda_{m,j}$. If the multiplicities of these eigenvalues are equal to 2, then the root subspace consists either of two eigenfunctions or of Jordan chains comprising one eigenfunction and one associated function. First, if the multiple eigenvalue $\lambda_{m,1} = \lambda_{m,2}$ has geometric multiplicity 2, we take the normalized eigenfunctions $\Psi_{m,1}(x)$, $\Psi_{m,2}(x)$. Secondly, if there is one eigenfunction $\Psi_{m,1}(x)$ corresponding to the multiple eigenvalue $\lambda_{m,1} = \lambda_{m,2}$, then we take the Jordan chain consisting of a normalized eigenfunction $\Psi_{m,1}(x)$ and corresponding associated function denoted again by $\Psi_{m,2}(x)$ and orthogonal to $\Psi_{m,1}(x)$. Thus the system of root functions obtained in this way will be called a *normal system*.

Moreover, for the other interesting results about the Riesz basis property of root functions of the periodic and antiperiodic problems, we refer in particular to [13–18].

In this paper, we prove the following main result.

Theorem 1. *Let $q \in L_1[0,1]$ be arbitrary complex-valued function and suppose that at least one of the conditions*

$$\lim_{m\to\infty} \frac{\rho(m)}{m q_{2m}} = 0, \qquad \lim_{m\to\infty} \frac{\rho(m)}{m q_{-2m}} = 0 \tag{9}$$

is satisfied, where $\rho(m)$, defined in (40), is a common order of the Fourier coefficients q_{2m} and q_{-2m} of q.

Then a normal system of root functions of the operator L forms a Riesz basis if and only if

$$q_{2m} \sim q_{-2m}. \tag{10}$$

This form of Theorem 1 is not novel (see, e.g., [12]). The novelty is in the term $\rho(m)$ defined in (40) (see also Lemma 4). Indeed, if we take $p = 0$ in the Sobolev space $W_1^p[0,1]$ given above in [12], that is, if $q \in L_1[0,1]$, then the nonnegative integer s in the conditions in (8) must be zero and the assertion on the Riesz basis property remains valid with a less restrictive condition in (9) instead of (8). For example, let $\rho(m) = o(m^{-1/2})$. If instead of (9) we suppose that at least one of the following conditions holds

$$\left|q_{2m}\right| > \varepsilon m^{-3/2} \quad \text{or} \quad \left|q_{-2m}\right| > \varepsilon m^{-3/2} \quad \forall \text{large } m \tag{11}$$

with some ε, then the assertion of Theorem 1 is obvious.

It is well known (see, e.g., [19], Theorem 2 in page 64) that the periodic eigenvalues $\lambda_{m,1}, \lambda_{m,2}$ are located in pairs satisfying the following asymptotic formula:

$$\lambda_{m,1} = \lambda_{m,2} + O\left(m^{1/2}\right) = (2m\pi)^2 + O\left(m^{1/2}\right), \tag{12}$$

for $m \geq N$. Here, by $N \gg 1$ we denote large enough positive integer. From this formula, the pair of the eigenvalues $\{\lambda_{m,1}, \lambda_{m,2}\}$ is close to the number $(2m\pi)^2$ and is isolated from the remaining eigenvalues of L by a distance m. That is, we have, for $j = 1,2$,

$$\left|\lambda_{m,j} - (2(m-k)\pi)^2\right| > |k|\,|2m-k| > Cm, \tag{13}$$

for all $k \neq 0, 2m$ and $k \in \mathcal{Z}$, where $m \geq N$ and, here and in subsequent relations, C is some positive constant whose exact value is not essential. For the potential $q = 0$ and $m \geq 1$, clearly, the system $\{e^{-i2m\pi x}, e^{i2m\pi x}\}$ is a basis of the eigenspace corresponding to the eigenvalue $(2m\pi)^2$ of the periodic boundary value problems.

Finally, let us state the following relevant theorem which will be used in the proof of Theorem 1.

Theorem 2 (see [12]). *The following assertions are equivalent.*

(i) A normal system of root functions of the operator L forms a Riesz basis in the space $L_2[0,1]$.

(ii) The number of Jordan chains is finite and the relation

$$u_{m,j} \sim v_{m,j} \tag{14}$$

holds for all indices m and j corresponding only to the simple eigenvalues $\lambda_{m,j}$ for $j = 1,2$, where $u_{m,j}$, $v_{m,j}$ are the Fourier coefficients defined in (21).

(iii) The number of Jordan chains is finite and the relation (14) for either $j = 1$ or $j = 2$ holds.

2. Preliminaries

The following well-known relation will be used to obtain, for large m, the asymptotic formulas for periodic eigenvalues $\lambda_{m,j}$ corresponding to the normalized eigenfunctions $\Psi_{m,j}(x)$:

$$\Lambda_{m-k,j}\left(\Psi_{m,j}, e^{i2(m-k)\pi x}\right) = \left(q\,\Psi_{m,j}, e^{i2(m-k)\pi x}\right), \tag{15}$$

where $\Lambda_{m-k,j} = \lambda_{m,j} - (2(m-k)\pi)^2$, $j = 1,2$. From Lemma 1 in [20], we iterate (15) by using the following relations:

$$\left(q\Psi_{m,j}, e^{i2m\pi x}\right) = \sum_{m_1=-\infty}^{\infty} q_{m_1}\left(\Psi_{m,j}, e^{i2(m-m_1)\pi x}\right), \tag{16}$$

$$\left|\left(q\Psi_{m,j}, e^{i2(m-m_1)\pi x}\right)\right| < 3M, \tag{17}$$

where for all $m \geq N$, $m_1 \in \mathcal{Z}$ and $j = 1,2$, where $M = \sup_{m\in\mathcal{Z}}|q_m|$.

Hence, substituting (16) in (15) for $k = 0$ and then isolating the terms with indices $m_1 = 0, 2m$, we deduce, in view of $q_0 = 0$, that

$$\begin{aligned}\Lambda_{m,j}\left(\Psi_{m,j}, e^{i2m\pi x}\right) &= q_{2m}\left(\Psi_{m,j}, e^{-i2m\pi x}\right) \\ &\quad + \sum_{m_1\neq 0,2m} q_{m_1}\left(\Psi_{m,j}, e^{i2(m-m_1)\pi x}\right).\end{aligned} \tag{18}$$

First, we use (15) for $k = m_1$ in the right-hand side of (18). Then, considering (16) with the indices m_2 and isolating the terms with indices $m_1 + m_2 = 0, 2m$, we get

$$\left[\Lambda_{m,j} - a_1\left(\lambda_{m,j}\right)\right] u_{m,j} = \left[q_{2m} + b_1\left(\lambda_{m,j}\right)\right] v_{m,j} + R_1(m), \tag{19}$$

by repeating this procedure once again, and

$$\begin{aligned}\left[\Lambda_{m,j} - a_1\left(\lambda_{m,j}\right) - a_2\left(\lambda_{m,j}\right)\right] u_{m,j} \\ = \left[q_{2m} + b_1\left(\lambda_{m,j}\right) + b_2\left(\lambda_{m,j}\right)\right] v_{m,j} + R_2(m),\end{aligned} \tag{20}$$

where $j = 1, 2$,

$$u_{m,j} = \left(\Psi_{m,j}, e^{i2m\pi x}\right), \qquad v_{m,j} = \left(\Psi_{m,j}, e^{-i2m\pi x}\right), \tag{21}$$

$$\begin{aligned} a_1\left(\lambda_{m,j}\right) &= \sum_{m_1} \frac{q_{m_1} q_{-m_1}}{\Lambda_{m-m_1,j}}, \\ a_2\left(\lambda_{m,j}\right) &= \sum_{m_1,m_2} \frac{q_{m_1} q_{m_2} q_{-m_1-m_2}}{\Lambda_{m-m_1,j}\,\Lambda_{m-m_1-m_2,j}}, \end{aligned} \tag{22}$$

$$\begin{aligned} b_1\left(\lambda_{m,j}\right) &= \sum_{m_1} \frac{q_{m_1} q_{2m-m_1}}{\Lambda_{m-m_1,j}}, \\ b_2\left(\lambda_{m,j}\right) &= \sum_{m_1,m_2} \frac{q_{m_1} q_{m_2} q_{2m-m_1-m_2}}{\Lambda_{m-m_1,j}\,\Lambda_{m-m_1-m_2,j}}, \end{aligned} \tag{23}$$

$$R_1(m) = \sum_{m_1,m_2} \frac{q_{m_1} q_{m_2}\left(q\,\Psi_{m,j}, e^{i2(m-m_1-m_2)\pi x}\right)}{\Lambda_{m-m_1,j}\,\Lambda_{m-m_1-m_2,j}}, \tag{24}$$

$$\begin{aligned} R_2(m) \\ = \sum_{m_1,m_2,m_3} \frac{q_{m_1} q_{m_2} q_{m_3}\left(q\,\Psi_{m,j}, e^{i2(m-m_1-m_2-m_3)\pi x}\right)}{\Lambda_{m-m_1,j}\,\Lambda_{m-m_1-m_2,j}\,\Lambda_{m-m_1-m_2-m_3,j}}, \end{aligned} \tag{25}$$

$$m_i \neq 0,\ \forall i; \quad \sum_{i=1}^{k} m_i \neq 0, 2m, \quad \forall k = 1, 2, 3. \tag{26}$$

Using (13), (17), and the relation

$$\sum_{m_1 \neq 0, 2m} \frac{1}{\left|m_1\right|\left|2m - m_1\right|} = O\left(\frac{\ln|m|}{m}\right) \tag{27}$$

one can prove the estimates

$$R_i(m) = O\left(\left(\frac{\ln|m|}{m}\right)^{i+1}\right), \quad i = 1, 2. \tag{28}$$

In the same way, by using the eigenfunction $e^{-i2m\pi x}$ of the operator L for $q = 0$, we can obtain the relations

$$\left[\Lambda_{m,j} - a'\left(\lambda_{m,j}\right)\right] v_{m,j} = \left[q_{-2m} + b_1'\left(\lambda_{m,j}\right)\right] u_{m,j} + R_1'(m), \tag{29}$$

$$\begin{aligned}\left[\Lambda_{m,j} - a_1'\left(\lambda_{m,j}\right) - a_2'\left(\lambda_{m,j}\right)\right] v_{m,j} \\ = \left[q_{2m} + b_1'\left(\lambda_{m,j}\right) + b_2'\left(\lambda_{m,j}\right)\right] u_{m,j} + R_2'(m),\end{aligned} \tag{30}$$

where

$$\begin{aligned} a_1'\left(\lambda_{m,j}\right) &= \sum_{m_1} \frac{q_{m_1} q_{-m_1}}{\Lambda_{m+m_1,j}}, \\ a_2'\left(\lambda_{m,j}\right) &= \sum_{m_1,m_2} \frac{q_{m_1} q_{m_2} q_{-m_1-m_2}}{\Lambda_{m+m_1,j}\,\Lambda_{m+m_1+m_2,j}}, \end{aligned} \tag{31}$$

$$\begin{aligned} b_1'\left(\lambda_{m,j}\right) &= \sum_{m_1} \frac{q_{m_1} q_{-2m-m_1}}{\Lambda_{m+m_1,j}}, \\ b_2'\left(\lambda_{m,j}\right) &= \sum_{m_1,m_2} \frac{q_{m_1} q_{m_2} q_{-2m-m_1-m_2}}{\Lambda_{m+m_1,j}\,\Lambda_{m+m_1+m_2,j}}, \end{aligned} \tag{32}$$

$$R_1'(m) = \sum_{m_1,m_2} \frac{q_{m_1} q_{m_2}\left(q\Psi_{m,j}, e^{i2(m+m_1+m_2)\pi x}\right)}{\Lambda_{m+m_1,j}\,\Lambda_{m+m_1+m_2,j}}, \tag{33}$$

$$R_2'(m) = \sum_{m_1,m_2,m_3} \frac{q_{m_1} q_{m_2} q_{m_3}\left(q\Psi_{m,j}, e^{i2(m+m_1+m_2+m_3)\pi x}\right)}{\Lambda_{m+m_1,j}\,\Lambda_{m+m_1+m_2,j}\,\Lambda_{m+m_1+m_2+m_3,j}}, \tag{34}$$

$$m_i \neq 0, \quad \sum_{i=1}^{k} m_i \neq 0, -2m, \ \forall k = 1, 2, 3. \tag{35}$$

Here the similar estimates as in (28) are valid for $R_i'(m)$, $i = 1, 2$.

In addition, by using (13), (15), and (17), we get

$$\sum_{k \in \mathscr{Z}; k \neq \pm m} \left|\left(\Psi_{m,j}, e^{i2k\pi x}\right)\right|^2 = O\left(\frac{1}{m^2}\right). \tag{36}$$

Thus, we obtain that the normalized eigenfunction $\Psi_{m,j}(x)$ by the basis $\{e^{i2k\pi x} : k \in \mathscr{Z}\}$ on $[0, 1]$ has the following expansion:

$$\Psi_{m,j}(x) = u_{m,j} e^{i2m\pi x} + v_{m,j} e^{-i2m\pi x} + h_m(x), \tag{37}$$

where

$$\left(h_m, e^{\mp i2m\pi x}\right) = 0, \qquad \left\|h_m(x)\right\| = O\left(m^{-1}\right), \tag{38}$$

$$\left|u_{m,j}\right|^2 + \left|v_{m,j}\right|^2 = 1 + O\left(m^{-2}\right). \tag{39}$$

Now, let us consider the following form of the Riemann-Lebesgue lemma. By this we set

$$\begin{aligned}\rho(m) =: \max\left\{\sup_{0 \le x \le 1}\left|\int_0^x q(t)\, e^{-i2(2m)\pi t} dt\right|,\right. \\ \left.\sup_{0 \le x \le 1}\left|\int_0^x q(t)\, e^{i2(2m)\pi t} dt\right|\right\},\end{aligned} \tag{40}$$

and clearly $\rho(m) \to 0$ as $m \to \infty$. As the proof of lemma is similar to that of Lemma 6 in [21], we pass to the proof.

Lemma 3. *If $q \in L^1[0, 1]$ then $\int_0^x q(t)\, e^{i2m\pi t} dt \to 0$ as $|m| \to \infty$ uniformly in x.*

3. Main Results

To prove the main results of the paper we need the following lemmas.

Lemma 4. *The eigenvalues $\lambda_{m,j}$ of the operator L for $m \geq N$ and $j = 1, 2$ satisfy*

$$\lambda_{m,j} = (2m\pi)^2 + O(\rho(m)), \quad (41)$$

where $\rho(m)$ is defined in (40).

Proof. For the proof we have to estimate the terms of (19) and (29). It is easily seen that

$$\sum_{m_1 \neq 0, \pm 2m} \left| \frac{1}{\Lambda_{m \mp m_1, j}} - \frac{1}{\Lambda^0_{m \mp m_1}} \right| = O\left(\frac{\Lambda_{m,j}}{m^2}\right), \quad (42)$$

where $\Lambda^0_{m \mp m_1} = (2m\pi)^2 - (2(m \mp m_1)\pi)^2$. Thus, we get

$$a_1(\lambda_{m,j}) = \frac{1}{4\pi^2} \sum_{m_1 \neq 0, 2m} \frac{q_{m_1} q_{-m_1}}{m_1(2m - m_1)} + O\left(\frac{\Lambda_{m,j}}{m^2}\right). \quad (43)$$

From the argument in Lemma 2(a) of [18] we deduce, with our notations,

$$a_1(\lambda_{m,j}) = \frac{1}{2\pi^2} \sum_{m_1 > 0, m_1 \neq 2m} \frac{q_{m_1} q_{-m_1}}{(2m + m_1)(2m - m_1)} + O\left(\frac{\Lambda_{m,j}}{m^2}\right) = \int_0^1 (G(x,m) - G_0(m))^2 e^{i2(4m)\pi x} dx + O\left(\frac{\Lambda_{m,j}}{m^2}\right), \quad (44)$$

where

$$G(x,m) = \int_0^x q(t) e^{-i2(2m)\pi t} dt - q_{2m} x, \quad (45)$$

$$G_{m_1}(m) =: (G(x,m), e^{i2m_1\pi x}) = \frac{q_{2m+m_1}}{i2\pi m_1} \quad (46)$$

for $m_1 \neq 0$ and

$$G(x,m) - G_0(m) = \sum_{m_1 \neq 2m} \frac{q_{m_1}}{i2\pi(m_1 - 2m)} e^{i2(m_1 - 2m)\pi x}. \quad (47)$$

Thus, from the equalities

$$G(x,m) - G_0(m) = O(\rho(m)), \qquad G(1,m) = G(0,m) = 0 \quad (48)$$

(see (40) and (45)) and since $q \in L^1[0,a]$, integration by parts gives for the integral in (44) the estimate

$$a_1(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{\Lambda_{m,j}}{m^2}\right) \quad (49)$$

for large m. It is easily seen by substituting $m_1 = -k$ into the relation for $a_1'(\lambda_{m,j})$ (see (29)) that

$$a_1(\lambda_{m,j}) = a_1'(\lambda_{m,j}). \quad (50)$$

In a similar way, by (42), and so forth, we get

$$b_1(\lambda_{m,j}) = \frac{1}{4\pi^2} \sum_{m_1 \neq 0, 2m} \frac{q_{m_1} q_{2m - m_1}}{m_1(2m - m_1)} + O\left(\frac{\Lambda_{m,j}}{m^2}\right) = -\int_0^1 (Q(x) - Q_0)^2 e^{-i2(2m)\pi x} dx + O\left(\frac{\Lambda_{m,j}}{m^2}\right) = \frac{-1}{i2\pi(2m)} \int_0^1 2(Q(x) - Q_0) q(x) e^{-i2(2m)\pi x} dx + O\left(\frac{\Lambda_{m,j}}{m^2}\right), \quad (51)$$

where $Q(x) = \int_0^x q(t)dt$, $Q_{m_1} =: (Q(x), e^{i2m_1\pi x}) = q_{m_1}/i2\pi m_1$ if $m_1 \neq 0$,

$$Q(x) - Q_0 = \sum_{m_1 \neq 0} Q_{m_1} e^{i2m_1\pi x}. \quad (52)$$

Thus, by using $Q(1) = q_0 = 0$ and (40), integration by parts again gives for the integral in (51) the following estimate:

$$b_1(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{\Lambda_{m,j}}{m^2}\right). \quad (53)$$

Similarly

$$b_1'(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{\Lambda_{m,j}}{m^2}\right). \quad (54)$$

To estimate $R_1(m) = o(\rho(m))$ (see (24)), let us show that

$$\rho(m) > Cm^{-1} \quad (55)$$

for $m \geq N$ and some $C > 0$. Since $q(x) \neq 0$ is summable function on $[0,1]$, there exists $x \in [0,1]$ such that

$$\int_0^x q(t) dt \neq 0 \quad (56)$$

and the integral (56) is bounded for all $x \in [0,1]$. Hence, multiplying the integrand of (56) by $e^{-i2(2m)\pi x} e^{i2(2m)\pi x}$ and then using integration by parts, we get

$$\sup_{0 \leq x \leq 1} \left| \int_0^x q(t) dt \right| \leq C(\rho(m) + m\rho(m)) \leq Cm\rho(m) \quad (57)$$

which implies (55).

Thus by (13), (17), and relation (27), we deduce that

$$|R_1(m)| \leq C \frac{(\ln |m|)^2}{m^2} = o(\rho(m)). \quad (58)$$

Also $R_1'(m) = o(\rho(m))$.

From relation (39), for large m, it follows that either $|u_{m,j}| > 1/2$ or $|v_{m,j}| > 1/2$. We first consider the case when $|u_{m,j}| > 1/2$. Hence, by using (19), (49), and (53) with $R_1(m) = o(\rho(m))$ we obtain

$$\Lambda_{m,j}\left(1+O\left(m^{-2}\right)\right) = q_{2m}\frac{v_{m,j}}{u_{m,j}} + o\left(\rho\left(m\right)\right). \quad (59)$$

This with definition (40) gives $\Lambda_{m,j} = O(\rho(m))$. Similarly, for the other case $|v_{m,j}| > 1/2$, by using (29), (49), (54), and $R_1'(m) = o(\rho(m))$, we get (41). The lemma is proved. □

Lemma 5. *For all large m, we have the following estimates (see, resp., (23), (32) and (25), (34)):*

$$b_2\left(\lambda_{m,j}\right), b_2'\left(\lambda_{m,j}\right) = O\left(\rho\left(m\right)m^{-2}\right),$$
$$R_2\left(m\right), R_2'\left(m\right) = O\left(\rho\left(m\right)m^{-1}\right). \quad (60)$$

Proof. Let us estimate the sum $R_2(m)$. By using estimate (28) and inequality (55) for large m, we deduce that

$$\left|R_2\left(m\right)\right| \le C\frac{\left(\ln|m|\right)^3}{m^3} = O\left(\rho\left(m\right)m^{-1}\right). \quad (61)$$

In the same way $R_2'(m) = O(\rho(m)m^{-1})$.

Arguing as in [12] (see the proof of Lemma 6), let us now estimate the sum $b_2(\lambda_{m,j})$. Taking into account (42) and Lemma 4, we have

$$b_2\left(\lambda_{m,j}\right) = \frac{1}{\left(2\pi\right)^4}I\left(m\right) + O\left(\frac{\rho\left(m\right)}{m^3}\right), \quad (62)$$

where

$$I\left(m\right) = \sum_{m_1,m_2}\frac{q_{m_1}q_{m_2}q_{2m-m_1-m_2}}{m_1\left(2m-m_1\right)\left(m_1+m_2\right)\left(2m-m_1-m_2\right)}. \quad (63)$$

By using the identity

$$\frac{1}{k\left(2m-k\right)} = \frac{1}{2m}\left(\frac{1}{k}+\frac{1}{2m-k}\right) \quad (64)$$

and the substitutions $k_1 = m_1$, $k_2 = 2m - m_1 - m_2$ in the formula $I(m)$, we obtain $I(m)$ with the indices m_1, m_2 in the following form:

$$I\left(m\right) = \frac{1}{\left(2m\right)^2}\left(I_1 + 2I_2 + I_3\right), \quad (65)$$

where

$$I_1 = \sum_{m_1,m_2}\frac{q_{m_1}q_{m_2}q_{2m-m_1-m_2}}{m_1m_2},$$
$$I_2 = \sum_{m_1,m_2}\frac{q_{m_1}q_{m_2}q_{2m-m_1-m_2}}{m_2\left(2m-m_1\right)}, \quad (66)$$
$$I_3 = \sum_{m_1,m_2}\frac{q_{m_1}q_{m_2}q_{2m-m_1-m_2}}{\left(2m-m_1\right)\left(2m-m_2\right)}.$$

From (46)–(48), (52), and $2I_2(m) = I_1(m)$ and using integration by parts only in I_1, we obtain the following estimates:

$$I_1 = -4\pi^2\int_0^1\left(Q\left(x\right)-Q_0\right)^2 q\left(x\right)e^{-i2(2m)\pi x}dx = O\left(\rho\left(m\right)\right),$$
$$I_3 = -4\pi^2\int_0^1\left(G\left(x,m\right)-G_0\left(m\right)\right)^2 q\left(x\right)e^{i2(2m)\pi x}dx = O\left(\rho\left(m\right)\right). \quad (67)$$

Then, in view of (65) and (67), $I(m) = O(\rho(m)m^{-2})$. This with equality (62) implies that $b_2(\lambda_{m,j}) = O(\rho(m)m^{-2})$. In the same way $b_2'(\lambda_{m,j})$ satisfies the same estimate. The lemma is proved. □

Thus by using Lemmas 4 and 5, Theorem 2, and an argument similar to that of Theorem 2 in [12] under the conditions in (9), let us prove the following main result.

Proof of Theorem 1. In view of Lemma 4, substituting the values of

$$b_1\left(\lambda_{m,j}\right), b_1'\left(\lambda_{m,j}\right) = O\left(\rho\left(m\right)m^{-1}\right),$$
$$b_2\left(\lambda_{m,j}\right), b_2'\left(\lambda_{m,j}\right) R_2\left(m\right), R_2'\left(m\right) = O\left(\rho\left(m\right)m^{-2}\right) \quad (68)$$

given by (53), (54), and (60) in relations (20) and (30), we get the following reversion of the relations

$$\left[\Lambda_{m,j} - a_1\left(\lambda_{m,j}\right) - a_2\left(\lambda_{m,j}\right)\right]u_{m,j} = \left[q_{2m} + O\left(\rho\left(m\right)m^{-1}\right)\right]v_{m,j} + O\left(\rho\left(m\right)m^{-2}\right), \quad (69)$$

$$\left[\Lambda_{m,j} - a_1'\left(\lambda_{m,j}\right) - a_2'\left(\lambda_{m,j}\right)\right]v_{m,j} = \left[q_{-2m} + O\left(\rho\left(m\right)m^{-1}\right)\right]u_{m,j} + O\left(\rho\left(m\right)m^{-2}\right) \quad (70)$$

for $j = 1, 2$.

It is easily seen again by substituting $m_1 + m_2 = -k_1$, $m_2 = k_2$, in the sum $a_2'(\lambda_{m,j})$ (see (29)) and using (50) that $a_i(\lambda_{m,j}) = a_i'(\lambda_{m,j})$ for $i = 1, 2$. Hence, multiplying (69) by $v_{m,j}$ and (70) by $u_{m,j}$ and subtracting we obtain the following equality:

$$q_{2m}v_{m,j}^2 - q_{-2m}u_{m,j}^2 = O\left(\rho\left(m\right)m^{-1}\right). \quad (71)$$

Suppose, for example, that q_{2m} satisfies the condition in (9). Then using this equality we get

$$v_{m,j}^2 - \kappa_m u_{m,j}^2 = o\left(1\right), \quad \kappa_m =: \frac{q_{-2m}}{q_{2m}}, \quad (72)$$

for $j = 1, 2$. In addition, for large m, the condition in (9) for q_{2m} implies that the geometric multiplicity of the eigenvalue $\lambda_{m,j}$ is 1. Arguing as in Lemma 4 of [12], if there exist mutually orthogonal two eigenfunctions $\Psi_{m,j}(x)$ corresponding to $\lambda_{m,1} = \lambda_{m,2}$, then one can choose an eigenfunction $\Psi_{m,j}(x)$ such that $u_{m,j} = 0$. Thus combining this with (39) and (71), we get $q_{2m} = O(\rho(m)m^{-1})$ which contradicts (9).

Let the normal system of root functions form a Riesz basis. To prove $\kappa_m \sim 1$, from (72) it is enough to show that all the large periodic eigenvalues $\lambda_{m,j}$ are simple, since in this case we have, by Theorem 2,

$$u_{m,j} \sim v_{m,j} \sim 1 \tag{73}$$

for $j = 1, 2$. For large m, again by Theorem 2 and the condition in (9) for q_{2m}, respectively, the number of Jordan chains and the eigenvalues of geometric multiplicity 2 are finite; that is, all large eigenvalues are simple.

Now let $q_{2m} \sim q_{-2m}$. From (72), we obtain the relation (73) for $j = 1$ which implies that the number of Jordan chains is finite. In fact, if there exists a Jordan chain consisting of an eigenfunction $\Psi_{m,1}(x)$ and an associated function $\Psi_{m,2}(x)$ corresponding to the eigenvalue $\lambda_{m,1} = \lambda_{m,2}$, then, for example, for $\lambda_{m,1}$, using the eigenfunction $\overline{\Psi_{m,1}(x)}$ of the adjoint operator L^* and the relation

$$(L - \lambda_{m,1}) \Psi_{m,2}(x) = \Psi_{m,1}(x), \tag{74}$$

we obtain that $(\Psi_{m,1}, \overline{\Psi_{m,1}}) = 0$. Thus, from expansion (37) for $j = 1$, we get $u_{m,1}v_{m,1} = O(m^{-2})$ which contradicts (73) for $j = 1$. Thus, using Theorem 2, we prove that a normal system of root functions of the operator L forms a Riesz basis. □

Arguing as in the proof of Theorem 1, we obtain a similar result established below for the antiperiodic problems.

Theorem 6. *Let $q \in L_1[0,1]$ be arbitrary complex-valued function and suppose that at least one of the conditions*

$$\lim_{m\to\infty} \frac{\rho(m)}{mq_{2m+1}} = 0, \qquad \lim_{m\to\infty} \frac{\rho(m)}{mq_{-2m-1}} = 0 \tag{75}$$

is satisfied, where $\rho(m)$ is obtained from (40) by replacing $2m$ with $2m+1$ and a common order of both Fourier coefficients q_{2m+1} and q_{-2m-1} of q.

Then a normal system of root functions of the operator L with antiperiodic boundary conditions forms a Riesz basis if and only if $q_{2m+1} \sim q_{-2m-1}$.

Remark 7. Clearly if instead of (9) we assume that at least one of the conditions

$$\rho(m) \sim q_{2m}, \qquad \rho(m) \sim q_{-2m} \tag{76}$$

holds, then the assertion of Theorem 1 is satisfied. In this way one can easily write a similar result for the antiperiodic problem.

In addition to all the above results, we note that if either the first condition of (9) and (10) or the second condition of (9) and (10) holds then all the periodic eigenvalues are asymptotically simple. We can write a similar result for the antiperiodic problem.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] N. Dunford and J. T. Schwartz, *Linear Operators Part 3: Spectral Operators*, John Wiley & Sons, New York, NY, USA, 1988.

[2] V. P. Mikhailov, "On Riesz bases in L_2 (0 ; 1)," *Doklady Akademii Nauk SSSR*, vol. 144, no. 5, pp. 981–984, 1962.

[3] A. Shkalikov, "On the basis property of eigenfunctions of an ordinary differential operator," *Uspekhi Matematicheskikh Nauk*, vol. 34, no. 5, pp. 235–236, 1979.

[4] A. Shkalikov, "On the basis property of eigenfunctions of an ordinary differential operator with integral boundary conditions," *Vestnik Moskovskogo Universiteta. Seriya Matematiki, Mekhaniki*, vol. 6, pp. 12–21, 1982.

[5] N. B. Kerimov and K. R. Mamedov, "On the Riesz basis property of root functions of some regular boundary value problems," *Mathematical Notes*, vol. 64, no. 4, pp. 483–487, 1998.

[6] N. Dernek and O. A. Veliev, "On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operator," *Israel Journal of Mathematics*, vol. 145, pp. 113–123, 2005.

[7] A. S. Makin, "Convergence of expansions in the root functions of periodic boundary value problems," *Doklady Mathematics*, vol. 73, no. 1, pp. 71–76, 2006.

[8] P. Djakov and B. Mityagin, "Instability zones of periodic 1-dimensional Schrödinger and Dirac operators," *Russian Mathematical Surveys*, vol. 61, no. 4, pp. 663–766, 2006.

[9] P. B. Dzhakov and B. S. Mityagin, "Convergence of spectral decompositions of hill operators with trigonometric polynomials as potentials," *Doklady Mathematics*, vol. 83, no. 1, pp. 5–7, 2011.

[10] P. Djakov and B. Mityagin, "Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials," *Mathematische Annalen*, vol. 351, no. 3, pp. 509–540, 2011.

[11] A. A. Kıraç, "Riesz basis property of the root functions of non-selfadjoint operators with regular boundary conditions," *International Journal of Mathematical Analysis*, vol. 3, no. 21–24, pp. 1101–1109, 2009.

[12] O. A. Veliev and A. A. Shkalikov, "On the Riesz basis property of eigen- and associated functions of periodic and anti-periodic Sturm-Liouville problems," *Mathematical Notes*, vol. 85, no. 5, pp. 647–660, 2009.

[13] P. Djakov and B. Mityagin, "Criteria for existence of Riesz bases consisting of root functions of Hill and 1D DIRac operators," *Journal of Functional Analysis*, vol. 263, no. 8, pp. 2300–2332, 2012.

[14] F. Gesztesy and V. Tkachenko, "A schauder and riesz basis criterion for non-self-adjoint Schrodinger operators with periodic and antiperiodic boundary conditions," *Journal of Differential Equations*, vol. 253, no. 2, pp. 400–437, 2012.

[15] R. Khanlar and H. M. Mamedov, "On the basisness in L_2 (0; 1) of the root functions in not strongly regular boundary value problems," *European Journal of Pure and Applied Mathematics*, vol. 1, no. 2, pp. 51–60, 2008.

[16] K. R. Mamedov, "On the basis property in $L_p(0, 1)$ of the root functions of a class non self adjoint Sturm-Lioville operators," *European Journal of Pure and Applied Mathematics*, vol. 3, no. 5, pp. 831–838, 2010.

[17] O. A. Veliev, "On the nonself-adjoint ordinary differential operators with periodic boundary conditions," *Israel Journal of Mathematics*, vol. 176, pp. 195–207, 2010.

[18] O. A. Veliev, "Asymptotic analysis of non-self-adjoint Hill operators," *Central European Journal of Mathematics*, vol. 11, no. 12, pp. 2234–2256, 2013.

[19] M. A. Naimark, *Linear Differential Operators*, vol. 1, George G. Harrap and Co., London, UK, 1967.

[20] O. A. Veliev and M. T. Duman, "The spectral expansion for a nonself-adjoint Hill operator with a locally integrable potential," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 1, pp. 76–90, 2002.

[21] B. J. Harris, "The form of the spectral functions associated with Sturm-Liouville problems with continuous spectrum," *Mathematika*, vol. 44, no. 1, pp. 149–161, 1997.