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Review Article

$W^{2,p}$-Solvability of the Dirichlet Problem for Elliptic Equations with Singular Data

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We give an overview on some results concerning the unique solvability of the Dirichlet problem in $W^{2,p}$, $p > 1$, for second-order linear elliptic partial differential equations in nondivergence form and with singular data in weighted Sobolev spaces. We also extend such results to the planar case.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. Consider in Ω the Dirichlet problem

$$\begin{gathered} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{gathered} \tag{1}$$

where $p \in]1, +\infty[$ and L is the second-order linear elliptic differential operator in nondivergence form defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{2}$$

If $n \geq 3$ and Ω is a bounded set with a suitable regularity property, the well-posedness of the Dirichlet problem (1) has been largely studied by several authors under various hypotheses of discontinuity on the coefficients. It must be mentioned the classical contribution by Miranda [1], where the author assumed that the a_{ij} belong to $W^{1,n}(\Omega)$ and considered the case $p = 2$. This result was later on generalized in [2, 3] by considering the coefficients a_{ij} belonging to the wider class of spaces VMO and $p \in]1, +\infty[$.

In the framework of open sets, nonnecessarily bounded, whose boundary has various singularities, for example, corners or edges, in accordance with the linear theory, it is natural to assume that the lower order coefficients and the right-hand side of the equation of problem (1) belong to some weighted Sobolev spaces, where the weight is usually a power of the distance function from the "singular set" of the boundary of domain. In these cases, near to the "singular set," the solution u of the boundary value problem may have a singularity which can be often characterized by a weight of mentioned type. For instance, if ρ is a bounded weight function related to the distance function from nonempty subset S_ρ of the boundary of an arbitrary domain Ω, not necessarily bounded and regular (see Section 2 for the definition of such weight function), a problem similar to (1) has been studied, in the weighted case, by several authors under suitable hypotheses on the weight function ρ and when the coefficients of lower order terms are singular near to S_ρ. This kind of Dirichlet problem has been dealt, for example, in [4–6] under hypotheses as those in [2, 3] with $p \in]1, +\infty[$. To be more precise, in [4–6], the authors consider the problem

$$\begin{gathered} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \\ Lu = f, \quad f \in L_s^p(\Omega), \end{gathered} \tag{3}$$

where $n \geq 3$, $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)$ and $L_s^p(\Omega)(p \in]1, +\infty[, s \in \mathbb{R})$ are some weighted Sobolev spaces whose weight functions are suitable powers of ρ. The existence and uniqueness of problem (3) have been firstly proved in [4, 5] when $p \geq 2$. Such results have been later on used in [6] to get the unique solvability also for $1 < p < 2$. We point out that in this last case we needed to employ some variational results (see [6] and its bibliography).

The aim of this work is to give an overview on the above-mentioned results concerning the solvability of (3) when $p \in]1,+\infty[$ as well as to extend such results to the planar case. Here we note also that, for $p = 2$, if $n \geq 3$ the required summability on $(a_{ij})_{x_h}$ can be equal to n while if $n = 2$ we have to take a summability greater than n (see hypothesis (h_2)).

At last, we observe that the existence and uniqueness result for the problem (3) are based on the unique solvability of the problem (3) for $p = 2$ and of a problem similar to (3) whose associated operator L_{s-1} differs from that in (2) for a compact operator (see Section 4).

For further results concerning elliptic boundary value problems similar to (3), involving different classes of weighted Sobolev spaces, we refer the reader also to [7, 8].

2. Notation and Function Spaces

In this section we recall the definitions and the main properties of the class of weights we are interested in and of certain classes of function spaces where the coefficients of our operator belong. Thus, from now on, let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. By $\Sigma(F)$ we denote the σ-algebra of all Lebesgue measurable subsets F of Ω. For $E \in \Sigma(F)$, χ_E is its characteristic function, $|E|$ is its Lebesgue measure and $E(x,r) = E \cap B(x,r)$ $(x \in \mathbb{R}^n, r \in \mathbb{R}_+)$, where $B(x,r)$ is the open ball with center in x and radius r. The class of restrictions to E of functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ (resp., $\zeta \in C_0^0(\mathbb{R}^n)$) with $\overline{E} \cap \operatorname{supp}\zeta \subseteq E$ is $\mathfrak{D}(E)$ (resp., $\mathfrak{D}^0(E)$). For $p \in [1,+\infty]$, $L^p_{\mathrm{loc}}(E)$ is the class of all functions g, defined on E, such that $\zeta\, g \in L^p(E)$ for all $\zeta \in \mathfrak{D}(E)$.

We denote by $\mathscr{A}(\Omega)$ the class of all measurable weight functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\gamma^{-1}\rho(y) \leq \rho(x) \leq \gamma\rho(y) \quad \forall y \in \Omega, \quad \forall x \in \Omega(y,\rho(y)), \tag{4}$$

where $\gamma \in \mathbb{R}_+$ is independent on x and y. Given $\rho \in \mathscr{A}(\Omega)$, we put

$$S_\rho = \left\{z \in \partial\Omega \mid \lim_{x\to z}\rho(x) = 0\right\}. \tag{5}$$

It is known that

$$\rho \in L^\infty_{\mathrm{loc}}\left(\overline{\Omega}\right), \qquad \rho^{-1} \in L^\infty_{\mathrm{loc}}\left(\overline{\Omega} \setminus S_\rho\right), \tag{6}$$

and if $S_\rho \neq \emptyset$,

$$\rho(x) \leq \operatorname{dist}\left(x, S_\rho\right) \quad \forall x \in \Omega, \tag{7}$$

(see [9, 10] for further details).

For $k \in \mathbb{N}_0$, $1 \leq p \leq +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathscr{A}(\Omega)$, the related weighted Sobolev space $W^{k,p}_s(\Omega)$ is made up of all the distributions u on Ω such that $\rho^{s+|\alpha|-k}\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$. We observe that $W^{k,p}_s(\Omega)$ is a Banach space with the norm defined by

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\leq k}\left\|\rho^{s+|\alpha|-k}\partial^\alpha u\right\|_{L^p(\Omega)}. \tag{8}$$

Moreover, it is separable if $1 \leq p < +\infty$, reflexive if $1 < p < +\infty$, and, in particular, $W^{k,2}_s(\Omega)$ is an Hilbert space. We also denote by $\mathring{W}^{k,p}_s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}_s(\Omega)$ and we put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$.

Clearly the following embeddings hold:

$$\mathring{W}^{k,p}_s(\Omega) \hookrightarrow W^{k,p}_s(\Omega) \hookrightarrow L^p_{s-k}(\Omega). \tag{9}$$

A more detailed account of the properties of the above-defined weighted Sobolev spaces can be found in [11–13].

For $p \in [1,+\infty[$, $s \in \mathbb{R}$ and $\rho \in \mathscr{A}(\Omega)$, let $K^p_s(\Omega)$ be the set of all the functions $g \in L^p_{\mathrm{loc}}(\overline{\Omega} \setminus S_\rho)$ such that

$$\|g\|_{K^p_s(\Omega)} = \sup_\Omega\left(\rho^{s-(n/p)}(x)\,\|g\|_{L^p(\Omega(x,\rho(x)))}\right) < +\infty, \tag{10}$$

equipped with the norm defined in (10). Obviously, the space $K^p_s(\Omega)$ is a Banach space containing $L^\infty_s(\Omega)$ and $C_0^\infty(\Omega)$ as well (see [10]). Therefore, we denote by $\widetilde{K}^p_s(\Omega)$ (resp., $\mathring{K}^p_s(\Omega)$) the closure of $L^\infty_s(\Omega)$ (resp., $C_0^\infty(\Omega)$) in $K^p_s(\Omega)$.

Now, let us define the moduli of continuity of functions belonging to $\widetilde{K}^p_s(\Omega)$ or $\mathring{K}^p_s(\Omega)$.

Given a function g in $K^p_s(\Omega)$, the following characterization holds:

$$g \in \widetilde{K}^p_s(\Omega) \Longleftrightarrow \lim_{h\to+\infty}\left(\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}(|\Omega(x,\rho(x))\cap E|/\rho^n(x))\leq 1/h}}\|g\chi_E\|_{K^p_s(\Omega)}\right) = 0, \tag{11}$$

(see [10]).

Thus, if g is a function in $\widetilde{K}^p_s(\Omega)$, a *modulus of continuity* of g in $\widetilde{K}^p_s(\Omega)$ is a map $\widetilde{\omega}^p_s[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}(|\Omega(x,\rho(x))\cap E|/\rho^n(x))\leq 1/h}}\|g\chi_E\|_{K^p_s(\Omega)} \leq \widetilde{\omega}^p_s[g](h), \qquad \lim_{h\to+\infty}\widetilde{\omega}^p_s[g](h) = 0. \tag{12}$$

Next, we introduce a class of mappings needed to define a *modulus of continuity* of g in $\mathring{K}^p_s(\Omega)$. Fix f in $\mathfrak{D}(\overline{\mathbb{R}}_+)$ satisfying the conditions

$$0 \leq f \leq 1, \quad f(t) = 1 \quad \text{if} \quad t \leq \frac{1}{2}, \quad f(t) = 0 \quad \text{if} \quad t \geq 1, \tag{13}$$

and $\alpha \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ equivalent to dist $(\cdot,\partial\Omega)$ (for more details on the existence of such an α see, for instance, Theorem 2, Chapter VI in [14] and Lemma 3.6.1 in [15]). Hence, for $k \in \mathbb{N}$ we define the functions

$$\psi_k : x \in \overline{\Omega} \longrightarrow (1 - f(k\alpha(x)))\, f\left(\frac{|x|}{2k}\right). \tag{14}$$

It is easy to prove that each ψ_k belongs to $\mathfrak{D}(\overline{\Omega} \setminus S_\rho)$ and

$$0 \le \psi_k \le 1, \quad (\psi_k)_{|\overline{\Omega}_k} = 1, \quad \operatorname{supp} \psi_k \subset \overline{\Omega}_{2k}, \tag{15}$$

where $\Omega_k = \{x \in \Omega \mid |x| < k, \alpha(x) > 1/k\}$.

Given g in $K_s^p(\Omega)$, it is known (see [10]) that

$$g \in \overset{\circ}{K}{}_s^p(\Omega) \Longleftrightarrow \lim_{h \to +\infty} \|(1 - \psi_h) g\|_{K_s^p(\Omega)} = 0. \tag{16}$$

Thus, a *modulus of continuity* of g in $\overset{\circ}{K}{}_s^p(\Omega)$ is a map $\overset{\circ}{\omega}{}_s^p[g] : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{gathered} \|(1 - \psi_h) g\|_{K_s^p(\Omega)} \\ + \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x\in\Omega} |\Omega(x,\rho(x)) \cap E| \le 1/h}} \|\psi_h g \chi_E\|_{K_s^p(\Omega)} \le \overset{\circ}{\omega}{}_s^p [g](h), \\ \lim_{h \to +\infty} \overset{\circ}{\omega}{}_s^p [g](h) = 0. \end{gathered} \tag{17}$$

Further properties of the spaces $K_s^p(\Omega)$, $\widetilde{K}_s^p(\Omega)$, and $\overset{\circ}{K}{}_s^p(\Omega)$ can be found in [10].

In the end, we recall the definitions of two other classes of function spaces where the leading coefficients of our operator belong.

If Ω has the property

$$|\Omega(x, r)| \ge A r^n \quad \forall x \in \Omega, \quad \forall r \in]0, 1], \tag{18}$$

where A is a positive constant independent of x and r, it is possible to consider the space $\mathrm{BMO}(\Omega, t)$ ($t \in \mathbb{R}_+$) of functions $g \in L^1_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\begin{aligned} &[g]_{\mathrm{BMO}(\Omega,t)} \\ &\quad = \sup_{\substack{x\in\Omega \\ r \in]0,t]}} ⨍_{\Omega(x,r)} \left| g - ⨍_{\Omega(x,r)} g \right| \\ &\quad < +\infty, \end{aligned} \tag{19}$$

where

$$⨍_{\Omega(x,r)} g = |\Omega(x, r)|^{-1} ⨍_{\Omega(x,r)} g. \tag{20}$$

If $g \in \mathrm{BMO}(\Omega) = \mathrm{BMO}(\Omega, t_A)$, where

$$t_A = \sup_{t \in \mathbb{R}_+} \left(\sup_{\substack{x\in\Omega \\ r \in]0,t]}} \frac{r^n}{|\Omega(x, r)|} \le \frac{1}{A} \right), \tag{21}$$

we say that $g \in \mathrm{VMO}(\Omega)$ if $[g]_{\mathrm{BMO}(\Omega,t)} \to 0$ for $t \to 0^+$. A function $\eta[g] : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *modulus of continuity* of g in $\mathrm{VMO}(\Omega)$ if

$$[g]_{\mathrm{BMO}(\Omega,t)} \le \eta[g](t) \quad \forall t \in \mathbb{R}_+, \quad \lim_{t \to 0^+} \eta[g](t) = 0. \tag{22}$$

A more detailed account of properties of the above-defined spaces $\mathrm{BMO}(\Omega, t)$ and $\mathrm{VMO}(\Omega)$ can be found in [16] or in some general reference books, for example, [17].

3. Preliminary Results

In this section we recall some embedding and compactness estimates for a multiplication operator as well as a regularity result which will be also extended to the planar case.

Let us assume that Ω has the segment property (for all definitions of regularity properties of open subsets of $\mathbb{R}^n$ we will refer to [18]) and fix $\rho \in \mathscr{A}(\Omega) \cap L^\infty(\Omega)$ such that $S_\rho \ne \emptyset$.

For our purposes, we suppose that the following condition on Ω holds.

> (h_0) There exists an open subset Ω_o of $\mathbb{R}^n$ with the uniform $C^{1,1}$-regularity property such that

$$\Omega \subset \Omega_o, \qquad \partial\Omega \setminus S_\rho \subset \partial\Omega_o. \tag{23}$$

We note that since the required segment property gives that Ω lies on one side of the "singular" part S_ρ of its boundary the hypothesis (h_o), roughly speaking, means that it is possible to "widen" suitably Ω on the other side of S_ρ.

Remark 1. We observe that, as a result of condition (h_0), there exists $\theta \in]0, \pi/2[$ such that

$$\forall x \in \Omega \quad \exists C_\theta(x) : \overline{C_\theta(x, \rho(x))} \subset \Omega, \tag{24}$$

where $C_\theta(x)$ is an open infinite cone with vertex in x and opening θ and $C_\theta(x, \rho(x)) = C_\theta(x) \cap B(x, \rho(x))$ (see Remark 5.1 in [10]). As a consequence, there exists a function $\sigma \in \mathscr{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ which is equivalent to ρ and such that

$$|\partial^\alpha \sigma(x)| \le c_\alpha \sigma^{1-|\alpha|}(x) \quad \forall x \in \Omega, \ \forall \alpha \in \mathbb{N}_0^n, \tag{25}$$

where $c_\alpha \in \mathbb{R}_+$ is independent of x (see [9]).

Let us start collecting two results of [10] which provide the boundedness and compactness of the multiplication operator

$$u \longrightarrow gu, \tag{26}$$

where the function g belongs to suitable spaces $K_s^p(\Omega)$.

Theorem 2. *Let r, s, p, q be numbers such that*

$$\begin{gathered} r \in \mathbb{N}, \quad s \in \mathbb{R}, \quad 1 \le p \le q < +\infty, \quad q \ge \frac{n}{r}, \\ q > \frac{n}{r} \text{ if } \frac{n}{r} = p > 1. \end{gathered} \tag{27}$$

If condition (h_0) holds, then for all $u \in W_s^{r,p}(\Omega)$ and for any $g \in K^q_{-s+r}(\Omega)$ we have $gu \in L^p(\Omega)$ and

$$\|gu\|_{L^p(\Omega)} \le c \|g\|_{K^q_{-s+r}(\Omega)} \|u\|_{W_s^{r,p}(\Omega)}, \tag{28}$$

where the constant $c \in \mathbb{R}_+$ is independent of g and u.

Furthermore, if $g \in \overset{\circ}{K}{}^q_{-s+r}(\Omega)$ then for any $\epsilon \in \mathbb{R}_+$ there exist a constant $c(\epsilon) \in \mathbb{R}_+$ and a bounded open set $\Omega_\epsilon \subset\subset \Omega$ with the cone property such that

$$\|gu\|_{L^p(\Omega)} \le \epsilon \|u\|_{W_s^{r,p}(\Omega)} + c(\epsilon) \|u\|_{L^p(\Omega_\epsilon)} \quad \forall u \in W_s^{r,p}(\Omega), \tag{29}$$

and we have that the multiplication operator

$$u \in W_s^{r,p}(\Omega) \longrightarrow gu \in L^p(\Omega) \tag{30}$$

is compact.

We go on collecting a density result, an a priori estimate, and a regularity result, which will be some of the crucial analytic tools of our main results.

Let $p \in]1,+\infty[$, $s \in \mathbb{R}$. At first we recall a density result (see Lemma 3.2 in [6]).

Lemma 3. *If Ω verifies condition (h_0), then for every $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)$ there exists a sequence $(u_k)_{k\in\mathbb{N}}$ such that*

$$\begin{gathered} u_k \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,2}(\Omega) \cap \mathfrak{D}^0\left(\overline{\Omega} \setminus S_\rho\right), \\ u_k \longrightarrow u \text{ in } W_s^{2,p}(\Omega). \end{gathered} \tag{31}$$

Now consider in Ω the second-order linear differential operator in nondivergence form

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{32}$$

Assume that the leading coefficients satisfy the hypothesis.

(h_1) There exist extensions a_{ij}^o of a_{ij} to Ω_o such that

$$\begin{gathered} a_{ij}^o = a_{ji}^o \in L^\infty(\Omega_o) \cap \mathrm{VMO}(\Omega_o), \quad i,j = 1,\dots,n, \\ \exists \nu_o \in \mathbb{R}_+ : \sum_{i,j=1}^{n} a_{ij}^o \xi_i \xi_j \geq \nu_o |\xi|^2 \text{ a.e. in } \Omega_o, \quad \forall \xi \in \mathbb{R}^n. \end{gathered} \tag{33}$$

We note that the assumption (h_1) holds if the coefficients a_{ij} are restrictions to Ω of functions in VMO (see also [2, 3]).

For the lower order terms coefficients suppose that

(i_2)

$$\begin{gathered} a_i \in \widetilde{K}_1^{t_1}(\Omega), \quad i = 1,\dots,n, \text{ where } t_1 \geq p,\ t_1 \geq n, \\ t_1 > p \text{ if } p = n, \\ a \in \widetilde{K}_2^{t_2}(\Omega), \text{ where } t_2 \geq p,\ t_2 \geq \frac{n}{2}, \\ t_2 > p \text{ if } p = \frac{n}{2}. \end{gathered} \tag{34}$$

We observe that, in view of Theorem 2, under the assumptions (h_0), (h_1), and (i_2), the operator $L : W_s^{2,p}(\Omega) \to L_s^p(\Omega)$ is bounded.

The following a priori estimate holds.

Lemma 4. *Let L be defined in (32). If hypotheses (h_0), (h_1), and (i_2) are satisfied, then there exists $c \in \mathbb{R}_+$ such that*

$$\begin{gathered} \|u\|_{W_s^{2,p}(\Omega)} \leq c\left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)}\right) \\ \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \end{gathered} \tag{35}$$

where c depends on Ω, n, p, t_1, t_2, s, ρ, ν_o, $\|a_{ij}^o\|_{L^\infty(\Omega_o)}$, $\|a_i\|_{K_1^{t_1}(\Omega)}$, $\|a\|_{K_2^{t_2}(\Omega)}$, $\eta[a_{ij}^o]$, $\widetilde{\omega}_1^{t_1}[a_i]$, and $\widetilde{\omega}_2^{t_2}[a]$.

The proof of (35) can be found in [19] in the case $n \geq 3$ and it can be easily extended to the planar case, taking care to use Lemma 3.1 in [20] in place of Theorem 5.1 in [21].

In the next Lemma we recall a regularity result of [5] (see Theorem 3.4) and we extend it to the case $n = 2$.

Lemma 5. *Let L be defined in (32). Suppose that conditions (h_0), (h_1), and (i_2) hold with $t_1 > n$ and $t_2 > n/2$. Then any solution u of the problem*

$$\begin{gathered} u \in W_{\mathrm{loc}}^{2,q}\left(\overline{\Omega} \setminus S_\rho\right) \cap \overset{\circ}{W}{}_{\mathrm{loc}}^{1,q}\left(\overline{\Omega} \setminus S_\rho\right) \cap L_{s_o-2}^{p_o}(\Omega), \\ Lu \in L_s^p(\Omega), \end{gathered} \tag{36}$$

with $q, p_o \in]1,p]$ and $s_o = s - n((1/p_o) - (1/p))$, belongs to the space $W_s^{2,p}(\Omega)$ and verifies the bound

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c\left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_{s_o-2}^{p_o}(\Omega)}\right), \tag{37}$$

where $c \in \mathbb{R}_+$ depends only on Ω, n, p, p_o, t_1, t_2, s, ρ, ν_o, $\|a_{ij}^o\|_{L^\infty(\Omega_o)}$, $\|a_i\|_{K_1^{t_1}(\Omega)}$, $\|a\|_{K_2^{t_2}(\Omega)}$, $\eta[a_{ij}^o]$, $\widetilde{\omega}_1^{t_1}[a_i]$, and $\widetilde{\omega}_2^{t_2}[a]$.

Let us give an overview of the proof of Lemma 5. The idea is to use a local regularity result which is based on an analogous result for solutions of boundary value problems in classical Sobolev spaces defined on regular domains. The mentioned regularity result has been proved in [19] when $n \geq 3$ (see, Theorem 5.1 of [19]) and it can be easily extended to the planar case, taking into account to apply, at right time, Lemma 4.2 in [20] in place of Lemma 4.2 in [22].

4. Main Results

Let L be the operator defined in (32). Suppose that the coefficients of operator L satisfy the assumptions (h_1),

(h_2)

$$(a_{ij})_{x_h}, \quad a_i \in \overset{\circ}{K}{}_1^{t_1}(\Omega), \ i,j,h = 1,\dots,n, \tag{38}$$

(h_3)

$$\begin{gathered} a = a' + b, \quad a' \in \overset{\circ}{K}{}_2^{t_2}(\Omega), \ b \in \widetilde{K}_2^{t_2}(\Omega), \\ \operatorname*{ess\,inf}_{\Omega}\left(\sigma^2 b\right) = b_o > 0, \end{gathered} \tag{39}$$

where t_1, t_2 are as in (i_2) and σ is the function defined in Remark 1.

Moreover, suppose that the following condition on ρ holds:

(h_4)

$$\lim_{x \to x_o} \sigma_x(x) = \lim_{|x| \to +\infty} \sigma_x(x) = 0 \quad \forall x_o \in S_\rho. \tag{40}$$

We are interested in the study of the Dirichlet problem:

$$u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \quad Lu = f, \quad f \in L_s^p(\Omega), \tag{41}$$

with $p \in]1, +\infty[$ and $s \in \mathbb{R}$.

We point out that the unique solvability of (41) has been firstly proved in [4, 5] for $p \geq 2$ and $n \geq 3$. Later on, the results of [4, 5] have been used in [6] to get the existence and uniqueness of solution also for $1 < p < 2$. Here, we collect the main results of the above-mentioned papers and we also extend them to the case $n = 2$.

For the case where the assumptions (h_0)–(h_4) are taken into account, with $t_1 > n$, $t_2 > n/2$ and $a' = 0$, and for $p = 2$, the unique solvability of (41) has been proved in Lemma 4.1 of [5], which can be easily extended to the planar case, applying our Lemmas 4 and 5 at the right time. If $p \neq 2$, the idea is to exploit the previous case and the unique solvability of a problem similar to (41), whose associated differential operator L_{s-1} differs from that in (32) for a compact operator.

Let $p \in]1, +\infty[$, $s \in \mathbb{R}$ and consider in Ω the differential operator

$$\begin{aligned} L_{s-1} = &- \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} \\ &- \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(\left(a_{ij}\right)_{x_j} + 2(s-1) \right. \right. \\ &\left. \left. \times a_{ij} \, \sigma^{-1} \, \sigma_{x_j} \right) \right) \frac{\partial}{\partial x_i} + b. \end{aligned} \tag{42}$$

In the following Lemma we put together some results of [4–6], showing an a priori estimate for the operator L_{s-1} and the unique solvability of its associated Dirichlet problem.

Lemma 6. *Suppose that conditions (h_0)–(h_4) hold. Then for any $p \in]1, +\infty[$ and $s \in \mathbb{R}$ there exists $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \left\| L_{s-1} u \right\|_{L_s^p(\Omega)} \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \tag{43}$$

where $c \in \mathbb{R}_+$ depends on Ω, n, p, t_1, t_2, s, ρ, ν_o, $\left\|a_{ij}^o\right\|_{L^\infty(\Omega_o)}$, $\left\|(a_{ij})_{x_h}\right\|_{K_1^{t_1}(\Omega)}$, $\|b\|_{K_2^{t_2}(\Omega)}$, $\eta[a_{ij}^o]$, $\tilde{\omega}_1^{t_1}[(a_{ij})_{x_h}]$, and $\tilde{\omega}_2^{t_2}[b]$.

Furthermore, if $t_1 > n$ and $t_2 > n/2$ then the problem

$$u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \quad L_{s-1} u = f, \quad f \in L_s^p(\Omega), \tag{44}$$

is uniquely solvable.

Here, we only give an overview of the proof, pointing out to the crucial aspects. Since the operator L_{s-1} verifies the same assumptions of the operator L, we can write for it the estimate (35). Now a crucial point is to provide a bound for $\|u\|_{L_{s-2}^p(\Omega)}$ in terms of $\left\|L_{s-1}u\right\|_{L_s^p(\Omega)}$, when the function u is more regular; that is, $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,2}(\Omega) \cap \mathfrak{D}^0(\overline{\Omega} \setminus S_\rho)$. To this aim, we build the following bilinear form a_{s-1} related, in appropriate way, to the operator L_{s-1}:

$$a_{s-1}(u, w) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} u_{x_i} w_{x_j} + buw \right) \sigma^{2(s-1)} dx, \tag{45}$$

where $u, w \in \overset{\circ}{W}{}_{s-1}^{1,2}(\Omega)$.

By simple computations, we get

$$a_{s-1}(u, w) = \int_\Omega L_{s-1} u w \sigma^{2(s-1)} \, dx. \tag{46}$$

Now we study separately the cases $p \geq 2$ and $p < 2$. If $p \geq 2$, applying Lemma 4.1 of [4] in (46), with a suitable choice of function w, we easily deduce the claimed bound on $\|u\|_{L_{s-2}^p(\Omega)}$. Thus, from Lemma 3 we get the estimate (43). Furthermore, in this case, the uniqueness of the solution of problem (44) easily follows from (43). For the existence of solution, we refer to Theorem 4.2 in [5], whose analytic technique exploits the bound (43), which also gives the closure of the range of operator L_{s-1}, the unique solvability of our problem when $p = 2$, and the regularity result of Lemma 5, which allow us to go up in summability.

Suppose now $1 < p < 2$. In this case, in order to get a bound on $\|u\|_{L_{s-2}^p(\Omega)}$, when u is more regular, we use a variational result (see the proof of Lemma 3.3 in [6]) whose main analytic tools exploit the existence and uniqueness result of previous case and Lemma 5. The mentioned variational result gives us a bound for $\|w\|_{L_{s-2}^{p'}(\Omega)}$, where w is the unique solution of the Dirichlet problem associated with $a_{s-1}(u, w)$ and p' is the conjugate exponent of p. From such estimate and (46), with a suitable choice of function w, we obtain our claim. Using again Lemma 3, we get (43) also in this case. Finally, for the existence and uniqueness of problem (44) we can refer to Lemma 3.4 of [6], whose proof can be easily extended also to the planar case, in view of previous considerations.

In the following Theorem we collect two results of [4, 6], giving an a priori bound for the operator L when $n \geq 2$.

Theorem 7. *Suppose that conditions (h_0)–(h_4) hold. Then for any $p \in]1, +\infty[$ and $s \in \mathbb{R}$ there exist $c \in \mathbb{R}_+$ and a bounded open set $\Omega_1 \subset\subset \Omega$, with the cone property, such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_1)} \right) \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \tag{47}$$

where $c \in \mathbb{R}_+$ and Ω_1 depend on Ω, n, p, t_1, t_2, s, ρ, ν_o, $\|a_{ij}^o\|_{L^\infty(\Omega_o)}$, $\|(a_{ij})_{x_h}\|_{K_1^{t_1}(\Omega)}$, $\|a_i\|_{K_1^{t_1}(\Omega)}$, $\|a'\|_{K_2^{t_2}(\Omega)}$, $\|b\|_{K_2^{t_2}(\Omega)}$, $\eta[a_{ij}^o]$, $\overset{\circ}{\omega}{}_1^{t_1}\left[\left(a_{ij}\right)_{x_h}\right]$, $\overset{\circ}{\omega}{}_1^{t_1}[a_i]$, $\overset{\circ}{\omega}{}_2^{t_2}[a']$, and $\tilde{\omega}_2^{t_2}[b]$.

Proof. Fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)$. We observe that

$$L = L_{s-1} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + a', \tag{48}$$

where

$$b_i = \sum_{j=1}^{n} \left(\left(a_{ij}\right)_{x_j} + 2(s-1) a_{ij}\, \sigma^{-1} \sigma_{x_j} \right) + a_i, \qquad i = 1, \dots, n. \tag{49}$$

Thus, from estimate (43) we deduce that there exists $c \in \mathbb{R}_+$, depending on Ω, n, p, t_1, t_2, s, ρ, ν_o, $\|a_{ij}^o\|_{L^\infty(\Omega_o)}$, $\|(a_{ij})_{x_h}\|_{K_1^{t_1}(\Omega)}$, $\|b\|_{K_2^{t_2}(\Omega)}$, $\eta[a_{ij}^o]$, $\widetilde{\omega}_1^{t_1}\left[\left(a_{ij}\right)_{x_h}\right]$, and $\widetilde{\omega}_2^{t_2}[b]$, such that

$$\|u\|_{W_s^{2,p}(\Omega)} \le c \left(\|Lu\|_{L_s^p(\Omega)} + \left\| \sum_{i=1}^{n} b_i u_{x_i} + a' u \right\|_{L_s^p(\Omega)} \right). \tag{50}$$

On the other hand, by Theorem 2 it follows that for any $\epsilon \in \mathbb{R}_+$ there exist $c(\epsilon) \in \mathbb{R}_+$ and a bounded open set $\Omega_\epsilon \subset\subset \Omega$, with the cone property, such that

$$\left\| \sum_{i=1}^{n} b_i u_{x_i} + a' u \right\|_{L_s^p(\Omega)} \le \epsilon \|u\|_{W_s^{2,p}(\Omega)} + c(\epsilon) \left(\left\|u_x\right\|_{L^p(\Omega_\epsilon)} + \|u\|_{L^p(\Omega_\epsilon)} \right), \tag{51}$$

where $c(\epsilon)$ and Ω_ϵ depend on ϵ, Ω, n, p, t_1, t_2, s, ρ, $\|(a_{ij})_{x_h}\|_{K_1^{t_1}(\Omega)}$, $\|a_i\|_{K_1^{t_1}(\Omega)}$, $\|a'\|_{K_2^{t_2}(\Omega)}$, $\mathring{\omega}_1^{t_1}[(a_{ij})_{x_h}]$, $\mathring{\omega}_1^{t_1}[a_i]$, and $\mathring{\omega}_2^{t_2}[a']$. The result easily follows from relations (50) and (51). □

Finally, in the next theorem we put together Theorem 4.2 of [5] with Theorem 4.2 of [6], showing the existence and uniqueness of problem (41) for $p \in]1, +\infty[$ and $n \ge 2$.

Theorem 8. *Under the same hypotheses of Theorem 7 with $t_1 > n$ and $t_2 > n/2$, for any $p \in]1, +\infty[$ and $s \in \mathbb{R}$, the problem (41) is an index problem with index equal to zero. Moreover, if $a' = 0$, then the problem is uniquely solvable.*

Proof. First we prove that the problem (41) is an index problem with index equal to zero. In fact, from Lemma 6 we deduce that the operator L_{s-1} is a Fredholm operator with index zero. On the other hand, by Theorem 2 it follows that the operator

$$u \in W_s^{2,p}(\Omega) \longrightarrow \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(\left(a_{ij}\right)_{x_j} + 2(s-1) a_{ij}\, \sigma^{-1}\, \sigma_{x_j} \right) + a_i \right) \times u_{x_i} + a' u \in L_s^p(\Omega) \tag{52}$$

is a compact operator. Thus, from (48), (49), and well known results of the classical Fredholm index theory, we deduce that the problem (41) is an index problem with index equal to zero.

Assume now $a' = 0$. We prove only the uniqueness of solution of problem (41); the existence will easily follow from uniqueness and from what we have proved in the case $a' \ne 0$. At first, suppose $p \ge 2$. Then, by the unique solvability of problem (41) for $p = 2$ there exists a unique $u \in W_t^{2,2}(\Omega) \cap \mathring{W}_{t-1}^{1,2}(\Omega)$ such that $Lu = 0$. Moreover, by our Lemma 5 and by Lemma 3.2 of [5] it follows that u belongs to $W_s^{2,p}(\Omega) \cap \mathring{W}_{s-1}^{1,p}(\Omega)$ and thus we deduce that $u = 0$. Now let $1 < p < 2$ and $u \in W_s^{2,p}(\Omega) \cap \mathring{W}_{s-1}^{1,p}(\Omega)$ such that $Lu = 0$. Using again our Lemma 5 and Lemma 3.2 of [5] we get $u \in W_t^{2,2}(\Omega) \cap \mathring{W}_{t-1}^{1,2}(\Omega)$ with $t = s + n(1/p - 1/2)$ and $Lu = 0$. Exploiting again the existence and uniqueness of (41) for $p = 2$, we obtain that $u = 0$. This concludes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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