

Research Article

Extremal Trees with respect to Number of $(A, B, 2C)$ -Edge Colourings

Krzysztof Piejko

Faculty of Mathematics and Applied Physics, Rzeszów University of Technology, Aleja Powstańców Warszawy 12, 35-959 Rzeszów, Poland

Correspondence should be addressed to Krzysztof Piejko; piejko@prz.edu.pl

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We determine the smallest and the largest number of $(A, B, 2C)$ -edge colourings in trees. We prove that the star is a unique tree that maximizes the number of all of the $(A, B, 2C)$ -edge colourings and that the path is a unique tree that minimizes it.

1. Introduction and Preliminary Results

For a general concept, see [1]. The Fibonacci sequence $\{F_n\}$ is defined recursively by the second-order recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = F_1 = 1$. A related sequence is the Pell sequence $\{P_n\}$ defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with $P_0 = 0, P_1 = 1$. Table 1 includes first terms of the sequence $\{P_n\}$. The terms of Fibonacci and Pell sequences are called Fibonacci numbers and Pell numbers, respectively. The numbers of the Fibonacci type play an important role in distinct areas of mathematics and they have many different applications and interpretations. Some of them are closely related to the Hosoya index $Z(G)$ (defined as a number of all matchings in the graph G , including the empty matching) and the Merrifield-Simmons index $\sigma(G)$ (defined as a number of all independent sets in G , including the empty set); see [2] and its references. It is well-known that $\sigma(\mathcal{P}_n) = F_n$ and $Z(\mathcal{P}_n \circ \mathcal{K}_1) = P_{n+1}$, for $n \geq 1$, where \mathcal{P}_n is an n -vertex path, \mathcal{K}_n is an n -vertex complete graph, and $G \circ H$ denotes the corona of two graphs. The numbers of the Fibonacci type in the graph theory were studied intensively also in [3–15].

Consider a simple, undirected graph G with the vertex set $V(G)$ and the edge set $E(G)$. In [11] we introduced an $(a_1A_1, a_2A_2, \dots, a_pA_p)$ -edge colouring of a graph G defined in the following way. Let G be p -edge coloured graph with the set of colours $\{A_1, A_2, \dots, A_p\}$, where $p \geq 2$. Moreover, let a_1, a_2, \dots, a_p be positive integers. We say that a subgraph of G is M -monochromatic if all its edges are coloured alike

by colour $M \in \{A_1, A_2, \dots, A_p\}$. The graph G is said to be $(a_1A_1, a_2A_2, \dots, a_pA_p)$ -edge coloured, if every maximal (with respect to set inclusion) A_i -monochromatic subgraph of G can be partitioned into edge-disjoint paths of the length a_i , $i = 1, 2, \dots, p$. This type of edge colouring of graph generalizes the edge colouring introduced by Piejko and Włoch in [10] and the edge colouring by monochromatic paths introduced by Trojnar-Spelina and Włoch in [13]. Many interesting results concerning some special kinds of $(a_1A_1, a_2A_2, \dots, a_pA_p)$ -edge colouring of graphs can be found in [10, 11]. We recall some of them.

Let $\{A, B, C\}$ be the set of colours. By $(A, B, 2C)$ -edge colouring we denote the 3-edge colouring of graph G , such that every C -monochromatic subgraph of G can be partitioned into edge-disjoint paths of the even length. Let $\delta(G)$ be the number of all $(A, B, 2C)$ -edge colourings of the graph G . The following result was given in [10].

Theorem 1 (see [10]). *Let $n \geq 2$ be an integer. Then $\delta(\mathcal{P}_n) = P_n$.*

Let $\{s_n\}$ be the sequence defined by the relation $s_n = 2s_{n-1} + (n-1)s_{n-2}$ for $n \geq 2$ with the initial conditions $s_0 = 1, s_1 = 2$. We can find a few first terms of $\{s_n\}$ in Table 1.

The sequence $\{s_n\}$ has many distinct interpretations also in graphs. It is worth mentioning that s_n is the Hosoya index of the corona of the complete graphs \mathcal{K}_n and \mathcal{K}_1 ; that is, $Z(\mathcal{K}_n \circ \mathcal{K}_1) = s_n$ for $n \geq 1$. For more interpretations see [16, 17].

TABLE 1: The first terms of $\{P_n\}$ and $\{s_n\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741
s_n	1	2	5	14	43	142	499	1850	7193	123109	538078	2430355

Another interpretation of the sequence $\{s_n\}$ in graphs, which is closely related to $(A, B, 2C)$ -edge colouring of n -edge star $\mathcal{K}_{1,n}$, was given in [11].

Theorem 2 (see [11]). *Let n be a positive integer. Then $\delta(\mathcal{K}_{1,n}) = s_n$.*

In [12] Prodinger and Tichy proved that the star is a tree that maximizes the Merrifield-Simmons index, while the path is a tree that minimizes it. In this paper we obtain an analogous result for the number of $(A, B, 2C)$ -edge colourings in trees.

Let G and H be given graphs with distinguished vertices $x \in V(G)$ and $y \in V(H)$. By $G^x * H^y$ we denote the graph obtained from G and H by identifying vertices x and y (see Figure 1) and by $G^x + H^y$ we denote the graph obtained from G and H by adding the edge xy (see Figure 2).

For $e \in E(G)$ the notation $G \setminus \{e\}$ means the graph obtained from G by deleting the edge e . We prove the following.

Theorem 3. *Let $x \in V(G)$, $y \in V(H)$ and let $x_1, x_2, \dots, x_k \in V(G)$ be neighbours of x and let $y_1, y_2, \dots, y_l \in V(H)$ be neighbours of y , where k and l are positive integers. Then*

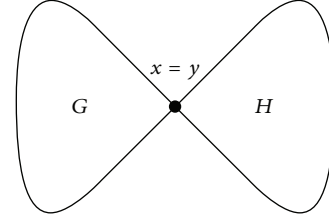
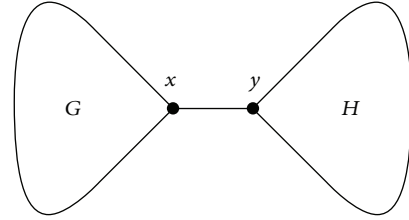
$$\begin{aligned} \delta(G^x + H^y) &= 2\delta(G)\delta(H) + \delta(H) \sum_{i=1}^k \delta(G \setminus \{xx_i\}) \\ &\quad + \delta(G) \sum_{j=1}^l \delta(H \setminus \{yy_j\}), \end{aligned} \quad (1)$$

$$\begin{aligned} \delta(G^x * H^y) &\geq \delta(G)\delta(H) \\ &\quad + \sum_{i=1}^k \sum_{j=1}^l \delta(H \setminus \{yy_j\}) \delta(G \setminus \{xx_i\}). \end{aligned} \quad (2)$$

Furthermore, if $l = 1$ then

$$\begin{aligned} \delta(G^x * H^y) &= \delta(G)\delta(H) \\ &\quad + \delta(H \setminus \{yy_1\}) \sum_{i=1}^k \delta(G \setminus \{xx_i\}). \end{aligned} \quad (3)$$

Proof. By δ_A , δ_B , and δ_C we denote the number of all $(A, B, 2C)$ -edge colourings of the graph $G^x + H^y$ such that an edge xy has a colour A , B , or C , respectively. It can be easily seen that δ_A and δ_B are equal to the number of all $(A, B, 2C)$ -edge colourings of the graph G , multiplied by the number of all $(A, B, 2C)$ -edge colourings of the graph H . Moreover, δ_C is equal to the number of all $(A, B, 2C)$ -edge colourings of the graph H multiplied by the number of all $(A, B, 2C)$ -edge colourings of graphs $G \setminus \{xx_i\}$, where $i = 1, 2, \dots, k$,

FIGURE 1: The graph $G^x * H^y$.FIGURE 2: The graph $G^x + H^y$.

plus the number of all $(A, B, 2C)$ -edge colourings of the graph G multiplied by the number of all $(A, B, 2C)$ -edge colourings of graphs $H \setminus \{yy_j\}$, where $j = 1, 2, \dots, l$. In other words, $\delta_A = \delta_B = \delta(G)\delta(H)$ and

$$\delta_C = \delta(H) \sum_{i=1}^k \delta(G \setminus \{xx_i\}) + \delta(G) \sum_{j=1}^l \delta(H \setminus \{yy_j\}). \quad (4)$$

Since $\delta(G^x + H^y) = \delta_A + \delta_B + \delta_C$, then we obtain equality (1).

By e_i we denote the edge $xx_i \in E(G)$, where $i = 1, 2, \dots, k$ and by e'_j we denote the edge $yy_j \in E(H)$, where $j = 1, 2, \dots, l$. Assume that $r = \min\{k, l\}$ and for $t \in \{0, 1, 2, \dots, r\}$ let $\zeta(t)$ be the number of all $(A, B, 2C)$ -edge colourings of the graph $G^x * H^y$ with exactly t C -monochromatic paths $\{e, e'\} \subset E(G^x * H^y)$, such that $e \in \{e_1, e_2, \dots, e_k\}$ and $e' \in \{e'_1, e'_2, \dots, e'_l\}$. Observe that $\zeta(0) = \delta(G)\delta(H)$ and

$$\zeta(1) = \sum_{i=1}^k \sum_{j=1}^l \delta(G \setminus \{xx_i\}) \delta(H \setminus \{yy_j\}). \quad (5)$$

It should be noted that $\delta(G^x * H^y) = \sum_{t=0}^r \zeta(t)$ and so $\delta(G^x * H^y) \leq \zeta(0) + \zeta(1)$, which gives inequality (2). Moreover, if $l = 1$ then also $r = 1$ and we obtain equality (3). This completes the proof. \square

2. The Largest Number of $(A, B, 2C)$ -Edge Colourings in Trees

In this section we will show that, among all trees with the given number of vertices n , the star $\mathcal{K}_{1,n-1}$ maximizes the

number of $(A, B, 2C)$ -edge colourings. Moreover the star $\mathcal{K}_{1,m-1}$ is the unique tree with such property. To prove it we need the following.

Theorem 4. *Let $m \geq 2$ be an integer. Then for a graph G and arbitrary $x \in V(G)$ one has*

$$\delta(G^x * \mathcal{K}_{1,m}^v) < \delta(G^x * \mathcal{K}_{1,m}^{v_0}), \quad (6)$$

where v is the leaf of $\mathcal{K}_{1,m}$ and v_0 is the center of $\mathcal{K}_{1,m}$.

Proof. Let $V(\mathcal{K}_{1,m}) = \{v_0, v_1, \dots, v_m\}$ and $E(\mathcal{K}_{1,m}) = \{v_0v_1, v_0v_2, \dots, v_0v_m\}$. Let $x \in V(G)$ be the vertex of degree $k \geq 1$ and let $x_1, x_2, \dots, x_k \in V(G)$ be neighbours of x . By (3) in Theorem 3 we have

$$\begin{aligned} \delta(G^x * \mathcal{K}_{1,m}^v) &= \delta(G) \delta(\mathcal{K}_{1,m}) \\ &+ \delta(\mathcal{K}_{1,m-1}) \sum_{i=1}^k \delta(G \setminus \{xx_i\}) \end{aligned} \quad (7)$$

and by inequality (2) in Theorem 3 we have

$$\begin{aligned} \delta(G^x * \mathcal{K}_{1,m}^{v_0}) &\geq \delta(G) \delta(\mathcal{K}_{1,m}) \\ &+ \sum_{i=1}^k \sum_{j=1}^m \delta(\mathcal{K}_{1,m} \setminus \{v_0v_j\}) \delta(G \setminus \{xx_i\}). \end{aligned} \quad (8)$$

It should be noted that $\sum_{j=1}^m \delta(\mathcal{K}_{1,m} \setminus \{v_0v_j\}) = m\delta(\mathcal{K}_{1,m-1})$ and so (8) gives

$$\begin{aligned} \delta(G^x * \mathcal{K}_{1,m}^{v_0}) &\geq \delta(G) \delta(\mathcal{K}_{1,m}) \\ &+ m\delta(\mathcal{K}_{1,m-1}) \sum_{i=1}^k \delta(G \setminus \{xx_i\}). \end{aligned} \quad (9)$$

Since $m \geq 2$, then from (7) and (9) we have $\delta(G^x * \mathcal{K}_{1,m}^v) < \delta(G^x * \mathcal{K}_{1,m}^{v_0})$, which completes the proof. \square

Theorem 5. *Let $n \geq 2$ be an integer and let T_n be a tree with n vertices. Then*

$$\delta(T_n) \leq \delta(\mathcal{K}_{1,n-1}). \quad (10)$$

Proof (by induction on the number of vertices of degree $k \geq 2$ in the tree T_n). Let $T_{n,t}$ be an n -vertex tree with exactly t vertices of degree $k \geq 2$. If $t = 1$ then the result is obvious, because $T_{n,1}$ is an n -vertex star $\mathcal{K}_{1,n-1}$. Assume that inequality (10) holds for $T_{n,t}$ with arbitrary $t \geq 1$. We will prove that it holds for $T_{n,t+1}$. Note that for each tree $T_{n,t+1}$ there exists $x \in V(T_{n,t+1})$, such that $T_{n,t+1}$ is isomorphic to $T_{n-m+1,t}^x * \mathcal{K}_{1,m-1}^v$, where v is the leaf of the star $\mathcal{K}_{1,m-1}$ and $m \geq 3$. Applying Theorem 4 we have

$$\begin{aligned} \delta(T_{n,t+1}) &= \delta(T_{n-m+1,t}^x * \mathcal{K}_{1,m-1}^v) \\ &< \delta(T_{n-m+1,t}^x * \mathcal{K}_{1,m-1}^{v_0}), \end{aligned} \quad (11)$$

where v_0 is the center of the star $\mathcal{K}_{1,m-1}$. Note that $T_{n-m+1,t}^x * \mathcal{K}_{1,m-1}^{v_0}$ is the n -vertex tree with t vertices of the degree $k \geq 2$. Thus, by the induction hypothesis we have $\delta(T_{n,t+1}) \leq \delta(\mathcal{K}_{1,n-1})$, which completes the proof. \square

Remark 6. From Theorems 4 and 5 we can see that the star $\mathcal{K}_{1,n-1}$ is a unique graph which maximizes the number of $(A, B, 2C)$ -edge colourings in trees of given order n .

3. The Smallest Number of $(A, B, 2C)$ -Edge Colourings in Trees

Now we show that, among all trees with the given number of vertices n , the path \mathcal{P}_n minimizes the number of all $(A, B, 2C)$ -edge colourings and that it is the unique tree with such property. To prove it we need some initial results. First we prove the following property of the Pell numbers.

Theorem 7. *Let $q \geq 2$ and $m_i \geq 1$ be integers for $i = 1, 2, \dots, q$. Then*

$$\sum_{i=1}^q P_{m-m_i} P_{m_i} > P_{m-1}, \quad (12)$$

where $m = m_1 + m_2 + \dots + m_q + 1$.

Proof (by induction on q). For $q = 2$ we have

$$\sum_{i=1}^2 P_{m_1+m_2+1-m_i} P_{m_i} = P_{m_1} P_{m_2+1} + P_{m_1+1} P_{m_2} > P_{m_1+m_2}. \quad (13)$$

We can check the above inequality using the well-known identity for the Pell numbers

$$P_n P_{m+1} + P_{n-1} P_m = P_{n+m}. \quad (14)$$

Assume that inequality (12) holds for an arbitrary $q \geq 2$. We show that it holds for $q + 1$; namely,

$$\sum_{i=1}^{q+1} P_{m+m_{q+1}-m_i} P_{m_i} > P_{m+m_{q+1}-1}, \quad (15)$$

where m_{q+1} is a positive integer and $m = m_1 + m_2 + \dots + m_q + 1$. Using (14) and the induction hypothesis we obtain

$$\begin{aligned} \sum_{i=1}^{q+1} P_{m+m_{q+1}-m_i} P_{m_i} &= \sum_{i=1}^q P_{m+m_{q+1}-m_i} P_{m_i} + P_m P_{m_{q+1}} \\ &= \sum_{i=1}^q \left[P_{m-m_i+1} P_{m_{q+1}} + P_{m-m_i} P_{m_{q+1}-1} \right] P_{m_i} + P_m P_{m_{q+1}} \\ &= P_{m_{q+1}} \sum_{i=1}^q P_{m-m_i+1} P_{m_i} + P_{m_{q+1}-1} \sum_{i=1}^q P_{m-m_i} P_{m_i} \\ &\quad + P_m P_{m_{q+1}} \\ &> P_{m_{q+1}} P_{m-1} + P_{m_{q+1}-1} P_{m-1} + P_m P_{m_{q+1}} \\ &> P_{m-1} P_{m_{q+1}-1} + P_m P_{m_{q+1}} = P_{m+m_{q+1}-1}, \end{aligned} \quad (16)$$

which ends the proof. \square

Let $m_1, m_2, \dots, m_q, q \geq 2$, be positive integers. By $\mathcal{S}_{m_1, m_2, \dots, m_q}$ we denote the subdivision of a star $\mathcal{K}_{1, q}$ in such way that in the resulting graph an i th edge of $\mathcal{K}_{1, q}$ is replaced by a path of length $m_i, i = 1, 2, \dots, q$. For $m_1 = m_2 = \dots = m_q = 1$ the graph $\mathcal{S}_{1, 1, \dots, 1}$ is a star $\mathcal{K}_{1, q}$ and for $q = 2$ the graph \mathcal{S}_{m_1, m_2} is a path $\mathcal{P}_{m_1+m_2+1}$. For $q = 3$ the star subdivision $\mathcal{S}_{m_1, m_2, m_3}$ is called a tripod. In [11] we proved that

$$\delta(\mathcal{S}_{m_1, m_2, m_3}) = P_{m_1+m_2+m_3+1} + 2P_{m_1}P_{m_2}P_{m_3} \quad (17)$$

for all positive integers m_1, m_2 , and m_3 . We will use the following notations:

$$\begin{aligned} V(\mathcal{S}_{m_1, m_2, \dots, m_q}) &= \{y_0\} \cup \{y_1^1, y_2^1, \dots, y_{m_1}^1\} \\ &\cup \{y_1^2, y_2^2, \dots, y_{m_2}^2\} \cup \dots \\ &\cup \{y_1^q, y_2^q, \dots, y_{m_q}^q\}, \\ E(\mathcal{S}_{m_1, m_2, \dots, m_q}) &= \{y_0y_1^1, y_1^1y_2^1, \dots, y_{m_1-1}^1y_{m_1}^1\} \\ &\cup \{y_0y_1^2, y_1^2y_2^2, \dots, y_{m_2-1}^2y_{m_2}^2\} \\ &\cup \dots \\ &\cup \{y_0y_1^q, y_1^qy_2^q, \dots, y_{m_q-1}^qy_{m_q}^q\}. \end{aligned} \quad (18)$$

Theorem 8. Let $m_1, m_2, \dots, m_q, q \geq 3$, be positive integers. Then

$$\delta(\mathcal{S}_{m_1, m_2, \dots, m_q}) > P_{m_1+m_2+\dots+m_q+1}. \quad (19)$$

Proof (by induction on q). If $q = 3$ then the result we have immediately from (17). Assume that the inequality holds for $t = 3, 4, \dots, q$, with arbitrary $q \geq 3$. We will prove that it holds for $t = q + 1$. Note that $\mathcal{S}_{m_1, \dots, m_q, m_{q+1}}$ is isomorphic to $\mathcal{S}_{m_1, \dots, m_q}^{y_0} * \mathcal{P}_{m_{q+1}+1}^v$, where y_0 is the center of $\mathcal{S}_{m_1, \dots, m_q}$, v is the leaf of $\mathcal{P}_{m_{q+1}+1}$, and m_{q+1} is a positive integer. Thus, applying (3) of Theorem 3 we obtain

$$\begin{aligned} &\delta(\mathcal{S}_{m_1, \dots, m_q, m_{q+1}}) \\ &= \delta(\mathcal{P}_{m_{q+1}+1}) \delta(\mathcal{S}_{m_1, \dots, m_q}) \\ &\quad + \delta(\mathcal{P}_{m_{q+1}}) \sum_{i=1}^q \delta(\mathcal{S}_{m_1, \dots, m_q} \setminus \{y_0y_1^i\}). \end{aligned} \quad (20)$$

Note that for all $i \in \{1, 2, \dots, q\}$

$$\begin{aligned} &\delta(\mathcal{S}_{m_1, \dots, m_q} \setminus \{y_0y_1^i\}) \\ &= \delta(\mathcal{S}_{m_1, m_2, \dots, m_{i-1}, m_{i+1}, \dots, m_q}) \delta(\mathcal{P}_{m_i}). \end{aligned} \quad (21)$$

By induction hypothesis we have $\delta(\mathcal{S}_{m_1, m_2, \dots, m_{i-1}, m_{i+1}, \dots, m_q}) > P_{m-m_i}$ for all $i \in \{1, 2, \dots, q\}$ and $m = m_1 + m_2 + \dots + m_q + 1$. Therefore (21) and Theorems 1 and 7 give

$$\sum_{i=1}^q \delta(\mathcal{S}_{m_1, \dots, m_q} \setminus \{y_0y_1^i\}) > \sum_{i=1}^q P_{m-m_i} P_{m_i} > P_{m-1}. \quad (22)$$

Thus by (20), (22), and Theorem 1 we have

$$\delta(\mathcal{S}_{m_1, \dots, m_q, m_{q+1}}) > P_{m_{q+1}+1} \delta(\mathcal{S}_{m_1, \dots, m_q}) + P_{m_{q+1}} P_{m-1}. \quad (23)$$

By the induction hypothesis we have $\delta(\mathcal{S}_{m_1, \dots, m_q}) > P_m$ and so (23) gives

$$\delta(\mathcal{S}_{m_1, \dots, m_q, m_{q+1}}) > P_{m_{q+1}+1} P_m + P_{m_{q+1}} P_{m-1}. \quad (24)$$

Using identity (14) we have $\delta(\mathcal{S}_{m_1, \dots, m_q, m_{q+1}}) > P_{m+m_{q+1}} = P_{m_1+\dots+m_q+m_{q+1}+1}$, which completes the proof. \square

Theorem 9. Let $m_1, m_2, \dots, m_q, q \geq 2$, be positive integers and let $m = m_1 + m_2 + \dots + m_q + 1$. Then for a graph G and arbitrary $x \in V(G)$ one has

$$\delta(G^x + \mathcal{S}_{m_1, m_2, \dots, m_q}^{y_0}) > \delta(G^x + \mathcal{P}_m^v), \quad (25)$$

where y_0 is the center of $\mathcal{S}_{m_1, m_2, \dots, m_q}$ and v is the leaf of the path \mathcal{P}_m .

Proof. Let $x \in V(G)$ be the vertex of degree $k \geq 1$ and let $x_1, x_2, \dots, x_k \in V(G)$ be neighbours of x . Note that $G^x + \mathcal{S}_{m_1, m_2, \dots, m_q}^{y_0}$ is isomorphic to $G^x * \mathcal{S}_{m_1, m_2, \dots, m_q, 1}^{y_0}$, where $\mathcal{S}_{m_1, m_2, \dots, m_q, 1}$ is the graph obtained from $\mathcal{S}_{m_1, m_2, \dots, m_q}$ by adding a vertex y and an edge y_0y . Therefore, equality (3) of Theorem 3 gives

$$\begin{aligned} &\delta(G^x + \mathcal{S}_{m_1, m_2, \dots, m_q}^{y_0}) \\ &= \delta(G) \delta(\mathcal{S}_{m_1, m_2, \dots, m_q, 1}) \\ &\quad + \delta(\mathcal{S}_{m_1, m_2, \dots, m_q}) \sum_{i=1}^k \delta(G \setminus \{xx_i\}). \end{aligned} \quad (26)$$

Moreover equality (1) of Theorem 3 gives

$$\begin{aligned} \delta(G^x + \mathcal{P}_m^v) &= 2\delta(G) \delta(\mathcal{P}_m) \\ &\quad + \delta(\mathcal{P}_m) \sum_{i=1}^k \delta(G \setminus \{xx_i\}) \\ &\quad + \delta(G) \delta(\mathcal{P}_{m-1}). \end{aligned} \quad (27)$$

By (27), Theorem 1, and the definition of (P_n) we obtain

$$\delta(G^x + \mathcal{P}_m^v) = \delta(G) P_{m+1} + P_m \sum_{i=1}^k \delta(G \setminus \{xx_i\}). \quad (28)$$

From Theorem 8 we have $\delta(\mathcal{S}_{m_1, m_2, \dots, m_q, 1}) > P_{m+1}$ and $\delta(\mathcal{S}_{m_1, m_2, \dots, m_q}) \geq P_m$ and so (26) and (28) give $\delta(G^x + \mathcal{S}_{m_1, m_2, \dots, m_q}^{y_0}) > \delta(G^x + \mathcal{P}_m^v)$, which ends the proof. \square

Theorem 10. Let $n \geq 2$ be an integer and let T_n be a tree with n vertices. Then

$$\delta(T_n) \geq \delta(\mathcal{P}_n). \quad (29)$$

Proof (by induction on the number of vertices of degree $k \geq 3$ in the tree T_n). Let $T_{n,t}$ be the n -vertex tree with exactly t vertices of degree $k \geq 3$. If $t = 0$ then the result is obvious, because $T_{n,0}$ is an n -vertex path \mathcal{P}_n . Assume that inequality (29) holds for $T_{n,t}$ with arbitrary $t \geq 0$. We will prove that it holds for $T_{n,t+1}$. Note that for each tree $T_{n,t+1}$ there exists $x \in V(T_{n,t+1})$, such that $T_{n,t+1}$ is isomorphic to $T_{n-m,t}^x + \mathcal{S}_{m_1, m_2, \dots, m_q}^{y_0}$, where y_0 is the center of $\mathcal{S}_{m_1, m_2, \dots, m_q}$, $m = m_1 + m_2 + \dots + m_q + 1$, and $q \geq 2$. Applying Theorem 9 we have

$$\begin{aligned} \delta(T_{n,t+1}) &= \delta\left(T_{n-m,t}^x + \mathcal{S}_{m_1, m_2, \dots, m_q}^{y_0}\right) \\ &> \delta\left(T_{n-m,t}^x + \mathcal{P}_m^v\right), \end{aligned} \quad (30)$$

where v is the leaf of the path \mathcal{P}_m . Note that $T_{n-m,t}^x + \mathcal{P}_m^v$ is the n -vertex tree with t vertices of degree $k \geq 3$. Thus by the induction hypothesis we have $\delta(T_{n,t+1}) \geq \delta(\mathcal{P}_n)$ and the proof is complete. \square

Remark 11. From Theorems 9 and 10 we can see that the path \mathcal{P}_n is a unique graph which minimizes the number of $(A, B, 2C)$ -edge colourings in trees of given order n .

From previous theorems we have also the following.

Corollary 12. *If T_n is a tree with the number of vertices $n \geq 2$, then*

$$P_n \leq \delta(T_n) \leq s_{n-1}. \quad (31)$$

Moreover, if T_n is different from \mathcal{P}_n and $\mathcal{K}_{1, n-1}$, then

$$P_n < \delta(T_n) < s_{n-1}. \quad (32)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

- [1] R. Diestel, *Graph Theory*, Springer, New York, NY, USA, 2005.
- [2] S. Wagner and I. Gutman, "Maxima and minima of the Hosoya index and the Merrifield-Simmons index, a survey of results and techniques," *Acta Applicandae Mathematicae*, vol. 112, no. 3, pp. 323–346, 2010.
- [3] U. Bednarz, D. Bród, K. Piejko, and A. Włoch, " (k, α_{n-1}) -Fibonacci numbers and P_k -mathings in multigraphs," *Ars Combinatoria*, vol. 111, pp. 354–355, 2013.
- [4] D. Bród, K. Piejko, and I. Włoch, "Distance Fibonacci numbers, distance Lucas numbers and their applications," *Ars Combinatoria*, vol. 112, pp. 397–409, 2013.
- [5] E. Kiliç, "On the usual Fibonacci and generalized order- k Pell numbers," *Ars Combinatoria*, vol. 88, pp. 33–45, 2008.
- [6] E. Kiliç, "The Binet formula, sums and representations of generalized Fibonacci p -numbers," *European Journal of Combinatorics*, vol. 29, no. 3, pp. 701–711, 2008.
- [7] E. Kiliç, "The generalized Pell (p, i) -numbers and their Binet formulas, combinatorial representations, sums," *Chaos, Solitons and Fractals*, vol. 40, no. 4, pp. 2047–2063, 2009.
- [8] E. Kiliç and P. Stanica, "General approach in computing sums of products of binary sequences," *Hacettepe Journal of Mathematics and Statistics*, vol. 42, no. 1, pp. 1–7, 2013.
- [9] E. Kiliç and D. Tasci, "On the generalized Fibonacci and Pell sequences by Hessenberg matrices," *Ars Combinatoria*, vol. 94, pp. 161–174, 2010.
- [10] K. Piejko and I. Włoch, "On k -distance Pell numbers in 3-edge-coloured graphs," *Journal of Applied Mathematics*, vol. 2014, Article ID 428020, 6 pages, 2014.
- [11] K. Piejko, "On the numbers of $(A, B, 2C)$ -edge colouring in graphs," *Ars Combinatoria*, In press.
- [12] H. Prodinger and R. F. Tichy, "Fibonacci numbers in graphs," *The Fibonacci Quarterly*, vol. 20, pp. 16–21, 1982.
- [13] L. Trojnar-Spelina and I. Włoch, "On Pell numbers and $(k_1 A_1, k_2 A_2, k_3 A_3)$ -edge colouring in graphs," *Ars Combinatoria*, In press.
- [14] I. Włoch and A. Włoch, "Generalized sequences and k -independent sets in graphs," *Discrete Applied Mathematics*, vol. 158, no. 17, pp. 1966–1970, 2010.
- [15] I. Włoch, "On generalized Pell numbers and their graph representations," *Commentationes Mathematicae*, vol. 48, no. 2, pp. 169–175, 2008.
- [16] P. Blasiak, K. A. Penson, A. I. Solomon, A. Horzela, and G. H. Duchamp, "Some useful combinatorial formulas for bosonic operators," *Journal of Mathematical Physics*, vol. 46, Article ID 052110, 6 pages, 2005.
- [17] The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/>.