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Research Article

Lightlike Hypersurfaces of Indefinite Generalized Sasakian Space Forms

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We study lightlike hypersurfaces M of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$, with indefinite trans-Sasakian structure of type (α, β), subject to the condition that the structure vector field of $\overline{M}$ is tangent to M. First we study the general theory for lightlike hypersurfaces of indefinite trans-Sasakian manifold of type (α, β). Next we prove several characterization theorems for lightlike hypersurfaces of an indefinite generalized Sasakian space form.

1. Introduction

Oubiña [1] introduced the notion of indefinite trans-Sasakian manifold of type (α, β). Indefinite Sasakian, Kenmotsu, and cosymplectic manifolds are three important kinds of indefinite trans-Sasakian manifold such that

$$\alpha = 1, \quad \beta = 0; \quad \alpha = 0, \quad \beta = 1; \quad \alpha = \beta = 0, \tag{1}$$

respectively. Alegre et al. [2] introduced indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Indefinite Sasakian, Kenmotsu, and cosymplectic space forms are some kinds of indefinite generalized Sasakian space form such that

$$\begin{gathered} f_1 = \frac{c+3}{4}, \qquad f_2 = f_3 = \frac{c-1}{4}; \\ f_1 = \frac{c-3}{4}, \qquad f_2 = f_3 = \frac{c+1}{4}; \\ f_1 = f_2 = f_3 = \frac{c}{4}, \end{gathered} \tag{2}$$

respectively, where c denotes constant J-sectional curvatures of each of them.

Recently author has been studying the geometry of lightlike hypersurfaces M of indefinite Sasakian [3], Kenmotsu [4], and cosymplectic [5] manifolds. In this paper, we study lightlike hypersurfaces M of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$, with indefinite trans-Sasakian structure of type (α, β), subject to the condition that the structure vector field of $\overline{M}$ is tangent to M. First we study lightlike hypersurfaces of indefinite trans-Sasakian manifold of type (α, β). Next we prove two characterization theorems for lightlike hypersurfaces of an indefinite generalized Sasakian space form such that the following hold.

(i) Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then α is a constant, $\beta = 0$, and

$$f_1 - f_2 = \alpha^2, \qquad f_1 - f_3 = \alpha^2, \qquad f_2 = f_3. \tag{3}$$

(ii) Let M be a screen conformal lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then $f_1 = f_2 = f_3 = 0$.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an *indefinite trans-Sasakian manifold* [1, 2] if there exist a $(1,1)$-type tensor field J, a vector field ζ which is called the *structure vector field,* and a 1-form θ such that

$$J^2X = -X + \theta(X)\zeta, \quad \theta(\zeta) = 1, \quad J\zeta = 0, \quad \theta \circ J = 0, \tag{4}$$

$$\overline{g}(JX, JY) = \overline{g}(X, Y) - \epsilon\theta(X)\theta(Y), \quad \epsilon = \overline{g}(\zeta, \zeta), \tag{5}$$

$$\begin{aligned}\left(\overline{\nabla}_X J\right) Y = \alpha\{\overline{g}(X, Y)\zeta - \epsilon\theta(Y)X\} \\ + \beta\{\overline{g}(JX, Y)\zeta - \epsilon\theta(Y)JX\},\end{aligned} \tag{6}$$

for any vector fields X and Y on $\overline{M}$, where $\epsilon = 1$ or -1 according to the fact that ζ is spacelike or timelike, respectively. In this case, the set $\{J, \zeta, \theta, \overline{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α, β).

In the entire discussion of this paper, we may assume that ζ is unit spacelike; that is, $\epsilon = 1$, without loss of generality. From (4) and (6), we get

$$\overline{\nabla}_X\zeta = -\alpha JX + \beta(X - \theta(X)\zeta), \qquad d\theta(X, Y) = g(X, JY). \tag{7}$$

Let (M, g) be a lightlike hypersurface, with a screen distribution $S(TM)$, of an indefinite trans-Sasakian manifold $\overline{M}$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E. Also donate by (Equation number)$_i$ the ith equation of several equations in (Equation number), for example, $(7)_1$ donates the first equation of the two equations in (7). We use same notations for any others.

We follow Duggal-Bejancu [6] for notations and structure equations used in this paper. It is well known that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique null section N of a unique vector bundle $\operatorname{tr}(TM)$ of rank 1 in $S(TM)^\perp$ satisfying

$$\begin{aligned}\overline{g}(\xi, N) = 1, \qquad \overline{g}(N, N) = \overline{g}(N, X) = 0, \\ \forall X \in \Gamma(S(TM)).\end{aligned} \tag{8}$$

In the following, let X, Y, Z, and W be the vector fields on M, unless otherwise specified. Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{9}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N; \tag{10}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{11}$$

$$\nabla_X\xi = -A^*_\xi X - \tau(X)\xi, \tag{12}$$

where ∇ and ∇^* are the liner connections on M and $S(TM)$, respectively, B and C are the local second fundamental forms on M and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on M and $S(TM)$, respectively, and τ is a 1-form on TM.

Since $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact that $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi)$, we show that B is independent of the choice of $S(TM)$ and satisfies

$$B(X, \xi) = 0. \tag{13}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{14}$$

where η is a 1-form such that

$$\eta(X) = \overline{g}(X, N). \tag{15}$$

But the connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms B and C are related to their shape operators by

$$B(X, Y) = g\left(A^*_\xi X, Y\right), \qquad \overline{g}\left(A^*_\xi X, N\right) = 0, \tag{16}$$

$$C(X, PY) = g(A_N X, PY), \qquad \overline{g}(A_N X, N) = 0. \tag{17}$$

Definition 1. A lightlike hypersurface M of $\overline{M}$ is said to be

(1) *totally umbilical* [6] if there is a smooth function ρ on any coordinate neighborhood $\mathscr{U}$ in M such that $A^*_\xi X = \rho PX$, or equivalently,

$$B(X, Y) = \rho g(X, Y). \tag{18}$$

In case $\rho = 0$ on $\mathscr{U}$, we say that M is *totally geodesic*;

(2) *screen totally umbilical* [6] if there exists a smooth function γ on $\mathscr{U}$ such that $A_N X = \gamma PX$, or equivalently,

$$C(X, PY) = \gamma g(X, Y). \tag{19}$$

In case $\gamma = 0$ on $\mathscr{U}$, we say that M is *screen totally geodesic*;

(3) *screen conformal* [7] if there exists a nonvanishing smooth function φ on $\mathscr{U}$ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y). \tag{20}$$

Denote by $\overline{R}$, R, and R^* the curvature tensors of the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$, the induced connection ∇ on M, and the induced connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that

$$\begin{aligned}&\overline{R}(X,Y)Z\\&\quad=R(X,Y)Z+B(X,Z)A_NY-B(Y,Z)A_NX\\&\qquad+\{(\nabla_XB)(Y,Z)-(\nabla_YB)(X,Z)\\&\qquad\quad+\tau(X)B(Y,Z)-\tau(Y)B(X,Z)\}N,\end{aligned}\tag{21}$$

$$\begin{aligned}&\overline{R}(X,Y)N\\&\quad=-\nabla_X(A_NY)+\nabla_Y(A_NX)+A_N[X,Y]\\&\qquad+\tau(X)A_NY-\tau(Y)A_NX\\&\qquad+\{B(Y,A_NX)-B(X,A_NY)+2d\tau(X,Y)\}N,\end{aligned}\tag{22}$$

$$\begin{aligned}&R(X,Y)PZ\\&\quad=R^*(X,Y)PZ+C(X,PZ)A^*_\xi Y\\&\qquad-C(Y,PZ)A^*_\xi X\\&\qquad+\{(\nabla_XC)(Y,PZ)-(\nabla_YC)(X,PZ)\\&\qquad\quad-\tau(X)C(Y,PZ)+\tau(Y)C(X,PZ)\tau(Y)\}\xi,\end{aligned}\tag{23}$$

$$\begin{aligned}&R(X,Y)\xi\\&\quad=-\nabla^*_X(A^*_\xi Y)+\nabla^*_Y(A^*_\xi X)+A^*_\xi[X,Y]\\&\qquad-\tau(X)A^*_\xi Y+\tau(Y)A^*_\xi X\\&\qquad+\{C(Y,A^*_\xi X)-C(X,A^*_\xi Y)-2d\tau(X,Y)\}\xi.\end{aligned}\tag{24}$$

3. Indefinite Trans-Sasakian Manifolds

Let M be a lightlike hypersurface of a indefinite trans-Sasakian manifold $\overline{M}$ such that ζ is tangent to M. Călin [8] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assume in this paper. It is well known [3, 6] that, for any lightlike hypersurface M of an indefinite almost contact metric manifold $\overline{M}$, $J(TM^\perp)$ and $J(\text{tr}(TM))$ are subbundles of $S(TM)$, of rank 1, and $J(TM^\perp)\cap J(\text{tr}(TM))=\{0\}$. Thus $J(TM^\perp)\oplus J(\text{tr}(TM))$ is a subbundle of $S(TM)$ of rank 2. First, we prove the following results.

Theorem 2. *(1) Let M be a totally umbilical lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$. Then $\alpha=0$ and M is totally geodesic.*

(2) Let M be a screen conformal or screen totally umbilical lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$. Then $\alpha=\beta=0$. In case M is screen totally umbilical, M is totally geodesic.

Proof. Applying $\overline{\nabla}_X$ to $g(\zeta,\xi)=0$ and $g(\zeta,N)=0$, we have

$$B(X,\zeta)=\alpha g(X,J\xi),\qquad C(X,\zeta)=\alpha g(X,JN)+\beta\eta(X).\tag{25}$$

(1) If M is totally umbilical, then, from (18) and $(25)_1$, we have

$$\rho g(X,\zeta)=\alpha g(X,J\xi),\quad\forall X\in\Gamma(TM).\tag{26}$$

Taking $X=\zeta$ and $X=JN$ by turns, we have $\rho=0$ and $\alpha=0$, respectively. As $\rho=0$, M is totally geodesic.

(2) If M is screen conformal, then, from (20) and $(25)_{1,2}$, we have

$$\alpha\varphi g(X,J\xi)=\alpha g(X,JN)+\beta\eta(X).\tag{27}$$

Taking $X=J\xi$ and $X=\xi$ by turns, we have $\alpha=0$ and $\beta=0$, respectively.

If M is screen totally umbilical, then, from (19) and $(25)_2$, we have

$$\gamma g(X,\zeta)=\alpha g(X,JN)+\beta\eta(X).\tag{28}$$

Taking $X=\zeta$, $X=J\xi$ and $X=\xi$ to this equation by turns, we have $\gamma=0$, $\alpha=0$, and $\beta=0$, respectively. As $\gamma=0$, M is screen totally geodesic. □

As $J(TM^\perp)\oplus J(\text{tr}(TM))$ is a subbundle of $S(TM)$ of rank 2, there exists a nondegenerate almost complex distribution D_o with respect to J; that is, $J(D_o)=D_o$, such that

$$\begin{aligned}&S(TM)=\{J(TM^\perp)\oplus J(\text{tr}(TM))\}\oplus_{\text{orth}}D_o,\\&TM=\{J(TM^\perp)\oplus J(\text{tr}(TM))\}\oplus_{\text{orth}}D_o\oplus_{\text{orth}}TM^\perp.\end{aligned}\tag{29}$$

Consider the 2-lightlike almost complex distribution D such that

$$\begin{aligned}&D=TM^\perp\oplus_{\text{orth}}J(TM^\perp)\oplus_{\text{orth}}D_o,\\&TM=D\oplus J(\text{tr}(TM))\end{aligned}\tag{30}$$

and the local lightlike vector fields U and V and their 1-forms such that

$$\begin{aligned}&U=-JN,\quad V=-J\xi,\quad u(X)=g(X,V),\\&v(X)=g(X,U).\end{aligned}\tag{31}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X=SX+u(X)U$. Applying J to this, we have

$$JX=FX+u(X)N,\tag{32}$$

where F is a tensor field of type (1, 1) globally defined on M by

$$FX=JSX.\tag{33}$$

Applying $\overline{\nabla}_X$ to the first two equations of (31) and (32) and using (9), (10), (12), (13), (6), (31), and (32), for any $X, Y \in \Gamma(TM)$, we have

$$B(X,U) = C(X,V), \quad (34)$$

$$\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta, \quad (35)$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \beta u(X)\zeta, \quad (36)$$

$$\begin{aligned}(\nabla_X F)(Y) &= u(Y)A_N X - B(X,Y)U \\ &\quad + \alpha\{g(X,Y)\zeta - \theta(Y)X\} \\ &\quad + \beta\{\overline{g}(JX,Y)\zeta - \theta(Y)FX\}.\end{aligned} \quad (37)$$

Theorem 3. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$. If V or U is parallel with respect to ∇, then $\alpha = \beta = 0$ and $\tau = 0$. If both V and U are parallel with respect to the induced connection ∇, then M is screen totally geodesic.*

Proof. (1) If U is parallel, then, from (32) and (35) we have

$$J(A_N X) - u(A_N X)N + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta = 0. \quad (38)$$

Taking the scalar product with V and ζ to (38) by turns and using (4), we have $\tau = 0$ and $\alpha\eta(X) + \beta v(X) = 0$, respectively. Taking $X = \xi$ and $X = V$ to the second result by turns, we have $\alpha = 0$ and $\beta = 0$, respectively.

(2) If V is parallel with respect to ∇, then, from (32) and (36), we have

$$J(A^*_\xi X) - u(A^*_\xi X)N - \tau(X)V - \beta u(X)\zeta = 0. \quad (39)$$

Taking the scalar product with U to (39) and using (4), we have $\tau = 0$. Taking the scalar product with ζ to (39) and using (4) and $\theta(N) = g(\zeta, N) = 0$, we get $\beta u(X) = 0$. Taking $X = U$ to this result, we have $\beta = 0$. From $(25)_1$ and $(31)_3$, we obtain

$$B(X,\zeta) = -\alpha u(X). \quad (40)$$

Applying J to (39) and using (4) and the fact $\tau = \beta = 0$, we have

$$A^*_\xi X = \theta(A^*_\xi X)\zeta + u(A^*_\xi X)U. \quad (41)$$

Taking the scalar product with U to this equation, we get

$$B(X,U) = g(A^*_\xi X, U) = v(A^*_\xi X) = 0. \quad (42)$$

Replacing X by U in (40) and using (42), we get

$$-\alpha = -\alpha u(U) = B(U,\zeta) = 0. \quad (43)$$

Thus $\alpha = \beta = 0$. Then we have

$$A^*_\xi X = u(A^*_\xi X)U. \quad (44)$$

(3) In case V and U are parallel with respect to ∇, as U is parallel, applying J to (38) and using (4), $(25)_2$ and the fact $\tau = \alpha = \beta = 0$, we obtain

$$A_N X = u(A_N X)U, \quad \forall X \in \Gamma(TM). \quad (45)$$

As V is parallel, from (34) and (42), we show that $u(A_N X) = v(A^*_\xi X) = 0$. Thus we obtain $A_N = 0$. Consequently M is screen totally geodesic. □

Theorem 4. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$. If F is parallel with respect to the connection ∇, then we have $\alpha = \beta = 0$. Furthermore D and $J(\mathrm{tr}(TM))$ are parallel distributions on M and M is locally a product manifold $\mathscr{C}_u \times M^\#$, where $\mathscr{C}_u$ is a null curve tangent to $J(\mathrm{tr}(TM))$ and $M^\#$ is a leaf of D.*

Proof. If F is parallel with respect to ∇, then, taking the scalar product with U to (37) and using the facts $g(\zeta, U) = 0$ and $g(FX, U) = -\eta(X)$, we get

$$u(Y)v(A_N X) - \theta(Y)\{\alpha v(X) - \beta\eta(X)\} = 0. \quad (46)$$

Taking $Y = U$ and $Y = \zeta$ by turns, we get $v(A_N X) = 0$ and $\alpha v(X) - \beta\eta(X) = 0$. Taking $X = V$ and $X = \xi$ to the second equation, we have $\alpha = \beta = 0$.

From (37) we have

$$u(Y)A_N X = B(X,Y)U, \quad B(X,Y) = u(Y)u(A_N X). \quad (47)$$

Taking $Y = V$ and $Y \in \Gamma(D_o)$ in $(47)_2$ by turns, we have $B(X,V) = 0$ and $B(X,Y) = 0$. These results and (13) imply that

$$B(X,Y) = 0, \quad \forall X \in \Gamma(TM), \ Y \in \Gamma(D). \quad (48)$$

By using (4), (9), (12), (14), (32), and (36), we derive

$$\begin{gathered}g(\nabla_X \xi, V) = -g(\xi, \overline{\nabla}_X V) = B(X,V) = 0, \\ g(\nabla_X V, V) = 0, \\ g(\nabla_X Y, V) = -g(Y, \nabla_X V) = g(A^*_\xi X, JY) = B(X, FY) = 0,\end{gathered} \quad (49)$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D_o)$, or equivalently, we get

$$\nabla_X Y \in \Gamma(D), \quad \forall X \in \Gamma(TM), \ \forall Y \in \Gamma(D). \quad (50)$$

This result implies that D is a parallel distribution on M.

Taking the scalar product with $Z \in \Gamma(D_o)$ to $(47)_1$, we get $u(Y)C(X,Z) = 0$ for all $X, Y \in \Gamma(TM)$. Taking $Y = U$ to this, we have

$$C(X,Y) = 0, \quad \forall X \in \Gamma(TM), \ Y \in \Gamma(D_o). \quad (51)$$

For all $X \in \Gamma(TM)$ and $Y \in \Gamma(D_o)$, using (35) we derive

$$\begin{aligned}&g(\nabla_X U, N) = v(A_N X) = 0, \\ &g(\nabla_X U, U) = -g(A_N X, N) = 0, \\ &g(\nabla_X U, Y) = g(F(A_N X), Y) \\ &\qquad = -g(A_N X, JY) = C(X, FY) = 0;\end{aligned} \quad (52)$$

that is, $\nabla_X U \in \Gamma(J(\mathrm{tr}(TM)))$ for all $X \in \Gamma(TM)$. Thus $J(\mathrm{tr}(TM))$ is also parallel. As $TM = D \oplus J(\mathrm{tr}(TM))$, and D and $J(\mathrm{tr}(TM))$ are parallel distributions, by the decomposition theorem of de Rham [9] we have $M = \mathscr{C}_u \times M^{\#}$, where $\mathscr{C}_u$ is a null curve tangent to $J(\mathrm{tr}(TM))$ and $M^{\#}$ is a leaf of D. □

Corollary 5. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\overline{M}$. If F and V are parallel with respect to ∇, then M is totally geodesic and screen totally geodesic.*

Proof. As F is parallel with respect to ∇, we get the two equations of (47). As V is also parallel with respect to ∇, substituting (34) to $(47)_2$ and using (42), we have $B = 0$. Thus M is totally geodesic. Replacing Y by U to $(47)_1$, we obtain $A_N = 0$. Thus M is also screen totally geodesic. □

4. Indefinite Generalized Sasakian Space Form

An indefinite almost contact metric manifold $(\overline{M}, J, \zeta, \theta, \overline{g})$ is said to be an *indefinite generalized Sasakian space form* [2] and denote it by $\overline{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2, and f_3 on $\overline{M}$ such that

$$\begin{aligned} &\overline{R}(X,Y)Z \\ &\quad = f_1\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ &\qquad + f_2\{\overline{g}(X,JZ)JY - \overline{g}(Y,JZ)JX + 2\overline{g}(X,JY)JZ\} \\ &\qquad + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &\qquad\quad + \overline{g}(X,Z)\theta(Y)\zeta - \overline{g}(Y,Z)\theta(X)\zeta\}, \end{aligned} \tag{53}$$

for any vector fields X, Y, and Z on $\overline{M}$.

Theorem 6. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then α is a constant, $\beta = 0$, and*

$$f_1 - f_2 = \alpha^2, \qquad f_1 - f_3 = \alpha^2, \qquad f_2 = f_3. \tag{54}$$

Proof. Comparing the tangential and transversal components of (21) and (53), and using (32), we get

$$\begin{aligned} &R(X,Y)Z \\ &\quad = f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &\qquad + f_2\{\overline{g}(X,JZ)FY - \overline{g}(Y,JZ)FX + 2\overline{g}(X,JY)FZ\} \\ &\qquad + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &\qquad\quad + \overline{g}(X,Z)\theta(Y)\zeta - \overline{g}(Y,Z)\theta(X)\zeta\} \\ &\qquad + B(Y,Z)A_N X - B(X,Z)A_N Y, \end{aligned} \tag{55}$$

$$\begin{aligned} &(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\qquad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\quad = f_2\{u(Y)\overline{g}(X,JZ) - u(X)\overline{g}(Y,JZ) \\ &\qquad + 2u(Z)\overline{g}(X,JY)\}. \end{aligned} \tag{56}$$

Taking the scalar product with N to (23), we have

$$\begin{aligned} g(R(X,Y)PZ,N) = {} & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ). \end{aligned} \tag{57}$$

Substituting (55) into the last equation and using $(17)_2$, we obtain

$$\begin{aligned} &(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\qquad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\ &\quad = f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\ &\qquad + f_2\{v(Y)\overline{g}(X,JPZ) - v(X)\overline{g}(Y,JPZ) \\ &\qquad\quad + 2v(PZ)\overline{g}(X,JY)\} \\ &\qquad + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ). \end{aligned} \tag{58}$$

Applying ∇_X to $(34)_1$: $B(Y,U) = C(Y,V)$, we have

$$\begin{aligned} &(\nabla_X B)(Y,U) \\ &\quad = (\nabla_X C)(Y,V) + g(A_N Y, \nabla_X V) - g(A^*_\xi Y, \nabla_X U). \end{aligned} \tag{59}$$

Using (25), (32), (34), (35), and (36), the above equation is reduced to

$$\begin{aligned} (\nabla_X B)(Y,U) = {} & (\nabla_X C)(Y,V) - 2\tau(X)C(Y,V) \\ & - \alpha^2 u(Y)\eta(X) - \beta^2 u(X)\eta(Y) \\ & + \alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\ & - g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)). \end{aligned} \tag{60}$$

Substituting this equation and (34) into (56) such that $Z = U$, we get

$$\begin{aligned} &(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V) - \tau(X)C(Y,V) \\ &\qquad + \tau(Y)C(X,V) + (\alpha^2 - \beta^2)\{u(X)\eta(Y) - u(Y)\eta(X)\} \\ &\qquad + 2\alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\ &\quad = f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\overline{g}(X,JY)\}. \end{aligned} \tag{61}$$

Comparing this equation with (58) such that $PZ = V$, we obtain

$$\begin{aligned} &\{f_1 - f_2 - (\alpha^2 - \beta^2)\}[u(Y)\eta(X) - u(X)\eta(Y)] \\ &\quad = 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}. \end{aligned} \tag{62}$$

Taking $X = \xi$ and $Y = U$ and $X = V$ and $Y = U$ by turns, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0. \tag{63}$$

Substituting (32) into (7) and using (9), we have

$$\nabla_X \zeta = -\alpha FX + \beta (X - \theta(X)\zeta), \quad \forall X \in \Gamma(TM). \tag{64}$$

Applying $\overline{\nabla}_X$ to $v(Y) = g(Y, U)$ and using (9), (32), (34), and (35), we get

$$(\nabla_X v)(Y) = v(Y)\tau(X) - \theta(Y)\{\alpha\eta(X) + \beta v(X)\} - g(A_N X, FY). \tag{65}$$

Applying $\overline{\nabla}_X$ to $\eta(Y) = \overline{g}(Y, N)$ and using (4) and (6) we have

$$(\nabla_X \eta)(Y) = -g(A_N X, Y) + \tau(X)\eta(Y). \tag{66}$$

Using (31), the equation $(25)_2$ is reduced to

$$C(Y, \zeta) = -\alpha v(Y) + \beta\eta(Y). \tag{67}$$

Applying ∇_X to this equation and using (64), (65), and (66), we have

$$\begin{aligned}(\nabla_X C)(Y, \zeta) &= -(X\alpha) v(Y) + (X\beta)\eta(Y) - \alpha\tau(X) v(Y) \\ &\quad + \alpha^2\theta(Y)\eta(X) + \beta^2\theta(X)\eta(Y) \\ &\quad - \beta\{g(X, A_N Y) + g(A_N X, Y) - \tau(X)\eta(Y)\} \\ &\quad + \alpha\{g(A_N X, FY) + g(A_N Y, FX)\}.\end{aligned} \tag{68}$$

Substituting this and (67) into (58) such that $PZ = \zeta$, we get

$$\{X\beta + A\theta(X)\}\eta(Y) - \{Y\beta + A\theta(Y)\}\eta(X) = (X\alpha) v(Y) - (Y\alpha) v(X), \tag{69}$$

where $A = f_1 - f_3 - (\alpha^2 - \beta^2)$. Taking $X = \xi$ and $Y = \zeta$ and then taking $X = U$ and $Y = V$ to this equation, we obtain

$$f_1 - f_3 = (\alpha^2 - \beta^2) - \zeta\beta, \qquad U\alpha = 0. \tag{70}$$

Applying $\overline{\nabla}_X$ to $u(Y) = g(Y, V)$ and using (9), (32), and (36), we get

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - \beta\theta(Y) u(X) - B(X, FY). \tag{71}$$

Applying ∇_Y to (40) and using (40) and (64) and (71), we have

$$(\nabla_X B)(Y, \zeta) = -(X\alpha) u(Y) - \beta B(X, Y) + \alpha\{u(Y)\tau(X) + B(X, FY) + B(Y, FX)\}. \tag{72}$$

Substituting this into (56) such that $Z = \zeta$ and using the fact that $U\alpha = 0$, we have $(X\alpha)u(Y) = 0$. Therefore the function α is a constant.

From the facts that α is a constant and $\alpha\beta = 0$, if $\alpha \neq 0$, then we get $\beta = 0$.

Assume that $\alpha = 0$. Then (64) is reduced to

$$\nabla_Y \zeta = \beta(Y - \theta(Y)\zeta). \tag{73}$$

By straightforward calculations form this equation, we obtain

$$\begin{aligned}R(X, Y)\zeta = &(X\beta) Y - (Y\beta) X - \{(X\beta)\theta(Y) - (Y\beta)\theta(X)\}\zeta \\ &+ \beta^2\{\theta(X) Y - \theta(Y) X\} - 2\beta d\theta(X, Y)\zeta.\end{aligned} \tag{74}$$

Comparing this equation with (55) such that $Z = \zeta$, we obtain

$$\begin{aligned}&(X\beta) Y - (Y\beta) X - \{(X\beta)\theta(Y) - (Y\beta)\theta(X)\}\zeta \\ &\quad + \beta^2\{\theta(X) Y - \theta(Y) X\} - 2\beta d\theta(X, Y)\zeta \\ &= (f_1 - f_3)\{\theta(Y) X - \theta(X) Y\}.\end{aligned} \tag{75}$$

Taking the scalar product with ζ to this equation, we get $\beta d\theta(X, Y) = 0$; that is,

$$\beta g(X, JY) = 0, \quad \forall X, Y \in \Gamma(TM), \tag{76}$$

due to $(32)_2$. Taking $X = U$ and $Y = \xi$ to this equation, we have $\beta = 0$.

As $\beta = 0$, (63) and (70) are reduced to $f_1 - f_2 = \alpha^2$ and $f_1 - f_3 = \alpha^2$, respectively. From these two results, we get $f_2 = f_3$. □

Corollary 7. *There exist no indefinite generalized Sasakian space forms, endowed with β-Kenmotsu structure, admitting a lightlike hypersurface.*

Corollary 8. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$, endowed with α-Sasakian structure. Then $\alpha = \pm 1$.*

Theorem 9. *Let M is lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. If M is screen totally umbilical, then $f_1 = f_2 = f_3 = 0$.*

Proof. As M is screen totally umbilical, $\alpha = \beta = C = 0$ by (2) of Theorem 2. Thus (58) is reduced to

$$\begin{aligned}&f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &\quad + f_2\{v(Y)\overline{g}(X, JPZ) - v(X)\overline{g}(Y, JPZ) \\ &\qquad + 2v(PZ)\overline{g}(X, JY)\} \\ &\quad + f_3\{\theta(X)\theta(PZ)\eta(Y) - \theta(Y)\theta(PZ)\eta(X)\} = 0,\end{aligned} \tag{77}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Y by ξ to this equation, we obtain

$$f_1 g(X, PZ) + f_2\{v(X) u(PZ) + 2u(X) v(PZ)\} - f_3\theta(X)\theta(PZ) = 0. \tag{78}$$

Taking $X = V$, $PZ = U$; $X = U$, $PZ = V$, and $X = PZ = \zeta$ by turns, we have

$$f_1 + f_2 = 0, \qquad f_1 + 2f_2 = 0, \qquad f_1 = f_3. \tag{79}$$

From the first two equations we show that $f_2 = 0$. As $\alpha = \beta = 0$, $\overline{M}$ is an indefinite cosymplectic manifold. Thus $f_1 = f_2 = f_3 = c/4$. This implies $f_1 = f_2 = f_3 = 0$. □

Theorem 10. *Let M be a screen conformal lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then $f_1 = f_2 = f_3 = 0$.*

Proof. Substituting (55) into (57) and using (56), we have

$$\begin{aligned} f_1 &\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\ &+ f_2\{[v(Y) - u(Y)]\overline{g}(X,JPZ) \\ &\quad - [v(X) - u(X)]\overline{g}(Y,JPZ) \\ &\quad + 2[v(PZ) - u(PZ)]\overline{g}(X,JY)\} \\ &+ f_3\{\theta(X)\theta(PZ)\eta(Y) - \theta(Y)\theta(PZ)\eta(X)\} \\ &= \{X[\varphi] - 2\varphi\tau(X)\}B(Y,PZ) \\ &\quad - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X,PZ). \end{aligned} \tag{80}$$

Replacing Y by ξ to the last equation, we obtain

$$\begin{aligned} &\{\xi[\varphi] - 2\varphi\tau(\xi)\}B(X,PZ) \\ &= f_1 g(X,PZ) + f_2\{v(X) - u(X)\}u(PZ) \\ &\quad + 2f_2\{v(PZ) - u(PZ)\}u(X) - f_3\theta(X)\theta(PZ). \end{aligned} \tag{81}$$

Taking $X = PZ = \zeta$ to this equation and using (40), we obtain $f_1 = f_3$. Also taking $X = V$, $PZ = U$, and $X = U$, $PZ = V$ by turns, we have

$$\begin{aligned} \{\xi[\varphi] - 2\varphi\tau(\xi)\}B(V,U) &= f_1 + f_2, \\ \{\xi[\varphi] - 2\varphi\tau(\xi)\}B(U,V) &= f_1 + 2f_2, \end{aligned} \tag{82}$$

respectively. Comparing these two equations, we obtain $f_2 = 0$.

As M is screen conformal, we obtain $\alpha = \beta = 0$ by Theorem 2. As $\alpha = \beta = 0$, we show that $\overline{M}$ is a cosymplectic manifold and $f_1 = f_2 = f_3 = c/4$. Therefore we get $f_1 = f_2 = f_3 = 0$.

Let $R^{(0,2)}$ denote the induced Ricci type tensor of M given by

$$R^{(0,2)}(X,Y) = \operatorname{trace}\{Z \longrightarrow R(Z,X)Y\}, \tag{83}$$

for any $X, Y \in \Gamma(TM)$. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $TM^{\perp} = \operatorname{Span}\{\xi\}$ and $S(TM) = \operatorname{Span}\{W_a\}$. Put $m = \operatorname{rank}(S(TM))$. Using this quasi-orthonormal frame field, we obtain

$$R^{(0,2)}(X,Y) = \sum_{a=1}^{m}\epsilon_a g(R(W_a,X)Y,W_a) + \overline{g}(R(\xi,X)Y,N), \tag{84}$$

for any $X, Y \in \Gamma(TM)$, where $\epsilon_a = g(W_a, W_a)$ is the causal character of W_a. In general, the induced Ricci type tensor $R^{(0,2)}$ is not symmetric [6, 7]. A tensor field $R^{(0,2)}$ of lightlike submanifolds M is called its *induced Ricci tensor* if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*. A lightlike manifold M equipped with an induced Ricci tensor is called *Ricci flat* if its Ricci tensor vanishes. M is called an *Einstein manifold* if the Ricci tensor of M satisfies $Ric = \gamma g$.

If M is a screen conformal lightlike hypersurface of $\overline{M}(f_1, f_2, f_3)$, then, using (55) and the fact that $f_1 = f_2 = f_3 = 0$, we have

$$R^{(0,2)}(X,Y) = \varphi\left\{B(X,Y)\operatorname{tr}A^{*}_{\xi} - g\left(A^{*}_{\xi}X, A^{*}_{\xi}Y\right)\right\}. \tag{85}$$

This implies that $R^{(0,2)}$ is a symmetric induced Ricci tensor *Ric*. □

Theorem 11. *Any screen conformal Einstein lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ is Ricci flat.*

Proof. As M is Einstein, from (85) and the fact $R^{(0,2)} = \gamma g$

$$g\left(A^{*}_{\xi}X, A^{*}_{\xi}Y\right) - \alpha g\left(A^{*}_{\xi}X, Y\right) - \gamma\varphi^{-1}g(X,Y) = 0, \tag{86}$$

where $\alpha = \operatorname{tr}A^{*}_{\xi}$ is trace of A^{*}_{ξ}. Define a nonnull vector field μ on $S(TM)$ by

$$\mu = U - \varphi V. \tag{87}$$

Then μ belongs to $J(TM^{\perp}) \oplus J(\operatorname{tr}(TM))$. Using (20) and (34), μ satisfies

$$B(X,\mu) = 0, \quad \forall X \in \Gamma(TM). \tag{88}$$

From this equation and (16), we show that

$$A^{*}_{\xi}\mu = 0. \tag{89}$$

Taking $X = Y = \mu$ to (86) and using (89), we get $\gamma = 0$. Therefore, M is Ricci flat. □

5. Parallel Structure Fields

Definition 12. Let $\nabla^{\perp}_{X}N = \pi(\overline{\nabla}_{X}N)$ for any $X \in \Gamma(TM)$, where π is the projection morphism of $T\overline{M}$ on $\operatorname{tr}(TM)$. Then $\nabla^{\perp}$ is a linear connection on $\operatorname{ltr}(TM)$. We say that $\nabla^{\perp}$ is the *transversal connection* of M. We define the curvature tensor $R^{\perp}$ of $\operatorname{tr}(TM)$ by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_{X}\nabla^{\perp}_{Y}N - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}N - \nabla^{\perp}_{[X,Y]}N. \tag{90}$$

The transversal connection of M is *flat* [3] if $R^{\perp}$ vanishes.

As $\nabla^{\perp}_{X}N = \tau(X)N$, we show that the transversal connection of M is flat if and only if the 1-form τ is closed; that is, $d\tau = 0$, on any $\mathcal{U} \subset M$ [3].

Denote λ and μ by the 1-forms such that

$$\lambda(X) = B(X,U) = C(X,V), \qquad \delta(X) = B(X,V). \tag{91}$$

Theorem 13. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. If one of the following conditions,*

(1) *F is parallel with respect to the connection ∇,*

(2) *U is parallel with respect to the connection ∇,*

(3) *V is parallel with respect to the connection ∇,*

is satisfied, then $\overline{M}(f_1, f_2, f_3)$ is a flat manifold with indefinite cosymplectic structure and the lightlike transversal connection of M is flat. In case (1), M is also a flat manifold.

Proof. (1) Assume that F is parallel with respect to ∇. Then we get $\alpha = \beta = 0$ by Theorem 4. Thus $f_1 = f_2 = f_3$ by Theorem 6 and (37) is reduced to

$$u(Y) A_N X - B(X,Y) U = 0. \tag{92}$$

Taking $Y = U$ to (92) and using (31), we have

$$A_N X = \lambda(X) U. \tag{93}$$

Taking the scalar product with V to (92) and using (17) and (31), we have

$$g\left(A^*_\xi X, Y\right) = g(\lambda(X) V, Y). \tag{94}$$

As $A^*_\xi X$ and V belong to $S(TM)$ and $S(TM)$ is nondegenerate, we have

$$A^*_\xi X = \lambda(X) V. \tag{95}$$

Taking the scalar product with U to (93), we obtain

$$C(X,U) = 0. \tag{96}$$

Applying ∇_X to $C(Y,U) = 0$ and using (37), (93) and $FU = 0$, we get

$$(\nabla_X C)(Y,U) = 0. \tag{97}$$

Replacing PZ by U to (58) and using the last two equations, we have

$$f_1\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0. \tag{98}$$

Taking $X = V$ and $Y = \xi$ to this equation, we get $f_1 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\overline{M}(f_1, f_2, f_3)$ is flat.

As $f_1 = f_2 = f_3 = 0$, substituting (93) and (95) into (55), we get

$$\begin{aligned} R(X,Y)Z = {} & \{\lambda(Y)\lambda(X) - \lambda(X)\lambda(Y)\} u(Z) U \\ & + \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\} w(Z) W = 0. \end{aligned} \tag{99}$$

Thus M is flat. From (37), (93) and the fact that $FU = \rho = 0$, we get

$$\nabla_X U = \tau(X) U. \tag{100}$$

Substituting this equation into $\nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U = 0$, we get $d\tau = 0$. Thus the transversal connection of M is flat.

(2) If U is parallel with respect to ∇, then, $\alpha = \beta = \tau = 0$ by Theorem 3. Thus $f_1 = f_2 = f_3$ by Theorem 6 and (35) is reduced to

$$J(A_N X) - u(A_N X) N = 0. \tag{101}$$

Applying J to (101) and using (4), (31), and (67), we have

$$A_N X = \lambda(X) U. \tag{102}$$

Taking the scalar product with U to (102), we get

$$C(X,U) = 0. \tag{103}$$

Applying ∇_Y to this and using (35), (102) and the fact that $FU = 0$, we get

$$(\nabla_X C)(Y,U) = 0. \tag{104}$$

Substituting the last two equation into (58) such that $PZ = U$, we have

$$f_1\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0. \tag{105}$$

Taking $X = V$ and $Y = \xi$ to this equation, we obtain $f_1 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\overline{M}(f_1, f_2, f_3)$ is flat. As $\tau = 0$, we obtain $d\tau = 0$. Thus the transversal connection of M is flat.

(3) If V is parallel with respect to ∇, then, $\alpha = \beta = \tau = 0$ by Theorem 3. Thus $f_1 = f_2 = f_3$ by Theorem 6 and (35) is reduced to

$$J\left(A^*_\xi X\right) - u\left(A^*_\xi X\right) N = 0. \tag{106}$$

Applying J to (106) and using (4) and (40), we have

$$A^*_\xi X = \mu(X) U. \tag{107}$$

Taking the scalar product with U to this equation, we get

$$B(X,U) = 0. \tag{108}$$

Applying ∇_Y to this equation and using (35), we have

$$(\nabla_X B)(Y,U) = -B(Y, F(A_N X)). \tag{109}$$

Substituting the last two equations into (56), we obtain

$$\begin{aligned} & B(X, F(A_N Y)) - B(Y, F(A_N X)) \\ & \quad = f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\overline{g}(X, JY)\}. \end{aligned} \tag{110}$$

Taking $X = \xi$ and $Y = U$ to this equation and using (14) and (108), we obtain $f_2 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\overline{M}(f_1, f_2, f_3)$ is flat. As $\tau = 0$, we obtain $d\tau = 0$. Thus the lightlike transversal connection of M is flat. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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