Research Article

Pareto Optimality Conditions and Duality for Vector Quadratic Fractional Optimization Problems

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One of the most important optimality conditions to aid in solving a vector optimization problem is the first-order necessary optimality condition that generalizes the Karush-Kuhn-Tucker condition. However, to obtain the sufficient optimality conditions, it is necessary to impose additional assumptions on the objective functions and on the constraint set. The present work is concerned with the constrained vector quadratic fractional optimization problem. It shows that sufficient Pareto optimality conditions and the main duality theorems can be established without the assumption of generalized convexity in the objective functions, by considering some assumptions on a linear combination of Hessian matrices instead. The main aspect of this contribution is the development of Pareto optimality conditions based on a similar second-order sufficient condition for problems with convex constraints, without convexity assumptions on the objective functions. These conditions might be useful to determine termination criteria in the development of algorithms.

1. Introduction

There are many contributions, concepts, and definitions that characterize and give the Pareto optimality conditions for solutions of a vector optimization problem (see, for instance, [1, 2]). One of the most important condition is the first-order necessary optimality condition that generalizes the Karush-Kuhn-Tucker (KKT) condition. However, to obtain the sufficient optimality conditions, it is necessary to impose additional assumptions (like convexity and its generalizations) on the objective functions and on the constraint set.

In this paper, we deal with a particular case of vector optimization problem (VOP), where each objective function consists of a ratio of two quadratic functions. Without generalized convexity assumptions on the objective functions, but by imposing some additional assumptions on a linear combination of Hessian matrices, Pareto optimality conditions are obtained and duality theorems are established.

Let us consider the following vector quadratic fractional optimization problem (VQFP):

$$\text{Minimize} \quad \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right) \quad \text{(VQFP)}$$

$$\text{subject to} \quad h_j(x) \leqq 0 \quad j \in J, \ x \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, f_i, g_i, $i \in I \equiv \{1, \ldots, m\}$, and h_j, $j \in J \equiv \{1, \ldots, \ell\}$, are continuously differentiable real-valued functions defined on Ω. In addition, we assume that f_i, g_i, $i \in I$, are quadratic functions and $g_i(x) > 0$ for $x \in \Omega$ and $i \in I$. We denote by S the *feasible set* of elements $x \in \Omega$ satisfying $h_j(x) \leqq 0$. We say that x is a *feasible point* if $x \in S$. The value $f_i(x)/g_i(x)$ is the result of the ith objective function if the decision maker chooses the action $x \in S$.

Fractional optimization problems arise frequently in decision making applications, including science management, portfolio selection, cutting and stock, and game theory, in the optimization of the ratio performance/cost, or profit/investment, or cost/time.

There are many contributions dealing with the scalar (single-objective) fractional optimization problem (FP) and vector fractional optimization problem (VFP). In most of them, using convexity or generalized convexity, optimality conditions in the KKT sense and the main duality theorems for optimal points are obtained. With a parametric approach, which transforms the original problem in a simpler associated problem, Dinkelbach [3], Jagannathan [4], and Antczak [5] established optimality conditions, presented algorithms, and applied their approaches in an example (FP) consisting of quadratic functions. Using some known generalized convexity, Antczak [5], Khan and Hanson [6], Reddy and Mukherjee [7], Jeyakumar [8], and Liang et al. [9] established optimality conditions and theorems that relate the pair primal-dual of problem (FP). In Craven [10] and Weir [11], other results for the scalar optimization (FP) can be found.

Further, Liang et al. [12] extended their approach to the vector optimization case (VFP) considering the type duals of Mond and Weir [13], Schaible [14], and Bector [15]. Considering the parametric approach of Dinkelbach [3], Jagannathan [4], and Bector et al. [16] and two classes of generalized convexity, Osuna-Gómez et al. [17] established weak Pareto optimality conditions and the main duality theorems for the differentiable vector optimization case (VFP). Santos et al. [18] deepened these results to the more general nondifferentiable case (VFP). Jeyakumar and Mond [19] used generalized convexity to study the problem (VFP).

Few studies are found involving quadratic functions at both the numerator and denominator of the ratio objective function. Most of them involve the mixing of linear and quadratic functions. The most similar approaches to the scalar quadratic fractional optimization problem (QFP) were considered in [20–24]. On the other hand, Benson [25] considered a pure (QFP) consisting of convex functions, where some theoretical properties and optimality conditions are developed, and an algorithm and its convergence properties are presented.

The closest approaches to the vector optimization case (VQFP) were considered in [26–33]. Using an iterative computational test, Beato et al. [27, 28] characterized the Pareto optimal point for the problem (VQFP), consisting of linear and quadratic functions, and some theoretical results were obtained by using the function linearization technique of Bector et al. [16]. Arévalo and Zapata [26], Konno and Inori [29], and Rhode and Weber [33] analyzed the portfolio selection problem. Kornbluth and Steuer [32] used an adapted Simplex method in the problem (VFP) consisting of linear functions. Korhonen and Yu [30, 31] proposed an iterative computational method for solving the problem (VQFP), consisting of linear and quadratic functions, based on search directions and weighted sums.

The approach taken in this work is different from the previous ones. The main aspect of this contribution is the development of Pareto optimality conditions for a particular vector optimization problem based on a similar second-order sufficient condition for Pareto optimality for problems with convex constraints without the hypothesis of convexity on the objective functions. These conditions might be useful to determine termination criteria in the development of algorithms, and new extensions can be established from these, where more general vector optimization problems in which algorithms are based on quadratic approximations are used locally.

This paper is organized as follows. We start by defining some notations and basic properties in Section 2. In Section 3, the sufficient Pareto optimality conditions are established. In Section 4, the relationship among the associated problems is presented and duality theorems are established. Finally, comments and concluding remarks are presented in Section 5.

2. Preliminaries

Let $\mathbb{R}_+$ denote the nonnegative real numbers and let x^T denote the transpose of the vector $x \in \mathbb{R}^n$. Furthermore, we will adopt the following conventions for inequalities among vectors. If $x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$ and $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$, then

$$x = y \text{ if and only if } x_i = y_i, \forall i \in I;$$
$$x < y \text{ if and only if } x_i < y_i, \forall i \in I;$$
$$x \leqq y \text{ if and only if } x_i \leqq y_i, \forall i \in I;$$
$$x \leq y \text{ if and only if } x \leqq y \text{ and } x \neq y.$$

Similarly we consider the equivalent convention for inequalities $>$, $\geqq$, and $\geq$.

Different optimality definitions for the problem (VQFP) are referred to as Pareto optimal solutions [34], two of which are defined as follows.

Definition 1. Feasible point x^* is said to be a *Pareto optimal solution* of (VQFP), if there does not exist another $x \in S$ such that $f(x)/g(x) \leq f(x^*)/g(x^*)$.

Definition 2. Feasible point x^* is said to be a *weakly Pareto optimal solution* of (VQFP), if there does not exist another $x \in S$ such that $f(x)/g(x) < f(x^*)/g(x^*)$.

Hypotheses of convexity or generalized convexity on the objective functions will be avoided in this work, but we will use such hypotheses on the constraint set. We recall the definition of convexity, where $\nabla f(x)$ denotes the gradient of the function $f : \mathbb{R}^n \to \mathbb{R}$ at the point x.

Definition 3. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function defined on an open convex set Ω and differentiable at $x^* \in \Omega$. f is called *convex* at x^* if, for all $x \in \Omega$, $f(x) - f(x^*) \geqq \nabla f(x^*)^T(x - x^*)$. When f is convex on the set Ω, we simply say that f is convex.

Maeda [35] used the generalized Guignard constraint qualification (GGCQ) [36], to derive the following necessary Pareto optimality conditions for the problem (VOP) in the KKT sense. Assuming differentiability of the objective and the constraint functions, Maeda guarantees the existence of

Lagrange multipliers, all strictly positive, associated with the objective functions.

Lemma 4 (Maeda [35]). *Let x^* be a Pareto optimal solution of* (VQFP). *Suppose that* (GGCQ) *holds at* x^*; *then there exist vectors* $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$ *such that*

$$\sum_{i=1}^m \tau_i \nabla \frac{f_i(x^*)}{g_i(x^*)} + \sum_{j=1}^\ell \lambda_j \nabla h_j(x^*) = 0,$$

$$\sum_{j=1}^\ell \lambda_j h_j(x^*) = 0, \tag{1}$$

$$\tau > 0, \quad \lambda \geqq 0.$$

For each $i \in I$ and $x \in \mathbb{R}^n$, we consider the objective functions defined as $f_i(x) = x^T A_i x + a_i^T x + \bar{a}_i$ and $g_i(x) = x^T B_i x + b_i^T x + \bar{b}_i$, where $A_i, B_i \in \mathbb{R}^{n \times n}$, A_i is symmetric, B_i is symmetric and positive semidefinite, and $a_i, b_i \in \mathbb{R}^n$ and $\bar{a}_i$, $\bar{b}_i \in \mathbb{R}$, with $\bar{b}_i > -(w^{i^T} B_i w^i + b_i^T w^i)$, where w^i is the solution of the system $2B_i x + b_i = 0$; that is, w^i is the point in which the function $x^T B_i x + b_i^T x$ reaches its minimum and this ensures that $g_i(x) > 0$, $\forall x \in \mathbb{R}^n$. We cannot consider the cases where $2B_i x + b_i = 0$ has no solution.

3. Sufficient Optimality Conditions

Without assumptions of generalized convexity, but imposing some additional assumptions on a linear combination of Hessian matrices of the objective functions f_i and g_i, $i \in I$, we provide in the next theorem a sufficient condition that guarantees that a feasible point of (VQFP) is Pareto optimal point. Similar to a second-order sufficient condition for Pareto optimality, this condition explores the intrinsic characteristics of the problem (VQFP).

We assume, unlike the objective functions, that each h_j is convex. Also, given $x^* \in S$, for each $i \in I$, we define the scalar functions $u_i : S \times S \to \mathbb{R}_+ \setminus \{0\}$ and $s_i : S \times S \to \mathbb{R}$ by

$$u_i(x, x^*)$$
$$\equiv \frac{g_i(x^*)}{g_i(x)},$$
$$s_i(x, x^*) \tag{2}$$
$$\equiv \frac{1}{g_i(x)} \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\}.$$

Theorem 5. *Let x^* be a feasible point of* (VQFP). *Suppose that the constraint function h_j is convex for each $j \in J$ and there exist vectors $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$, such that*

$$\sum_{i=1}^m \tau_i \nabla \frac{f_i(x^*)}{g_i(x^*)} + \sum_{j=1}^\ell \lambda_j \nabla h_j(x^*) = 0, \tag{3}$$

$$\sum_{j=1}^\ell \lambda_j h_j(x^*) = 0, \tag{4}$$

$$\tau > 0, \quad \lambda \geqq 0. \tag{5}$$

If, for any $x \in S$, we obtain

$$\sum_{i=1}^m \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)} \geqq 0, \tag{6}$$

then x^* is a Pareto optimal solution for (VQFP).

Proof. Given $x \in S$, we obtain for each $i \in I$

$$f_i(x) - f_i(x^*)$$
$$= (x - x^*)^T A_i(x - x^*) + \nabla f_i(x^*)^T(x - x^*),$$

$$g_i(x) - g_i(x^*)$$
$$= (x - x^*)^T B_i(x - x^*) + \nabla g_i(x^*)^T(x - x^*),$$

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)}$$

$$= \frac{f_i(x) g_i(x^*) - g_i(x) f_i(x^*)}{g_i(x) g_i(x^*)}$$

$$= (f_i(x) g_i(x^*) - f_i(x^*) g_i(x^*) + f_i(x^*) g_i(x^*)$$
$$\quad - g_i(x) f_i(x^*))$$
$$\cdot (g_i(x) g_i(x^*))^{-1}$$

$$= \frac{g_i(x^*) \{f_i(x) - f_i(x^*)\} - f_i(x^*) \{g_i(x) - g_i(x^*)\}}{g_i(x) g_i(x^*)}$$

$$= \frac{1}{g_i(x)} \left\{ (x - x^*)^T A_i(x - x^*) + \nabla f_i(x^*)^T(x - x^*) \right\}$$

$$- \frac{f_i(x^*)}{g_i(x) g_i(x^*)}$$

$$\cdot \left\{ (x - x^*)^T B_i(x - x^*) + \nabla g_i(x^*)^T(x - x^*) \right\}$$

$$= \frac{1}{g_i(x)} \left\{ (x - x^*)^T A_i(x - x^*) \right\}$$

$$- \frac{f_i(x^*)}{g_i(x) g_i(x^*)} \left\{ (x - x^*)^T B_i(x - x^*) \right\}$$

$$+ \frac{1}{g_i(x)} \left\{ \nabla f_i(x^*)^T(x - x^*) \right\}$$

$$- \frac{f_i(x^*)}{g_i(x) g_i(x^*)} \left\{ \nabla g_i(x^*)^T(x - x^*) \right\}$$

$$= \frac{1}{g_i(x)} \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\}$$

$$+ \frac{g_i(x^*)}{g_i(x)}$$

$$\cdot \left\{ \left[\frac{\nabla f_i(x^*) g_i(x^*) - \nabla g_i(x^*) f_i(x^*)}{[g_i(x^*)]^2} \right]^T (x - x^*) \right\}$$

$$= \frac{1}{g_i(x)} \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\}$$
$$+ \frac{g_i(x^*)}{g_i(x)} \left\{ \left[\nabla \frac{f_i(x^*)}{g_i(x^*)} \right]^T (x - x^*) \right\}.$$

$$(7)$$

Thus, each function f_i/g_i satisfies

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} = u_i(x, x^*) \left[\nabla \frac{f_i(x^*)}{g_i(x^*)} \right]^T (x - x^*)$$
$$+ s_i(x, x^*), \quad \forall x \in S.$$

$$(8)$$

Suppose that x^* is not a Pareto optimal solution of (VQFP). Then there exists another point $x \in S$ such that

$$\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}.$$

$$(9)$$

Since $u_i(x, x^*) > 0$, $i \in I$, from (8) we obtain

$$\frac{1}{u_i(x, x^*)} \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right)$$
$$= \left[\nabla \frac{f_i(x^*)}{g_i(x^*)} \right]^T (x - x^*) + \frac{s_i(x, x^*)}{u_i(x, x^*)}, \quad i \in I.$$

$$(10)$$

From (9), we have

$$\frac{f(x)}{g(x)} - \frac{f(x^*)}{g(x^*)} \leq 0,$$

$$(11)$$

and we obtain m inequalities

$$\left[\nabla \frac{f_i(x^*)}{g_i(x^*)} \right]^T (x - x^*) + \frac{s_i(x, x^*)}{u_i(x, x^*)} \leq 0, \quad i \in I,$$

$$(12)$$

with at least one strict inequality. Multiplying the m inequalities above by their respective $\tau_i > 0$, $i \in I$, and summing all the products, we obtain

$$\sum_{i=1}^{m} \tau_i \left[\nabla \frac{f_i(x^*)}{g_i(x^*)} \right]^T (x - x^*) + \sum_{i=1}^{m} \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)} < 0.$$

$$(13)$$

Then, we have

$$\left[\sum_{i=1}^{m} \tau_i \nabla \frac{f_i(x^*)}{g_i(x^*)} \right]^T (x - x^*) + \sum_{i=1}^{m} \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)} < 0.$$

$$(14)$$

Substituting (3) into (14), we get

$$\left[-\sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) \right]^T (x - x^*) + \sum_{i=1}^{m} \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)} < 0.$$

$$(15)$$

Using (6) and (15), we obtain

$$0 \leq \sum_{i=1}^{m} \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)} < \left[\sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) \right]^T (x - x^*).$$

$$(16)$$

That is,

$$\left[\sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) \right]^T (x - x^*) > 0.$$

$$(17)$$

On the other hand, by convexity of h_j, we have, for each $j \in J$,

$$h_j(x) - h_j(x^*) \geq \nabla h_j(x^*)^T (x - x^*).$$

$$(18)$$

Since $\lambda_j \geq 0$, $j \in J$, we have

$$\sum_{j=1}^{\ell} \lambda_j \left(h_j(x) - h_j(x^*) \right) \geq \sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*)^T (x - x^*)$$
$$= \left[\sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) \right]^T (x - x^*).$$

$$(19)$$

However, since x is feasible point, condition (4) and $\lambda_j \geq 0$, $j \in J$, imply that

$$\sum_{j=1}^{\ell} \lambda_j \left(h_j(x) - h_j(x^*) \right) \leq 0.$$

$$(20)$$

We conclude that

$$\left[\sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) \right]^T (x - x^*) \leq 0,$$

$$(21)$$

which contradicts (17). Therefore x^* is a Pareto optimal solution for (VQFP). □

The expression $f_i(x)/g_i(x) - f_i(x^*)/g_i(x^*)$ in Theorem 5 is manipulated in a similar manner in [6, 7, 9, 12, 19]; however some generalized convexity on the functions f_i and g_i is imposed. In most of them, for each $i \in I$ and $x \in S$, the hypotheses $f_i(x) \geq 0$, $g_i(x) > 0$, and f_i, $-g_i$ satisfy some generalized convexity. This is not the purpose of this work, but the constraint functions can be assumed in a more general class of convex functions; for example, the generalized convexity of Liang et al. [9] can be used.

In the following, the Pareto optimal solution set is denoted by Eff (VQFP).

Corollary 6. *Let x^* be a feasible point of (VQFP). Suppose that the constraint function h_j is convex for each $j \in J$ and there exist vectors $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$, such that (3), (4), and (5) are valid. If $[A_i - (f_i(x^*)/g_i(x^*))B_i]$ are positive semidefinite matrices for each $i \in I$, then $x^* \in$ Eff (VQFP).*

Proof. By hypothesis, given $x \in S$ and $i \in I$, we obtain

$$(x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \geqq 0$$

$$\implies \tau_i \frac{1}{g_i(x^*)} \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\} \geqq 0$$

$$\implies \sum_{i=1}^m \tau_i \frac{g_i(x)}{g_i(x^*)} \frac{1}{g_i(x)}$$

$$\cdot \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\} \geqq 0$$

$$\implies \sum_{i=1}^m \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)} \geqq 0. \tag{22}$$

Therefore, inequality (6) is valid and the result follows from Theorem 5.

To ensure that inequality (6) is valid, we start exploring the features of the Hessian matrices of the objective functions of (VQFP).

Negative values can occur in each term $\tau_i(s_i(x, x^*)/u_i(x, x^*))$ of the sum $\sum_{i=1}^m \tau_i(s_i(x, x^*)/u_i(x, x^*))$, which depends on each matrix $[A_i - (f_i(x^*)/g_i(x^*))B_i]$, $i \in I$, and the vector $(x - x^*)$. Let us check new conditions for which (6) is satisfied; that is, we want to ensure the result of Theorem 5 by analysing the function

$$Z(x, x^*) \equiv \sum_{i=1}^m \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)}$$

$$= \sum_{i=1}^m \tau_i \frac{g_i(x)}{g_i(x^*)} \frac{1}{g_i(x)}$$

$$\cdot \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\}$$

$$= (x - x^*)^T \left\{ \sum_{i=1}^m \left[\frac{\tau_i}{g_i(x^*)} A_i - \frac{\tau_i f_i(x^*)}{[g_i(x^*)]^2} B_i \right] \right\} (x - x^*). \tag{23}$$

Note that $Z(\cdot, x^*)$ is a quadratic function without the linear part; thus in (VQFP) we obtain $Z(x, x^*) \geqq 0$ on S if and only if $\min_{x \in S} Z(x, x^*) \geqq 0$; that is, we can use the classical results on quadratic optimization to check if $\min_{x \in S} Z(x, x^*) \geqq 0$. The next corollary follows immediately from Theorem 5.

Corollary 7. *Let x^* be a feasible point of (VQFP). Suppose that the constraint function h_j is convex for each $j \in J$ and there exist vectors $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$, such that (3), (4), and (5) are valid. If $\min_{x \in S} Z(x, x^*) \geqq 0$, then $x^* \in \text{Eff}(VQFP)$.* □

Using the previous results to check whether a feasible point is a Pareto optimal solution of (VQFP), we propose the following computational test method.

Pareto Optimality Test

Step 1. Given $x^* \in S$, find the vectors $\tau > 0$ and $\lambda \geqq 0$ such that (3) and (4) are valid. If the vectors τ and λ do not exist, then $x^* \notin \text{Eff}(VQFP)$.

Step 2. Otherwise, solve $Z(\overline{x}, x^*) = \min_{x \in S} Z(x, x^*)$. If $Z(\overline{x}, x^*) \geqq 0$, we say that x^* has passed the Pareto optimality test and $x^* \in \text{Eff}(VQFP)$.

Pareto optimality test starts with a feasible point; then it seeks to solve a system of linear equations containing $m + \ell$ unknowns, τ and λ, the inequalities $\tau > 0$, $\lambda \geqq 0$, and two equalities (3) and (4). If this system has no solution, then the point x^* does not satisfy the first-order necessary condition for Pareto optimality, so the method terminates concluding that $x^* \notin \text{Eff}(VQFP)$. Otherwise, in Step 2, a quadratic optimization problem on S should be solved. If the minimum of the quadratic problem is nonnegative, then the procedure ends, concluding that $x^* \in \text{Eff}(VQFP)$. Otherwise, we say that x^* has not passed the Pareto optimality test. Its complexity lies in solving a system of linear inequalities plus a quadratic optimization problem.

The next results, which address a linear combination of the Hessian matrices, can be used to develop a computational search method.

Looking at the previous Pareto optimality test, if the fixed point x^* is assumed to be a variable y, then the linear system in Step 1 becomes a nonlinear system for the variables $\tau > 0$, $\lambda \geqq 0$, $y \in S$. And the quadratic optimization problem in Step 2 becomes a quadratic optimization problem of the type $\min_{x, y \in S} Z(x, y)$. This raises considerable difficulties. In order to reduce these difficulties, we further explore the characteristics of the matrix

$$\widehat{F}(y^*) \equiv \sum_{i=1}^m \left[\frac{\tau_i}{g_i(y^*)} A_i - \frac{\tau_i f_i(y^*)}{[g_i(y^*)]^2} B_i \right]. \tag{24}$$

One possibility is to search for points y^* such that $\widehat{F}(y^*)$ becomes positive semidefinite. In this case $Z(x, y^*) = (x - y^*)^T \widehat{F}(y^*)(x - y^*) \geqq 0$ depends only on $y^* \in S$.

Consider a fixed point x^*; the next theorem takes advantage of the symmetry and diagonalizations of the matrices A_i and B_i, $i \in I$, to give sufficient Pareto optimality conditions for a feasible point of (VQFP). Consider the usual inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$.

Theorem 8. *Let x^* be a feasible point of (VQFP). Suppose that the constraint function h_j is convex for each $j \in J$ and there exist vectors $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$, such that (3), (4), and (5) are*

valid. Consider also, for each $i \in I$ and $k \in K \equiv \{1, \ldots, n\}$, the following functions:

$$\gamma_i^k(x, x^*, \tau) \equiv \frac{\tau_i}{g_i(x^*)} \left\langle x - x^*, p_i^k \right\rangle^2,$$

$$\eta_i^k(x, x^*, \tau) \equiv \frac{\tau_i f_i(x^*)}{[g_i(x^*)]^2} \left\langle x - x^*, q_i^k \right\rangle^2, \tag{25}$$

where p_i^k and q_i^k are the columns of orthogonal matrices P_i and Q_i, constructed from the normalized eigenvectors of the matrices A_i and B_i, respectively. If for all $x \in S$ the following inequality

$$\mu_k^{A_i} \gamma_i^k(x, x^*, \tau) \geqq \mu_k^{B_i} \eta_i^k(x, x^*, \tau), \quad \forall i \in I, \ \forall k \in K, \tag{26}$$

is valid, where $\mu_k^{A_i}$ and $\mu_k^{B_i}$ are the eigenvalues of the matrices A_i and B_i associated with the eigenvectors p_i^k and q_i^k, respectively, then $x^* \in \text{Eff}(VQFP)$.

Proof. The matrices A_i and B_i, $i \in I$, are diagonalizable and can be rewritten as $A_i = P_i D_{A_i} P_i^T = \sum_{k=1}^n \mu_k^{A_i} p_i^k p_i^{k^T}$ and $B_i = Q_i D_{B_i} Q_i^T = \sum_{k=1}^n \mu_k^{B_i} q_i^k q_i^{k^T}$, where D_{A_i} and D_{B_i} are diagonal matrices, with their diagonal formed by the eigenvalues $\mu_k^{A_i}$ and $\mu_k^{B_i}$, $k \in K$, of the matrices A_i and B_i, respectively. Thus, we obtain

$$\sum_{i=1}^m \tau_i \frac{s_i(x, x^*)}{u_i(x, x^*)}$$

$$= \sum_{i=1}^m \tau_i \frac{g_i(x)}{g_i(x^*)} \frac{1}{g_i(x)}$$

$$\cdot \left\{ (x - x^*)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x - x^*) \right\}$$

$$= \sum_{i=1}^m \left\{ (x - x^*)^T \left[\frac{\tau_i}{g_i(x^*)} A_i - \frac{\tau_i f_i(x^*)}{[g_i(x^*)]^2} B_i \right] (x - x^*) \right\}$$

$$= \sum_{i=1}^m \left\{ (x - x^*)^T \left[\frac{\tau_i}{g_i(x^*)} \left(\sum_{k=1}^n \mu_k^{A_i} p_i^k p_i^{k^T} \right) \right. \right.$$
$$\left. \left. - \frac{\tau_i f_i(x^*)}{[g_i(x^*)]^2} \left(\sum_{k=1}^n \mu_k^{B_i} q_i^k q_i^{k^T} \right) \right] (x - x^*) \right\}$$

$$= \sum_{i=1}^m \left[\sum_{k=1}^n \mu_k^{A_i} \left(\frac{\tau_i}{g_i(x^*)} \left\langle x - x^*, p_i^k \right\rangle^2 \right) \right.$$
$$\left. - \sum_{k=1}^n \mu_k^{B_i} \left(\frac{\tau_i f_i(x^*)}{[g_i(x^*)]^2} \left\langle x - x^*, q_i^k \right\rangle^2 \right) \right]$$

$$= \sum_{i=1}^m \left[\sum_{k=1}^n \mu_k^{A_i} \gamma_i^k(x, x^*, \tau) - \sum_{k=1}^n \mu_k^{B_i} \eta_i^k(x, x^*, \tau) \right]$$

$$= \sum_{i=1}^m \sum_{k=1}^n \left[\mu_k^{A_i} \gamma_i^k(x, x^*, \tau) - \mu_k^{B_i} \eta_i^k(x, x^*, \tau) \right]. \tag{27}$$

Since, for all $x \in S$, we have $\mu_k^{A_i} \gamma_i^k(x, x^*, \tau) \geqq \mu_k^{B_i} \eta_i^k(x, x^*, \tau)$, for all $i \in I$ and $k \in K$, we conclude that $Z(x, x^*) \geqq 0$. Therefore, inequality (6) is valid and the result follows from Theorem 5. $\qquad\square$

Theorem 8 is not simple to use since (26) depends on all points of the feasible set; that is, it depends on the functions $\gamma_i^k(x, x^*, \tau)$, $\eta_i^k(x, x^*, \tau)$, $\forall i \in I$, $\forall k \in K$, and $x \in S$. However, even if, for some $i \in I$ and $k \in K$, $\mu_k^{A_i} \gamma_i^k(x, x^*, \tau) < \mu_k^{B_i} \eta_i^k(x, x^*, \tau)$ occurs, inequality (6) can still be satisfied. In order to obtain (26), we present the next corollary, which follows immediately from the previous theorem.

Corollary 9. *Let x^* be a feasible point of (VQFP). Suppose that the constraint function h_j is convex for each $j \in J$ and there exist vectors $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$, such that (3), (4), and (5) are valid. Consider also, for each $i \in I$ and $k \in K$,*

$$a_{i,k}^+ \equiv \sqrt{\mu_k^{A_i}} p_i^k + \sqrt{\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}} q_i^k,$$

$$a_{i,k}^- \equiv \sqrt{\mu_k^{A_i}} p_i^k - \sqrt{\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}} q_i^k,$$

$$\alpha_{i,k} \equiv \left\langle x^*, a_{i,k}^+ \right\rangle a_{i,k}^- + \left\langle x^*, a_{i,k}^- \right\rangle a_{i,k}^+, \tag{28}$$

$$\beta_{i,k} \equiv \left\langle x^*, a_{i,k}^+ \right\rangle \left\langle x^*, a_{i,k}^- \right\rangle,$$

$$H_{i,k}(x) \equiv x^T \left[a_{i,k}^+ a_{i,k}^{-T} \right] x - \alpha_{i,k}^T x + \beta_{i,k},$$

where p_i^k and q_i^k are the columns of orthogonal matrices P_i and Q_i, constructed from the normalized eigenvectors of the matrices A_i, B_i and $\mu_k^{A_i}$, $\mu_k^{B_i}$ are the eigenvalues of the matrices A_i and B_i associated with the eigenvectors p_i^k and q_i^k, respectively. If, for all $x \in S$, one obtains $H_{i,k}(x) \geqq 0$ for each $i \in I$ and $k \in K$, then $x^* \in \text{Eff}(VQFP)$.

Proof. According to Theorem 8, it is enough to show that, for every feasible point and for all $i \in I$ and $k \in K$, $\mu_k^{A_i} \gamma_i^k(x, x^*, \tau) \geqq \mu_k^{B_i} \eta_i^k(x, x^*, \tau)$ is valid. Given $x \in S$ and a pair $\{i, k\} \in I \times K$, we obtain

$$\mu_k^{A_i} \gamma_i^k(x, x^*, \tau) \geqq \mu_k^{B_i} \eta_i^k(x, x^*, \tau)$$

$$\Longleftrightarrow \frac{\tau_i \mu_k^{A_i}}{g_i(x^*)} \left\langle x - x^*, p_i^k \right\rangle^2 \geqq \frac{\tau_i \mu_k^{B_i} f_i(x^*)}{[g_i(x^*)]^2} \left\langle x - x^*, q_i^k \right\rangle^2$$

$$\Longleftrightarrow \mu_k^{A_i} \left\langle x - x^*, p_i^k \right\rangle^2 \geqq \mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)} \left\langle x - x^*, q_i^k \right\rangle^2$$

$$\Longleftrightarrow \left(\sqrt{\mu_k^{A_i}} \left\langle x - x^*, p_i^k \right\rangle + \sqrt{\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}} \left\langle x - x^*, q_i^k \right\rangle \right)$$

$$\cdot \left(\sqrt{\mu_k^{A_i}} \left\langle x - x^*, p_i^k \right\rangle - \sqrt{\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}} \left\langle x - x^*, q_i^k \right\rangle \right) \geqq 0$$

$$\Longleftrightarrow \left(\left\langle x - x^*, \sqrt{\mu_k^{A_i}}\, p_i^k + \sqrt{\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}}\, q_i^k \right\rangle \right)$$

$$\cdot \left(\left\langle x - x^*, \sqrt{\mu_k^{A_i}}\, p_i^k - \sqrt{\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}}\, q_i^k \right\rangle \right) \geqq 0$$

$$\Longleftrightarrow \left(\left\langle x - x^*, a_{i,k}^+ \right\rangle \right) \left(\left\langle x - x^*, a_{i,k}^- \right\rangle \right) \geqq 0$$

$$\Longleftrightarrow \left(\left\langle x, a_{i,k}^+ \right\rangle - \left\langle x^*, a_{i,k}^+ \right\rangle \right) \left(\left\langle x, a_{i,k}^- \right\rangle - \left\langle x^*, a_{i,k}^- \right\rangle \right) \geqq 0$$

$$\Longleftrightarrow \left\langle x, a_{i,k}^+ \right\rangle \left\langle x, a_{i,k}^- \right\rangle - \left\langle x^*, a_{i,k}^- \right\rangle \left\langle x, a_{i,k}^+ \right\rangle$$
$$- \left\langle x^*, a_{i,k}^+ \right\rangle \left\langle x, a_{i,k}^- \right\rangle + \left\langle x^*, a_{i,k}^+ \right\rangle \left\langle x^*, a_{i,k}^- \right\rangle \geqq 0$$

$$\Longleftrightarrow \left\langle x, a_{i,k}^+ \right\rangle \left\langle x, a_{i,k}^- \right\rangle$$
$$- \left\langle x, \left(\left\langle x^*, a_{i,k}^+ \right\rangle a_{i,k}^- + \left\langle x^*, a_{i,k}^- \right\rangle a_{i,k}^+ \right) \right\rangle$$
$$+ \left\langle x^*, a_{i,k}^+ \right\rangle \left\langle x^*, a_{i,k}^- \right\rangle \geqq 0$$

$$\Longleftrightarrow \left\langle x, a_{i,k}^+ \right\rangle \left\langle x, a_{i,k}^- \right\rangle - \left\langle x, \alpha_{i,k} \right\rangle + \beta_{i,k} \geqq 0$$

$$\Longleftrightarrow x^T \left[a_{i,k}^+ a_{i,k}^{-T} \right] x - \alpha_{i,k}^T x + \beta_{i,k} \geqq 0$$

$$\Longleftrightarrow H_{i,k}(x) \geqq 0.$$

$$(29)$$

Therefore, the result follows from Theorem 8. □

From Corollary 9, if each quadratic function $H_{i,k}(x)$, $\{i,k\} \in I \times K$, is nonnegative in the feasible set, then a feasible point satisfying (3), (4), and (5) is a Pareto optimal solution of (VQFP).

Let $[\overline{H}_{i,k}] \equiv [a_{i,k}^+ a_{i,k}^{-T}] \in \mathbb{R}^{n \times n}$. Then $\beta_{i,k} = x^{*T}[\overline{H}_{i,k}]x^*$, and the nonnegativity of the quadratic $H_{i,k}(x) = x^T[\overline{H}_{i,k}]x - \alpha_{i,k}^T x + \beta_{i,k}$ depends on each matrix $[\overline{H}_{i,k}]$ and each vector $\alpha_{i,k} \in \mathbb{R}^n$, where $\{i,k\} \in I \times K$ and $x^* \in S$. For example, the unconstrained (VQFP) requires that each matrix $[\overline{H}_{i,k}]$ be positive semidefinite and that $\beta_{i,k} \geqq (\alpha_{i,k} - w^T[\overline{H}_{i,k}]w)$, where w is a solution of the system $2[\overline{H}_{i,k}]x = \alpha_{i,k}$.

Corollary 10. *Let x^* be a feasible point of (VQFP). Suppose that the constraint function h_j is convex for each $j \in J$ and there exist vectors $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^\ell$, such that (3), (4), and (5) are valid. If, for each pair $\{i,k\} \in I \times K$, the matrix $[\overline{H}_{i,k}]$ is positive semidefinite and $\alpha_{i,k} = 0$ (see (28)), then $x^* \in \mathrm{Eff}\,(VQFP)$.*

Proof. By hypothesis, for all $x \in S$, we have $H_{i,k}(x) \geqq 0$ for each pair $\{i,k\} \in I \times K$. Therefore, the result follows from Corollary 9. □

Given a pair $\{i,k\} \in I \times K$, writing each entry of the matrix $[\overline{H}_{i,k}] = (\overline{H}_{i,k}(r,s))$ and each entry of the vector $\alpha_{i,k} = (\alpha_{i,k}(r))$ according to the entries of the eigenvectors $p_i^k = (p_i^k(r))$ and $q_i^k = (q_i^k(r))$, where $r, s \in K$, we obtain, for each pair $\{r, s\} \in K \times K$,

$$\overline{H}_{i,k}(r,s) = a_{i,k}^+(r)\, a_{i,k}^-(s)$$

$$= \mu_k^{A_i} p_i^k(r)\, p_i^k(s)$$

$$+ \sqrt{\mu_k^{A_i}\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}} \left(p_i^k(s)\, q_i^k(r) - p_i^k(r)\, q_i^k(s) \right)$$

$$- \mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)}\, q_i^k(r)\, q_i^k(s),$$

$$\overline{H}_{i,k}(r,r) = \mu_k^{A_i} \left(p_i^k(r) \right)^2 - \mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)} \left(q_i^k(r) \right)^2,$$

$$\alpha_{i,k}(r) = 2\mu_k^{A_i} \left\langle x^*, p_i^k \right\rangle p_i^k(r) - 2\mu_k^{B_i} \frac{f_i(x^*)}{g_i(x^*)} \left\langle x^*, q_i^k \right\rangle q_i^k(r).$$

$$(30)$$

We can draw some conclusions from (30). For example, for a fixed pair $\{i,k\} \in I \times K$, the vector $\alpha_{i,k}$ is a linear combination of the eigenvectors p_i^k and q_i^k. If $\mu_k^{A_i} = 0$, $\mu_k^{B_i} = 0$, or $f_i(x^*) = 0$, then $[\overline{H}_{i,k}]$ is a symmetric matrix. Moreover, if $\mu_k^{A_i}\mu_k^{B_i} f_i(x^*) < 0$ and there exists a pair $\{r, s\} \in K \times K$ such that $p_i^k(s) q_i^k(r) \neq p_i^k(r) q_i^k(s)$, then the matrix $[\overline{H}_{i,k}] \notin \mathbb{R}^{n \times n}$. In this case, if there exists $x \in S$ such that $H_{i,k}(x) \in \mathbb{C} \setminus \mathbb{R}$, $H_{i,k}(x) \geqq 0$ does not make sense. However, when (26) is required, it is possible to show that $H_{i,k}(x) \in \mathbb{C} \setminus \mathbb{R}$ is not possible.

The results of Theorems 5 and 8 and their corollaries can be used in order to develop a method of searching for Pareto optimal solutions of (VQFP), and it might be useful to determine the termination criteria in the development of algorithms.

4. Duality

Matrix (24) defines a specific function, and by adding some assumptions about it, we obtain new results, such as, a relationship between the problem (VQFP) and a scalar problem associated with it, and the main duality theorems.

In the scalar optimization problem case, Dinkelbach [3] and Jagannathan [4] used a parametric approach that transforms the fractional optimization problem in a new scalar optimization problem. Similarly, we consider the following *parameterized problem* $(VQFP)_{x^*}$ associated with the problem (VQFP):

$$\text{Minimize} \quad f(x) - \frac{f(x^*)}{g(x^*)} g(x)$$

$$= \left(f_1(x) - \frac{f_1(x^*)}{g_1(x^*)} g_1(x), \ldots, f_m(x) \right.$$

$$\left. - \frac{f_m(x^*)}{g_m(x^*)} g_m(x) \right)$$

$$\text{subject to} \quad h_j(x) \leqq 0 \quad j \in J, \ x \in \Omega,$$

$$(VQFP)_{x^*}$$

where $\Omega \subseteq \mathbb{R}^n$, f_i, g_i, $i \in I$, and h_j, $j \in J$, are defined in (VQFP), and $x^* \in S$.

Using assumptions of generalized convexity, Osuna-Gómez et al. [17] presented the problem (VFP)$_{x^*}$ and obtained necessary and sufficient conditions for weakly Pareto optimality and main duality theorems. The results presented in [3, 4, 17] considered each objective function as $f_i(x) - \alpha_i g_i(x)$, $i \in I$, and they studied the properties of the parameter $\alpha_i \in \mathbb{R}$. Following the ideas presented by Osuna-Gómez et al. [17], we obtain new results by considering directly $\alpha_i \equiv f_i(x^*)/g_i(x^*)$, $i \in I$, where $x^* \in \text{Eff}\,(\text{VQFP})$. However, by imposing hypothesis on the linear combination of matrices $[A_i - (f_i(x^*)/g_i(x^*))B_i]$, $i \in I$ and $x^* \in S$, we consider Pareto optimal solutions rather than weakly Pareto optimal solutions.

To characterize the solutions of the problems (VOP), Geoffrion [37] used the solutions of the associated scalar problems. Similarly, we consider the following *weighted scalar problem* (VQFP)$_{x^*}^w$ associated with the problem (VQFP):

$$\text{Minimize } \sum_{i=1}^m w_i \left(f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right) \qquad \text{(VQFP)}_{x^*}^w$$

$$\text{subject to } h_j(x) \leqq 0 \quad j \in J, \ x \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^n$, f_i, g_i, $i \in I$, and h_j, $j \in J$, are defined in (VQFP), $x^* \in S$ and $w = (w_1, \ldots, w_m)^T \in \mathbb{R}^m$, $w > 0$.

4.1. The Relationship between the Associated Problems. The next theorem and its proof are similar to Lemma 1.1 from [17], when Pareto optimal solutions (not necessarily weak) are considered.

Theorem 11. $x^* \in \text{Eff}\,(VQFP)$ *if and only if* $x^* \in \text{Eff}\,(VQFP)_{x^*}$.

Proof. See Lemma 1.1 in [17], considering "$\leq$" instead of "$<$." $\square$

In Section 3, we define the matrix $\widehat{F}(x^*) = \sum_{i=1}^m [(\tau_i/g_i(x^*))A_i - (\tau_i f_i(x^*)/[g_i(x^*)]^2)B_i]$, where $x^* \in S$ and $\tau \in \mathbb{R}^m$, $\tau > 0$. Let us define now the set $\mathcal{W} = \{w \in \mathbb{R}^m \mid w > 0\}$, the function $F : \mathcal{W} \times S \to \mathbb{R}^{n \times n}$ given by

$$F(w, x) \equiv \sum_{i=1}^m w_i \left[A_i - \frac{f_i(x)}{g_i(x)} B_i \right], \tag{31}$$

and, for each $i \in I$, the functions $F_i : S \to \mathbb{R}^{n \times n}$ given by

$$F_i(x) \equiv A_i - \frac{f_i(x)}{g_i(x)} B_i. \tag{32}$$

Then, we have $\widehat{F}(x^*) = F(\tau/g(x^*), x^*)$, where $\tau/g(x^*) = (\tau_1/g_1(x^*), \ldots, \tau_m/g_m(x^*))^T \in \mathcal{W}$, $F(w, x) = \sum_{i=1}^m w_i F_i(x)$, and we can establish some relations among the associated problems (VQFP), (VQFP)$_{x^*}$, and (VQFP)$_{x^*}^w$.

Theorem 12. *If x^* is an optimal solution of the weighted scalar problem* (VQFP)$_{x^*}^w$, *then* $x^* \in \text{Eff}\,(VQFP)$.

Proof. Suppose that $x^* \notin \text{Eff}\,(\text{VQFP})$; then there exists another point $x \in S$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)}$$

$$\implies f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \leq 0$$

$$\implies \sum_{i=1}^m w_i \left(f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right) < 0 \qquad (33)$$

$$\implies \sum_{i=1}^m w_i \left(f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right)$$

$$< \sum_{i=1}^m w_i \left(f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x^*) \right).$$

This contradicts the minimality of x^* in (VQFP)$_{x^*}^w$. $\square$

Lemma 13. *Let $x^* \in \text{Eff}\,(VQFP)$. Suppose that the constraint qualification (GGCQ) is satisfied at x^*; then there exist vectors $\tau^* > 0$ and $\lambda^* \geqq 0$ such that*

$$\sum_{i=1}^m \tau_i^* \left(\nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) \right) + \sum_{j=1}^\ell \lambda_j^* \nabla h_j(x^*) = 0,$$

$$\sum_{j=1}^\ell \lambda_j^* h_j(x^*) = 0.$$

$$(34)$$

Proof. Let $x^* \in S$, $\mu \in \mathbb{R}^m$, $\mu > 0$, and $\tau_i = \mu_i/g_i(x^*) > 0$, $i \in I$. Then

$$\sum_{i=1}^m \mu_i \nabla \frac{f_i(x^*)}{g_i(x^*)} = \sum_{i=1}^m \tau_i g_i(x^*) \nabla \frac{f_i(x^*)}{g_i(x^*)}$$

$$= \sum_{i=1}^m \tau_i g_i(x^*) \left(\frac{\nabla f_i(x^*) g_i(x^*) - \nabla g_i(x^*) f_i(x^*)}{[g_i(x^*)]^2} \right) \quad (35)$$

$$= \sum_{i=1}^m \tau_i \left(\nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) \right),$$

and if $x^* \in \text{Eff}\,(\text{VQFP})$, by Lemma 4, there exist $\mu^* > 0$ and $\lambda^* \geqq 0$ such that (x^*, μ^*, λ^*) is a critical point, in the KKT sense, of the problem (VQFP). That is,

$$\sum_{i=1}^m \mu_i^* \nabla \frac{f_i(x^*)}{g_i(x^*)} + \sum_{j=1}^\ell \lambda_j^* \nabla h_j(x^*) = 0,$$

$$(36)$$

$$\sum_{j=1}^\ell \lambda_j^* h_j(x^*) = 0.$$

From (35), there exist $\tau^* > 0$, $\tau_i^* = \mu_i^*/g_i(x^*) > 0$, $i \in I$, and $\lambda^* \geqq 0$ such that

$$\sum_{i=1}^{m} \tau_i^* \left(\nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) \right) + \sum_{j=1}^{\ell} \lambda_j^* \nabla h_j(x^*) = 0,$$

$$\sum_{j=1}^{\ell} \lambda_j^* h_j(x^*) = 0.$$

(37)

Therefore, the result is valid. $\square$

Lemma 14. *Let $x^* \in S$. If there exists $w \in \mathcal{W}$, such that the matrix $F(w, x^*)$ is positive semidefinite, then the objective function of* $(VQFP)_{x^*}^{w}$ *is convex.*

Proof. Given $x_1, x_2 \in S$, we have, for each $i \in I$,

$$f_i(x_1) - f_i(x_2) = (x_1 - x_2)^T A_i (x_1 - x_2)$$
$$+ \nabla f_i(x_2)^T (x_1 - x_2),$$

$$g_i(x_1) - g_i(x_2) = (x_1 - x_2)^T B_i (x_1 - x_2)$$
$$+ \nabla g_i(x_2)^T (x_1 - x_2).$$

(38)

Hence, for each objective function of $(VQFP)_{x^*}$, we have

$$\left(f_i(x_1) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x_1) \right)$$

$$- \left(f_i(x_2) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x_2) \right)$$

$$= (f_i(x_1) - f_i(x_2)) - \frac{f_i(x^*)}{g_i(x^*)} (g_i(x_1) - g_i(x_2))$$

$$= \left\{ (x_1 - x_2)^T A_i (x_1 - x_2) + \nabla f_i(x_2)^T (x_1 - x_2) \right\}$$

$$- \frac{f_i(x^*)}{g_i(x^*)}$$

$$\cdot \left\{ (x_1 - x_2)^T B_i (x_1 - x_2) + \nabla g_i(x_2)^T (x_1 - x_2) \right\}$$

$$= (x_1 - x_2)^T \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x_1 - x_2)$$

$$+ \left(\nabla f_i(x_2) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x_2) \right)^T (x_1 - x_2).$$

(39)

If there exists $w \in \mathcal{W}$ such that the matrix $F(w, x^*)$ is positive semidefinite, then

$$\sum_{i=1}^{m} w_i \left(f_i(x_1) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x_1) \right)$$

$$- \sum_{i=1}^{m} w_i \left(f_i(x_2) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x_2) \right)$$

$$= (x_1 - x_2)^T \sum_{i=1}^{m} w_i \left[A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right] (x_1 - x_2)$$

$$+ \sum_{i=1}^{m} w_i \left(\nabla f_i(x_2) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x_2) \right)^T (x_1 - x_2)$$

$$\geqq \left(\sum_{i=1}^{m} w_i \left(\nabla f_i(x_2) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x_2) \right) \right)^T (x_1 - x_2).$$

(40)

Therefore, the objective function of $(VQFP)_{x^*}^{w}$ is convex.

Note that the hypothesis of semidefiniteness on the matrix $F(w, x^*)$ or on the matrices $F_i(x^*)$, $i \in I$, $x^* \in S$, is punctual. However, in the next example, we draw a situation in which, for all $x \in S$ and $i \in I$, we have $y^T F_i(x) y \geqq 0$, for all $y \in S$, and then $y^T F(w, x) y \geqq 0$, for all $y \in S$.

Example. Consider the problem (VQFP), where $S = [-2, 2]$ and for all $x \in S$

$$\frac{f_1(x)}{g_1(x)} = \frac{x - 2}{x^2 + 2},$$

$$\frac{f_2(x)}{g_2(x)} = \frac{2x^2 - x - 1}{x^2 + 1},$$

$$\frac{f_3(x)}{g_3(x)} = \frac{-2x^2 - 2x - 5}{x^2 + x + 1}.$$

(41)

For all these functions, we obtain for all $y \in S$

$$y^T \left[A_1 - \frac{f_1(x)}{g_1(x)} B_1 \right] y = y^2 \left(0 - \frac{x - 2}{x^2 + 2} 1 \right)$$

$$= \frac{y^2 (2 - x)}{x^2 + 2} \geqq 0,$$

$$y^T \left[A_2 - \frac{f_2(x)}{g_2(x)} B_2 \right] y = y^2 \left(2 - \frac{2x^2 - x - 1}{x^2 + 1} 1 \right)$$

$$= \frac{y^2 (x + 3)}{x^2 + 1} \geqq 0,$$

$$y^T \left[A_3 - \frac{f_3(x)}{g_3(x)} B_3 \right] y = y^2 \left(-2 - \frac{-2x^2 - 2x - 5}{x^2 + x + 1} 1 \right)$$

$$= \frac{3y^2}{x^2 + x + 1} \geqq 0.$$

(42)

Therefore, for this example, each point x^* satisfying (3), (4), and (5) is Pareto optimal. For example, for $(\tau_1, \tau_2, \tau_3)^T = (0.5, 1, 0.25)^T$ and $(\lambda_1, \lambda_2)^T = (1, 1)$, we have that $x^* = 0$ is Pareto optimal solution. Likewise, for $(\tau_1, \tau_2, \tau_3)^T \approx (0.62, 1, 0.89)^T$ and $(\lambda_1, \lambda_2)^T = (0, 0)^T$, we have that $x^* = -0.25$ is Pareto optimal solution. $\square$

Theorem 11 shows an equivalence between the associated problems (VQFP) and $(VQFP)_{x^*}$. The next theorem shows a relation between the problems $(VQFP)_{x^*}$ and $(VQFP)_{x^*}^{w}$; then it provides a converse to the Theorem 12.

Theorem 15. *Let $x^* \in \text{Eff}(VQFP)_{x^*}$. Suppose that the constraint qualification (GGCQ) is satisfied at x^* and the constraint function h_j is convex for each $j \in J$. Then there exists $w \in \mathcal{W}$ such that if the matrix $F(w, x^*)$ is positive semidefinite, then x^* is the optimal solution for the weighted scalar problem $(VQFP)_{x^*}^w$.*

Proof. If $x^* \in \text{Eff}(VQFP)_{x^*}$ and satisfies (GGCQ), by Lemma 13, there exist $w > 0$ and $\lambda \geqq 0$, such that

$$\sum_{i=1}^{m} w_i^* \left(\nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) \right) + \sum_{j=1}^{\ell} \lambda_j^* \nabla h_j(x^*) = 0,$$
$$\sum_{j=1}^{\ell} \lambda_j^* h_j(x^*) = 0. \tag{43}$$

Therefore, x^* is a critical point of the weighted scalar problem $(VQFP)_{x^*}^w$, and since $F(w, x^*)$ is positive semidefinite, by Lemma 14, the objective function of $(VQFP)_{x^*}^w$ is convex. Since for each $j \in J$ the constraint function h_j is convex, it follows that x^* is an optimal solution for $(VQFP)_{x^*}^w$. $\square$

4.2. Duality Theorems. For a given mathematical optimization problem there are many types of duality. Two well-known duals are the Wolfe dual [38] and the Mond-Weir dual [13]. In this work, we consider the primal problem (VQFP) and discuss the Mond-Weir dual problem, but we use the associated problem $(VQFP)_{x^*}$ to generate the constraint set of the dual problem. Let us consider the following vector quadratic fractional dual optimization problem (VQFD):

$$\text{Maximize} \quad \frac{f(u)}{g(u)} = \left(\frac{f_1(u)}{g_1(u)}, \ldots, \frac{f_m(u)}{g_m(u)} \right)$$

$$\text{subject to} \quad \sum_{i=1}^{m} \tau_i \left(\nabla f_i(u) - \frac{f(u)}{g(u)} \nabla g_i(u) \right)$$
$$+ \sum_{j=1}^{\ell} \lambda_j \nabla h_j(u) = 0, \qquad \text{(VQFD)}$$
$$\sum_{j=1}^{\ell} \lambda_j h_j(u) \geqq 0,$$
$$\tau > 0, \quad \lambda \geqq 0, \quad \sum_{j=1}^{\ell} \lambda_j = 1,$$
$$u \in S,$$

where f_i and g_i, $i \in I$, are the same quadratic functions defined on (VQFP), and we denote its feasible set by Y.

Theorem 16 (weak duality). *Let $x \in S$ and $(u, \tau, \lambda) \in Y$. If $F(\tau, u)$ is positive semidefinite and the constraint function h_j is convex for each $j \in J$, then*

$$\frac{f(x)}{g(x)} \nleq \frac{f(u)}{g(u)}. \tag{44}$$

Proof. If there are $x \in S$ and $(u, \tau, \lambda) \in Y$ such that $f(x)/g(x) \leq f(u)/g(u)$, then

$$\frac{f(x)}{g(x)} \leq \frac{f(u)}{g(u)} \implies f(x) - \frac{f(u)}{g(u)} g(x) \leq 0$$
$$\implies \sum_{i=1}^{m} \tau_i \left(f_i(x) - \frac{f_i(u)}{g_i(u)} g_i(x) \right) < 0. \tag{45}$$

Since $x \in S$, then $\sum_{j=1}^{\ell} \lambda_j h_j(x) \leqq 0$, and $\sum_{j=1}^{\ell} \lambda_j h_j(u) \geqq 0$ implies that

$$\sum_{i=1}^{m} \tau_i \left(f_i(x) - \frac{f_i(u)}{g_i(u)} g_i(x) \right) + \sum_{j=1}^{\ell} \lambda_j h_j(x)$$
$$< \sum_{i=1}^{m} \tau_i \left(f_i(u) - \frac{f_i(u)}{g_i(u)} g_i(u) \right) + \sum_{j=1}^{\ell} \lambda_j h_j(u). \tag{46}$$

Once $F(\tau, u)$ is positive semidefinite and each constraint function h_j is convex, we can use Lemma 14 to conclude that the objective function of $(VQFP)_{x^*}^\tau$ is convex, and

$$0 > \sum_{i=1}^{m} \tau_i \left[\left(f_i(x) - \frac{f_i(u)}{g_i(u)} g_i(x) \right) - \left(f_i(u) - \frac{f_i(u)}{g_i(u)} g_i(u) \right) \right]$$
$$+ \sum_{j=1}^{\ell} \lambda_j \left(h_j(x) - h_j(u) \right)$$
$$\geqq \sum_{i=1}^{m} \tau_i \left(\nabla f_i(u) - \frac{f_i(u)}{g_i(u)} \nabla g_i(u) \right)^T (x - u)$$
$$+ \sum_{j=1}^{\ell} \lambda_j \nabla h_j(u)^T (x - u)$$
$$= (x - u)^T$$
$$\cdot \left(\sum_{i=1}^{m} \tau_i \left(\nabla f_i(u) - \frac{f_i(u)}{g_i(u)} \nabla g_i(u) \right) + \sum_{j=1}^{\ell} \lambda_j \nabla h_j(u) \right)$$
$$= 0, \tag{47}$$

which is a contradiction. $\square$

Theorem 17 (strong duality). *Let $x^* \in \text{Eff}(VQFP)$. Suppose that (GGCQ) holds at x^*; then there exists (τ^*, λ^*) such that (x^*, τ^*, λ^*) is feasible for (VQFD) and the values of the objective function of (VQFP) and (VQFD) are equal. Moreover, if $F(\tau^*, x^*)$ is positive semidefinite and the constraint function h_j is convex for each $j \in J$, then $(x^*, \tau^*, \lambda^*) \in \text{Eff}(VQFD)$.*

Proof. If $x^* \in \text{Eff (VQFP)}$, by Lemma 13, there are $\tau^* > 0$ and $\lambda^* \geqq 0$ such that (x^*, τ^*, λ^*) satisfies

$$\sum_{i=1}^{m} \tau_i^* \left(\nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) \right) + \sum_{j=1}^{\ell} \lambda_j^* \nabla h_j(x^*) = 0,$$
$$\sum_{j=1}^{\ell} \lambda_j^* h_j(x^*) = 0.$$

(48)

Then $(x^*, \tau^*, \lambda^*) \in Y$ and the values of the objective functions of (VQFP) and (VQFD) are equal. Moreover, if $F(\tau^*, x^*)$ is positive semidefinite, each constraint function h_j is convex, and $(x^*, \tau^*, \lambda^*) \notin \text{Eff (VQFD)}$, then there exists another point $(u, \tau, \lambda) \in Y$ such that

$$\frac{f(u)}{g(u)} \geq \frac{f(x^*)}{g(x^*)},$$

(49)

contradicting the weak duality. $\qquad\square$

Theorem 18 (converse duality). *Let $(u^*, \tau^*, \lambda^*) \in Y$ and u^* be feasible point of the primal problem (VQFP). If $F(\tau^*, u^*)$ is positive semidefinite and the constraint function h_j is convex for each $j \in J$, then $u^* \in \text{Eff (VQFP)}$.*

Proof. If $(u^*, \tau^*, \lambda^*) \in Y$ and $u^* \in S$, then $\sum_{j=1}^{\ell} \lambda_j^* h_j(u^*) \geqq 0$, $\sum_{j=1}^{\ell} \lambda_j^* h_j(u^*) \leqq 0$, and

$$\sum_{i=1}^{m} \tau_i^* \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right) + \sum_{j=1}^{\ell} \lambda_j^* \nabla h_j(u^*) = 0,$$
$$\sum_{j=1}^{\ell} \lambda_j^* h_j(u^*) = 0.$$

(50)

Therefore, u^* is a critical point for the weighted scalar problem $(\text{VQFP})_{u^*}^{\tau^*}$. Since $F(\tau^*, u^*)$ is positive semidefinite, by Lemma 14, the objective function of $(\text{VQFP})_{u^*}^{\tau^*}$ is convex. Moreover, if each constraint function h_j is convex, $j \in J$, then u^* is an optimal solution of $(\text{VQFP})_{u^*}^{\tau^*}$. Thus, by Theorem 12, we have $u^* \in \text{Eff (VQFP)}$. $\qquad\square$

We can obtain a second type of converse duality theorem requiring more of the matrix function F. Specifically, there must be vectors $(w, x) \in \mathcal{W} \times S$ such that $F(w, x)$ is positive definite; that is, $y^T F(w, x) y > 0$, $\forall y \in \mathbb{R}^n$, and $y \neq 0$.

Theorem 19 (strict converse duality). *Let $x^* \in S$ and $(u^*, \tau^*, \lambda^*) \in Y$ such that*

$$\sum_{i=1}^{m} \tau_i^* \frac{f_i(x^*)}{g_i(x^*)} = \sum_{i=1}^{m} \tau_i^* \frac{f_i(u^*)}{g_i(u^*)}.$$

(51)

If the matrix $F(\tau^*/g(x^*), u^*)$ is positive definite and the constraint function h_j is convex for each $j \in J$, then $x^* = u^*$.

Proof. Suppose $x^* \neq u^*$. Since $x^* \in S$ and $(u^*, \tau^*, \lambda^*) \in Y$, then $-\sum_{j=1}^{\ell} \lambda_j^* h_j(u^*) \leqq 0$ and $\sum_{j=1}^{\ell} \lambda_j^* h_j(x^*) \leqq 0$. If each constraint function h_j is convex, $j \in J$, we obtain

$$0 \geq \sum_{j=1}^{\ell} \lambda_j^* h_j(x^*) - \sum_{j=1}^{\ell} \lambda_j^* h_j(u^*)$$

$$\geq \left(\sum_{j=1}^{\ell} \lambda_j^* \nabla h_j(u^*) \right)^T (x^* - u^*)$$

$$= \left(-\sum_{i=1}^{m} \tau_i^* \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right) \right)^T (x^* - u^*)$$

$$\implies \sum_{i=1}^{m} \tau_i^* \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right)^T (x^* - u^*) \geqq 0.$$

(52)

Using the proof of Theorem 5, given $u^* \in S$ and $i \in I$, for all $x \in S$, we have

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(u^*)}{g_i(u^*)}$$

$$= \frac{1}{g_i(x)} \left\{ (x - u^*)^T \left[A_i - \frac{f_i(u^*)}{g_i(u^*)} B_i \right] (x - u^*) \right\}$$

$$+ \frac{1}{g_i(x)} \left\{ \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right)^T (x - u^*) \right\}.$$

(53)

Therefore, for $(\tau^*, x^*) \in \mathcal{W} \times S$, we obtain

$$\sum_{i=1}^{m} \tau_i^* \left(\frac{f_i(x^*)}{g_i(x^*)} - \frac{f_i(u^*)}{g_i(u^*)} \right)$$

$$= (x^* - u^*)^T \left[F\left(\frac{\tau^*}{g(x^*)}, u^* \right) \right] (x^* - u^*)$$

$$+ \sum_{i=1}^{m} \frac{\tau_i^*}{g_i(x^*)} \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right)^T (x^* - u^*),$$

(54)

and since $F(\tau^*/g(x^*), u^*)$ is positive definite and $x^* \neq u^*$, then by (51)

$$0 = \sum_{i=1}^{m} \tau_i^* \left(\frac{f_i(x^*)}{g_i(x^*)} - \frac{f_i(u^*)}{g_i(u^*)} \right)$$

$$> \sum_{i=1}^{m} \frac{\tau_i^*}{g_i(x^*)} \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right)^T (x^* - u^*)$$

$$\geqq \min_{i \in I} \left\{ \frac{1}{g_i(x^*)} \right\}$$

$$\cdot \sum_{i=1}^{m} \tau_i^* \left(\nabla f_i(u^*) - \frac{f_i(u^*)}{g_i(u^*)} \nabla g_i(u^*) \right)^T (x^* - u^*) \geqq 0,$$

(55)

which is a contradiction. $\qquad\square$

5. Conclusions

The main contribution of this work is the development of Pareto optimality conditions for a particular vector optimization problem, where each objective function consists of a ratio of two quadratic functions with convexity being only assumed on the constraint set. We took advantage of the diagonalization of Hessian matrices. We have shown the relationship between the particular problem and two problems associated with it, and we use some assumptions of the linear combination of Hessian matrices to show the main duality theorems. For the particular problem, the results presented in this work might be useful to determine the termination criteria in the development of algorithms, and new extensions can be established to more general vector optimization problems, in which algorithms based on quadratic approximations are used locally. In future work we plan to develop algorithms using the concepts presented here.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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