Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 978758, 10 pages
http://dx.doi.org/10.1155/2014/978758

Research Article

Stability Analysis of a Ratio-Dependent Predator-Prey Model Incorporating a Prey Refuge

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Received 24 November 2013; Revised 11 February 2014; Accepted 12 February 2014; Published 23 March 2014

Academic Editor: Shan Zhao

A ratio-dependent predator-prey model incorporating a prey refuge with disease in the prey population is formulated and analyzed. The effects of time delay due to the gestation of the predator and stage structure for the predator on the dynamics of the system are concerned. By analyzing the corresponding characteristic equations, the local stability of a predator-extinction equilibrium and a coexistence equilibrium of the system is discussed, respectively. Further, it is proved that the system undergoes a Hopf bifurcation at the coexistence equilibrium, when $\tau = \tau_0$. By comparison arguments, sufficient conditions are obtained for the global stability of the predator-extinction equilibrium. By using an iteration technique, sufficient conditions are derived for the global attractivity of the coexistence equilibrium of the proposed system.

1. Introduction

Since the pioneering work of Kermack-Mckendrick on SIRS [1], epidemiological models have received much attention from scientists. Mathematical models have become important tools in analyzing the spread and control of infectious disease. It is of more biological significance to consider the effect of interacting species when we study the dynamical behaviors of epidemiological models. Ecoepidemiology which is a relatively new branch of study in theoretical biology, tackles such situations by dealing with both ecological and epidemiological issues. It can be viewed as the coupling of an ecological predator-prey model and an epidemiological SI, SIS, or SIRS model. Following Anderso and May [2] who were the first to propose an ecoepidemiological model by merging the ecological predator-prey model introduced by Lotka and Volterra, the effect of disease in ecological system is an important issue from mathematical and ecological point of view. Many works have been devoted to the study of the effects of a disease on a predator-prey system [1–5]. In [5], Xiao and Chen have considered a ratio-dependent predator-prey system with disease in the prey. Consider

$$\begin{aligned}\frac{dS}{dt} &= rS\left(1-\frac{S+I}{K}\right)-\beta SI,\\ \frac{dI}{dt} &= \beta SI - dI - \frac{bIY}{aY+I},\\ \frac{dY}{dt} &= \frac{pbIY}{aY+I} - cY,\end{aligned} \tag{1}$$

where $S(t)$ and $I(t)$ represent the densities of susceptible and infected prey population at time t, respectively, and $Y(t)$ represents the density of the predator population at time t. The parameters r, K, β, d, b, a, p, and c are positive constants representing the prey intrinsic growth rate, carrying capacity, transmission rate, the infected prey death rate, capturing rate, half capturing saturation constant, conversion rate, and the predator death rate, respectively. A periodic solution can occur whether the system (1) is permanent or not; that is, there are solutions which tend to disease-free equilibrium while bifurcating periodic solution exists.

Recently, the qualitative analysis of predator-prey models incorporating a prey refuge has been done by many authors, see [3, 4]. In [3], Pal and Samanta incorporated a prey refuge

$(1-m)I$ into system (1). Sufficient conditions were derived for the stability of the equilibria of the system.

We note that it is assumed in system (1) that each individual predator admits the same ability to feed on prey. This assumption seems not to be realistic for many animals. In the natural world, there are many species whose individuals pass through an immature stage during which they are raised by their parents, and the rate at which they attack prey can be ignored. Moreover, it can be assumed that their reproductive rate during this stage is zero. Stage-structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. Stage-structured models have received great attention in recent years (see, e.g., [6–9]).

Time delays of one type or another have been incorporated into biological models by many researchers (see, e.g., [8–11]). In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause the population to fluctuate. Time delay due to gestation is a common example, because, generally, the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the effect of time delays.

Based on the above discussions, in this paper, we incorporate a prey refuge, stage structure for the predator, and time delay due to the gestation of predator into the system (1). To this end, we study the following differential equations:

$$\begin{aligned} \frac{dS}{dt} &= rS(t)\left(1-\frac{S(t)+I(t)}{K}\right)-\beta S(t)I(t), \\ \frac{dI}{dt} &= \beta S(t)I(t)-dI(t)-\frac{b(1-m)I(t)Y_2(t)}{aY_2(t)+(1-m)I(t)}, \\ \frac{dY_1}{dt} &= \frac{pb(1-m)I(t-\tau)Y_2(t-\tau)}{aY_2(t-\tau)+(1-m)I(t-\tau)}-(r_1+d_1)Y_1(t), \\ \frac{dY_2}{dt} &= r_1Y_1(t)-d_2Y_2(t), \end{aligned} \tag{2}$$

where $Y_1(t)$ and $Y_2(t)$ represent the densities of the immature and the mature predator population at time t, respectively, the parameters d_1, d_2, and r_1 are positive constants in which d_1 and d_2 are the death rates of the immature and the mature predator, respectively, r_1 denotes the rate of immature predator becoming mature predator, the constant proportion infected prey refuge is $(1-m)I$, where $m \in [0,1)$ is a constant, and $\tau \geq 0$ is a constant delay due to the gestation of the predator.

The initial conditions for system (2) take the form

$$\begin{gathered} S(\theta)=\phi_1(\theta)\geq 0, \qquad I(\theta)=\phi_2(\theta)\geq 0, \\ Y_1(\theta)=\varphi_1(\theta)\geq 0, \qquad Y_2(\theta)=\varphi_2(\theta)\geq 0, \\ \theta\in[-\tau,0), \quad \phi_1(0)>0, \quad \phi_2(0)>0, \\ \varphi_1(0)>0, \quad \varphi_2(0)>0, \\ (\phi_1(\theta),\phi_2(\theta),\varphi_1(\theta),\varphi_2(\theta))\in C\left([-\tau,0],R^4_{+0}\right), \end{gathered} \tag{3}$$

where $R^4_{+0}=\{(x_1,x_2,x_3,x_4): x_i\geq 0, i=1,2,3,4\}$.

It is well known by the fundamental theory of functional differential equations [12] that system (2) has a unique solution $(S(t), I(t), Y_1(t), Y_2(t))$ satisfying initial conditions (3).

The organization of this paper is as follows. In the next section, we show the positivity and the boundedness of solutions of system (2) with initial conditions (3). In Section 3, we investigate the global stability of the predator-extinction equilibrium. In Section 4, we establish the local stability and the global attractivity of the coexistence equilibrium of system (2). Further, we study the existence of Hopf bifurcation for system (2) at the positive equilibrium. A brief discussion is given in Section 5 to conclude this work.

2. Preliminaries

In this section, we show the positivity and the boundedness of solutions of system (2) with initial conditions (3).

Theorem 1. *Solutions of system* (2) *with initial conditions* (3) *are positive, for all* $t \geq 0$.

Proof. Let $(S(t), I(t), Y_1(t), Y_2(t))$ be a solution of system (2) with initial conditions (3). It follows from the first and the second equations of system (2) that

$$S(t)=S(0)\exp\left\{\int_0^t\left[r-\frac{r}{K}S(t)-\left(\frac{r}{K}+\beta\right)I(t)\right]dt\right\}>0, \tag{4}$$

$$\begin{aligned} I(t) &= I(0)\exp\left\{\int_0^t\left[\beta S(t)-d-\frac{b(1-m)Y_2(t)}{aY_2(t)+(1-m)I(t)}\right]dt\right\} \\ &> 0. \end{aligned} \tag{5}$$

Let us consider $Y_1(t)$ and $Y_2(t)$, for $t \in [0,\tau]$. Since $\psi_2(\theta)\geq 0$ for $\theta\in[-\tau,0]$, we derive from the third equation of system (2) that

$$\frac{dY_1}{dt}\geq -(r_1+d_1)Y_1(t). \tag{6}$$

Since $\psi_1(0)>0$, a standard comparison argument shows that

$$Y_1(t)\geq Y_1(0)e^{-(r_1+d_1)t}>0; \tag{7}$$

that is, $Y_1(t)>0$ for $t\in[0,\tau]$. For $t\in[0,\tau]$, it follows from the fourth equation of (2) that

$$\frac{dY_2}{dt}\geq -d_2Y_2(t). \tag{8}$$

Since $\psi_2(0)>0$, a standard comparison argument shows that

$$Y_2(t)\geq Y_2(0)e^{-d_2t}>0; \tag{9}$$

that is, $Y_2(t)>0$ for $t\in[0,\tau]$. In a similar way, we treat the intervals $[\tau,2\tau],\ldots,[n\tau,(n+1)\tau]$, $n\in N$. Thus, $S(t)>0$, $I(t)>0$, $Y_1(t)>0$, and $Y_2(t)>0$, for all $t\geq 0$. This completes the proof. □

Theorem 2. *Positive solutions of system* (2) *with initial conditions* (3) *are ultimately bounded.*

Proof. Let $(S(t), I(t), Y_1(t), Y_2(t))$ be any positive solution of system (2) with initial conditions (3). Denote $\hat{d} = \min\{d, d_1, d_2\}$. Define

$$V(t) = pS(t-\tau) + pI(t-\tau) + Y_1(t) + Y_2(t). \quad (10)$$

Calculating the derivative of $V(t)$ along positive solutions of system (2), it follows that

$$\begin{aligned} \frac{dV}{dt} &= \frac{pr}{K} S(t-\tau)[K - S(t-\tau) - I(t-\tau)] \\ &\quad - pdI(t-\tau) - d_1 Y_1(t) - d_2 Y_2(t) \\ &\leq -\hat{d}V(t) - \frac{pr}{K}\left[S(t-\tau) - \frac{K(r+\hat{d})}{2r}\right]^2 + \frac{pK(r+\hat{d})^2}{4r}, \end{aligned} \quad (11)$$

which yields

$$\limsup_{t\to\infty} V(t) \leq \frac{pK(r+\hat{d})^2}{4r\hat{d}}. \quad (12)$$

If we choose $M_1 = (K(r+\hat{d})^2)/4r\hat{d}$ and $M_2 = (pK(r+\hat{d})^2)/4r\hat{d}$, then

$$\begin{gathered} \limsup_{t\to\infty} S(t) \leq M_1, \qquad \limsup_{t\to\infty} I(t) \leq M_1, \\ \limsup_{t\to\infty} Y_i(t) \leq M_2, \quad (i = 1, 2). \end{gathered} \quad (13)$$

This completes the proof. □

3. Predator-Extinction Equilibrium and Its Stability

In this section, we discuss the stability of the predator-extinction equilibrium.

It is easy to show that if $K\beta > d$, system (2) admits a predator-extinction equilibrium $E_1(S_1, I_1, 0, 0)$, where

$$S_1 = \frac{d}{\beta}, \qquad I_1 = \frac{r(K\beta - d)}{\beta(K\beta + r)}. \quad (14)$$

The characteristic equation of system (2) at the equilibrium E_1 is of the form

$$\lambda^2 + g_1\lambda + g_0 + f_0 e^{-\lambda\tau} = 0, \quad (15)$$

where $g_1 = (r_1 + d_1 + d_2)$, $g_0 = d_2(r_1 + d_1)$, $f_0 = -pbr_1$. When $\tau = 0$, if $d_2(r_1 + d_1) > pbr_1$, then E_1 is locally asymptotically stable and if $d_2(r_1 + d_1) < pbr_1$, then E_1 is unstable. It is easily seen that

$$\begin{gathered} g_1^2 - 2g_0 = (r_1 + d_1)^2 + d_2^2 > 0, \\ g_0 - f_0 = d_2(r_1 + d_1) + pbr_1 > 0. \end{gathered} \quad (16)$$

Hence, if $d_2(r_1 + d_1) > pbr_1$, by Lemma B in [11], it follows that the equilibrium E_1 is locally asymptotically stable for all $\tau \geq 0$. If $d_2(r_1 + d_1) < pbr_1$, then E_1 is unstable for all $\tau \geq 0$.

Theorem 3. *Let $K\beta > d$ hold; the predator-extinction equilibrium E_1 is globally stable provided that*

$$d_2(r_1 + d_1) > pbr_1, \qquad 0 < (1-m) < \frac{a}{b}(K\beta - d). \quad (17)$$

Proof. Based on above discussions, we only prove the global attractivity of the equilibrium E_1. Let $(S(t), I(t), Y_1(t), Y_2(t))$ be any positive solution of system (2) with initial conditions (3). It follows from the first and the second equations of system (2) that

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \beta SI, \qquad \frac{dI}{dt} \leq \beta SI - dI. \quad (18)$$

Consider the following auxiliary equations:

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1\left(1 - \frac{x_1 + x_2}{K}\right) - \beta x_1 x_2, \\ \frac{dx_2}{dt} &= \beta x_1 x_2 - dx_2. \end{aligned} \quad (19)$$

If $K\beta > d$, then by Theorem 3.1 in [4], it follows from (19) that

$$\lim_{t\to+\infty} x_1(t) = \frac{d}{\beta}, \qquad \lim_{t\to+\infty} x_2(t) = \frac{r(K\beta - d)}{\beta(K\beta + r)}. \quad (20)$$

By comparison, we obtain that

$$\limsup_{t\to+\infty} S(t) \leq \frac{d}{\beta}, \qquad \limsup_{t\to+\infty} I(t) \leq \frac{r(K\beta - d)}{\beta(K\beta + r)} = I_1. \quad (21)$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, then $I(t) \leq I_1 + \varepsilon$.

It follows from the third and the fourth equations of system (2) that, for $t > T_1 + \tau$,

$$\begin{aligned} \frac{dY_1}{dt} &\leq \frac{pb(1-m)(I_1 + \varepsilon)Y_2(t-\tau)}{aY_2(t-\tau) + (1-m)(I_1 + \varepsilon)} - (r_1 + d_1)Y_1(t), \\ \frac{dY_2}{dt} &= r_1 Y_1(t) - d_2 Y_2(t). \end{aligned} \quad (22)$$

Consider the following auxiliary equations:

$$\begin{aligned} \frac{du_1}{dt} &= \frac{pb(1-m)(I_1 + \varepsilon)u_2(t-\tau)}{au_2(t-\tau) + (1-m)(I_1 + \varepsilon)} - (r_1 + d_1)u_1(t), \\ \frac{du_2}{dt} &= r_1 u_1(t) - d_2 u_2(t). \end{aligned} \quad (23)$$

If $d_2(r_1 + d_1) > r_1 pb$, then by Lemma 2.4 in [9], it follows from (23) that

$$\lim_{t\to+\infty} u_1(t) = 0, \qquad \lim_{t\to+\infty} u_2(t) = 0. \quad (24)$$

By comparison, we obtain that

$$\lim_{t\to+\infty} Y_1(t) = 0, \qquad \lim_{t\to+\infty} Y_2(t) = 0. \quad (25)$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > 0$ such that if $t > T_2$, then $Y_2(t) \le \varepsilon$.

It follows from the first and the second equations of system (2) that for $t > T_2$:

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \beta SI, \quad \frac{dI}{dt} \ge \beta SI - dI - \frac{b(1-m)\varepsilon I}{a\varepsilon + (1-m)I}. \tag{26}$$

Consider the following auxiliary equations:

$$\frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1+x_2}{K}\right) - \beta x_1 x_2, \quad \frac{dx_2}{dt} = \beta x_1 x_2 - dx_2 - \frac{b(1-m)\varepsilon x_2}{a\varepsilon + (1-m)x_2}. \tag{27}$$

If $K\beta > d$, and $(1-m) < (a/b)(K\beta - d)$, then by Theorem 3.1 in [4], it follows from (27) that

$$\lim_{t\to+\infty} x_2(t) = \frac{(1-m)r(K\beta-d) - \varepsilon a\beta(K\beta+r)}{2(1-m)\beta(K\beta+r)} + \Big(\big([(1-m)r(K\beta-d) - \varepsilon a\beta(K\beta+r)]^2 + 4\varepsilon(1-m)r\beta(K\beta+r) \times [a(K\beta-d) - b(1-m)]\big)^{1/2} \times \left(2(1-m)\beta(K\beta+r)\right)^{-1}\Big) := \underline{x}_2$$

$$\lim_{t\to+\infty} x_1(t) = K - \frac{K\beta+r}{r}\underline{x}_2. \tag{28}$$

By comparison, for ε sufficiently small, we obtain that

$$\liminf_{t\to+\infty} S(t) \ge \frac{d}{\beta}, \qquad \liminf_{t\to+\infty} I(t) \ge \frac{r(K\beta-d)}{\beta(K\beta+r)}, \tag{29}$$

which, together with (21), yields

$$\lim_{t\to+\infty} S(t) = \frac{d}{\beta}, \qquad \lim_{t\to+\infty} I(t) = \frac{r(K\beta-d)}{\beta(K\beta+r)}. \tag{30}$$

Hence, if $K\beta > d$, $d_2(r_1+d_1) > pbr_1$, $(1-m) < (a/b)(K\beta - d)$ hold, then the equilibrium $E_1(S_1, I_1, 0, 0)$ is globally stable. □

4. Coexistence Equilibrium and Its Stability

In this section, we discuss the stability of the coexistence equilibrium and the existence of a Hopf bifurcation. It is easy to show that if the following holds:

(H1) $d_2(r_1 + d_1) < r_1 pb$, $0 < (1-m) < (ar_1 p(K\beta - d))/(r_1 pb - d_2(r_1+d_1))$,

then system (2) has a unique coexistence equilibrium $E^*(S^*, I^*, Y_1^*, Y_2^*)$, where

$$S^* = \frac{dar_1 p + (1-m)[r_1 pb - d_2(r_1+d_1)]}{\beta a r_1 p}, \quad I^* = \frac{r}{r+K\beta}(K - S^*), \quad Y_1^* = \frac{(1-m)[r_1 pb - d_2(r_1+d_1)]}{ar_1(r_1+d_1)} I^*, \quad Y_2^* = \frac{(1-m)[r_1 pb - d_2(r_1+d_1)]}{ad_2(r_1+d_1)} I^*. \tag{31}$$

The characteristic equation of system (2) at the equilibrium E^* takes the form

$$\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + \left(q_2\lambda^2 + q_1\lambda + q_0\right)e^{-\lambda\tau} = 0, \tag{32}$$

where

$$\begin{aligned}
p_3 &= r_1 + d_1 + d_2 + \frac{r}{K}S^* - \alpha_1, \\
p_2 &= d_2(r_1+d_1) + (r_1+d_1+d_2)\left(\frac{r}{K}S^* - \alpha_1\right) + S^*\left[\beta\left(\frac{r}{K}+\beta\right)I^* - \frac{r}{K}\alpha_1\right], \\
p_1 &= S^*(r_1+d_1+d_2)\left[\beta\left(\frac{r}{K}+\beta\right)I^* - \frac{r}{K}\alpha_1\right] + d_2(r_1+d_1)\left(\frac{r}{K}S^* - \alpha_1\right), \\
p_0 &= d_2(r_1+d_1)S^*\left[\beta\left(\frac{r}{K}+\beta\right)I^* - \frac{r}{K}\alpha_1\right], \\
q_2 &= -pr_1\alpha_2, \qquad q_1 = -pr_1\alpha_2\left(\frac{r}{K}S^* - \alpha_1\right) + pr_1\alpha_2\alpha_3, \\
q_0 &= -pr_1\alpha_2 S^*\left[\beta\left(\frac{r}{K}+\beta\right)I^* - \frac{r}{K}\alpha_1\right] + pr_1\alpha_2\alpha_3\frac{r}{K}S^*, \\
\alpha_1 &= \frac{b(1-m)^2 I^* Y_2^*}{[aY_2^* + (1-m)I^*]^2}, \qquad \alpha_2 = \frac{b(1-m)^2 (I^*)^2}{[aY_2^* + (1-m)I^*]^2}, \\
\alpha_3 &= \frac{ab(1-m)(Y_2^*)^2}{[aY_2^* + (1-m)I^*]^2}.
\end{aligned} \tag{33}$$

When $\tau = 0$, (32) becomes

$$\lambda^4 + p_3\lambda^3 + (p_2 + q_2)\lambda + p_0 + q_0 = 0. \tag{34}$$

If the following holds:

(H2) $(r/K)S^* - \alpha_1 > 0$, $\beta((r/K)+\beta)I^* - (r/K)\alpha_1 > 0$,

then it is easy to show that

$$p_3 > 0, \qquad p_0 + q_0 > 0, \qquad p_1 + q_1 > 0, \qquad p_2 + q_2 > 0. \tag{35}$$

If $(p_1 + q_1)[p_3(p_2 + q_2) - (p_1 + q_1)] > p_3^2(p_0 + q_0)$, then, by the Routh-Hurwitz theorem, when $\tau = 0$, the coexistence equilibrium E^* of system (2) is locally asymptotically stable and E^* is unstable if $(p_1 + q_1)[p_3(p_2 + q_2) - (p_1 + q_1)] < p_3^2(p_0 + q_0)$.

If $i\omega(\omega > 0)$ is a solution of (34), separating real and imaginary parts, we have

$$\begin{aligned} \left(q_2\omega^2 - q_0\right)\sin\omega\tau + q_1\omega\cos\omega\tau &= p_3\omega^3 - p_1\omega, \\ \left(q_2\omega^2 - q_0\right)\cos\omega\tau - q_1\omega\sin\omega\tau &= \omega^4 - p_2\omega^2 + p_0. \end{aligned} \tag{36}$$

By squaring and adding the two equations of (36), it follows that

$$\omega^8 + h_3\omega^6 + h_2\omega^4 + h_1\omega^2 + p_0^2 - q_0^2 = 0, \tag{37}$$

where

$$\begin{gathered} h_3 = p_3^2 - 2p_2, \qquad h_2 = p_2^2 + 2p_0 - q_2^2 - 2p_1p_3, \\ h_1 = p_1^2 + 2q_0q_2 - q_1^2 - 2p_0p_2. \end{gathered} \tag{38}$$

If $h_3 > 0$, $h_2 > 0$, $h_1 > 0$ and $p_0 - q_0 > 0$, by the general theory on characteristic equation of delay differential equation from [13] (Theorem 4.1), E^* remains stable for all $\tau > 0$.

If $h_i > 0$, $(i = 1, 2, 3)$ and $p_0 - q_0 < 0$, then (37) has a unique positive root ω_0; that is, (34) admits a pair of purely imaginary roots of the form $\pm\omega_0$. From (36), we see that

$$\begin{aligned} \tau_n = \frac{2n\pi}{\omega_0} + \frac{1}{\omega_0}\arccos & \\ &\times\left(\left(q_2\omega_0^2 - q_0\right)\left(\omega_0^4 - p_2\omega_0^2 + p_0\right)\right. \\ &\quad \left. + q_1\omega_0\left(p_3\omega_0^3 - p_1\omega_0\right)\right) \\ &\times\left(\left(q_1\omega_0\right)^2 + \left(q_2\omega_0^2 - q_0\right)^2\right)^{-1}, \\ & \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{39}$$

By Theorem 3.4.1 in [13], we see that E^* remains stable for $\tau < \tau_0$.

In the following, we claim that

$$\left.\frac{d(\mathrm{Re}(\lambda))}{d\tau}\right|_{\tau=\tau_0} > 0. \tag{40}$$

This will show that there exists at least one eigenvalue with a positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation (Theorem 2.9.1 in [13]) are then satisfied yielding a periodic solution. To this end, by differentiating equation (34) with respect to τ, it follows that

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} = &\frac{4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1}{-\lambda\left(\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0\right)} \\ &+ \frac{2q_2\lambda + q_1}{\lambda\left(q_2\lambda^2 + q_1\lambda + q_0\right)} - \frac{\tau}{\lambda}. \end{aligned} \tag{41}$$

Hence, a direct calculation shows that

$$\begin{aligned} &\mathrm{sgn}\left\{\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} \\ &= \mathrm{sgn}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \\ &= \mathrm{sgn}\left\{\left(\left(3p_3\omega_0^2 - p_1\right)\left(p_3\omega_0^2 - p_1\right)\right.\right. \\ &\qquad \left. + 2\left(2\omega_0^2 - p_2\right)\left(\omega_0^4 - p_2\omega_0^2 + p_0\right)\right) \\ &\qquad \times\left(\omega_0^2\left(p_1 - p_3\omega_0^2\right)^2 + \left(\omega_0^4 - p_2\omega_0^2 + p_0\right)^2\right)^{-1} \\ &\qquad \left. + \frac{-q_1^2 + 2q_2\left(q_0 - q_2\omega_0^2\right)}{\left(q_1\omega_0\right)^2 + \left(q_2\omega_0^2 - q_0\right)^2}\right\}. \end{aligned} \tag{42}$$

We derive from (36) that

$$\begin{aligned} &\omega_0^2\left(p_1 - p_3\omega_0^2\right)^2 + \left(\omega_0^4 - p_2\omega_0^2 + p_0\right)^2 \\ &\quad = \left(q_1\omega_0\right)^2 + \left(q_2\omega_0^2 - q_0\right)^2. \end{aligned} \tag{43}$$

Hence, it follows that

$$\begin{aligned} &\mathrm{sgn}\left\{\frac{d\,(\mathrm{Re}\,\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} \\ &\quad = \mathrm{sgn}\left\{\frac{4\omega_0^6 + 3h_3\omega_0^4 + 2h_2\omega_0^2 + h_1}{\left(q_1\omega_0\right)^2 + \left(q_2\omega_0^2 - q_0\right)^2}\right\} > 0\,. \end{aligned} \tag{44}$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0, \tau = \tau_0$.

In conclusion, we have the following results.

Theorem 4. *For system (2), let (H1) and (H2) hold; we have the following:*

(i) *if $(p_1+q_1)[p_3(p_2+q_2)-(p_1+q_1)] > p_3^2(p_0+q_0)$, $h_i > 0$, and $p_0 - q_0 > 0$, then the coexistence equilibrium E^* is locally asymptotically stable, for all $\tau \geq 0$;*

(ii) *if $(p_1+q_1)[p_3(p_2+q_2)-(p_1+q_1)] > p_3^2(p_0+q_0)$, $h_i > 0$, and $p_0 - q_0 < 0$, then there exists a positive number τ_0, such that the coexistence equilibrium E^* is locally asymptotically stable if $0 \leq \tau < \tau_0$ and is unstable for $\tau > \tau_0$; further, system (2) undergoes a Hopf bifurcation at E^* when $\tau = \tau_n$, $n = 0, 1, 2, \ldots$;*

(iii) *if $(p_1 + q_1)[p_3(p_2 + q_2) - (p_1 + q_1)] < p_3^2(p_0 + q_0)$, then the coexistence equilibrium E^* is unstable, for all $\tau \geq 0$.*

We now give some examples to illustrate the main results above.

Example 5. In (2), we let $a = 2$, $p = 0.9$, $r = 10$, $r_1 = 0.5$, $k = 1, \beta = 1$, $d = d_1 = d_2 = 0.1$, $b = 1$, and

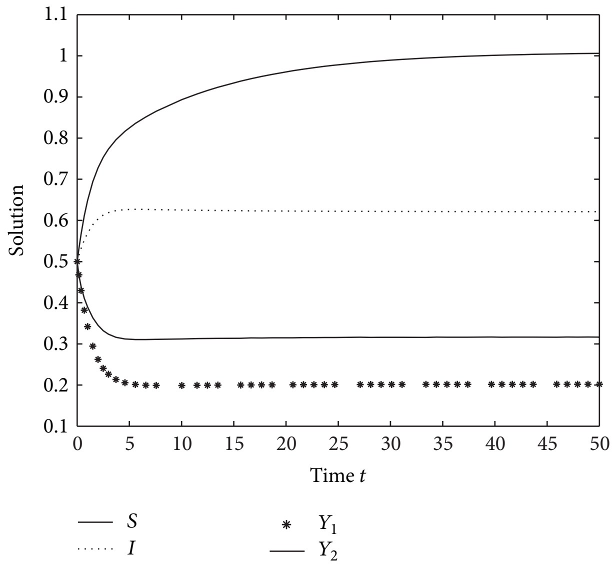

Figure 1: The temporal solution found by numerical integration of system (2) with $\tau = 1$.

$m = 0.5$. System (2), with the above coefficients, has a unique coexistence equilibrium $E^*(0.3167, 0.6212, 0.2019, 1.0095)$. It is easy to show that $(r/K)S^* - \alpha_1 \approx 3.1468 > 0$, $\beta((r/K) + \beta)I^* - (r/K)S^*\alpha_1 \approx 6.6343 > 0$, that is the condition (H2) holds. We can get $(p_1+q_1)[p_3(p_2+q_2)-(p_1+q_1)]-p_3^2(p_0+q_0) \approx 23.1510 > 0$, $h_3 \approx 6.0704 > 0$, $h_2 \approx 6.5263 > 0$, $h_1 \approx 1.6535 > 0$, and $p_0 - q_0 \approx 0.1310 > 0$. By Theorem 4(i), the coexistence equilibrium E^* is locally asymptotically stable, for all $\tau \geq 0$. Numerical simulation illustrates our result (see Figure 1).

Example 6. In (2), we let $a = 0.55$, $p = 0.95$, $r = 20$, $r_1 = 0.5$, $k = 1$, $\beta = 1$, $d = d_1 = d_2 = 0.1$, $b = 1$, and $m = 0.5$. System (2), with the above coefficients, has a unique coexistence equilibrium $E^*(0.8946, 0.1007, 0.1266, 0.6332)$. It is easy to show that $(r/K)S^* - \alpha_1 \approx 17.7848 > 0$, and $\beta((r/K) + \beta)I^* - (r/K)S^*\alpha_1 \approx 0.1083 > 0$; that is, the condition (H2) holds. We can get $(p_1+q_1)[p_3(p_2+q_2)-(p_1+q_1)]-p_3^2(p_0+q_0) \approx 199.2429 > 0$, $h_3 \approx 316.4769 > 0$, $h_2 \approx 116.9725 > 0$, $h_1 \approx 1.1233 > 0$, and $p_0 - q_0 \approx -0.0875 < 0$. By Theorem 4(ii), there exists a positive number $\tau_0 \approx 11.4092$, such that the coexistence equilibrium E^* is locally asymptotically stable if $0 \leq \tau < \tau_0$ and is unstable for $\tau > \tau_0$. Numerical simulations illustrate our results (see Figure 2).

Now, we are concerned with the global attractiveness of the coexistence equilibrium E^*.

Theorem 7. *The coexistence equilibrium $E^*(S^*, I^*, Y_1^*, Y_2^*)$ of system (2) is globally attractive provided that the following conditions hold:*

(i) $0 < (1 - m) < (a/b)(K\beta - d)$;

(ii) $d_2(r_1 + d_1) < r_1 pb < 2d_2(r_1 + d_1)$.

That is, the system (2) is persistent, if conditions (i) and (ii) hold.

Proof. Let $(S(t), I(t), Y_1(t), Y_2(t))$ be any positive solution of system (2) with initial conditions (3). Let

$$U_S = \limsup_{t\to+\infty} S(t), \qquad L_S = \liminf_{t\to+\infty} S(t),$$
$$U_I = \limsup_{t\to+\infty} I(t), \qquad L_I = \liminf_{t\to+\infty} I(t), \tag{45}$$
$$U_{Y_i} = \limsup_{t\to+\infty} Y_i(t), \qquad L_{Y_i} = \liminf_{t\to+\infty} Y_i(t).$$

We now claim that $U_S = L_S = S^*$, $U_I = L_I = I^*$, $U_{Y_i} = L_{Y_i} = Y_i^*$ $(i = 1, 2)$. The strategy of the proof is to use an iteration technique.

We derive from the first and the second equations of the system (2) that

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \beta SI, \qquad \frac{dI}{dt} \leq \beta SI - dI. \tag{46}$$

Consider the following auxiliary equations:

$$\frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1+x_2}{K}\right) - \beta x_1 x_2,$$
$$\frac{dx_2}{dt} = \beta x_1 x_2 - dx_2. \tag{47}$$

If $K\beta > d$, then by Theorem 3.1 in [4], it follows from (47) that

$$\lim_{t\to+\infty} x_1(t) = \frac{d}{\beta}, \qquad \lim_{t\to+\infty} x_2(t) = \frac{r(K\beta - d)}{\beta(K\beta + r)}. \tag{48}$$

By comparison, we obtain that

$$U_S = \limsup_{t\to+\infty} S(t) \leq \frac{d}{\beta} := M_1^S,$$
$$U_I = \limsup_{t\to+\infty} I(t) \leq \frac{r(K\beta - d)}{\beta(K\beta + r)} := M_1^I. \tag{49}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, then $I(t) \leq M_1^I + \varepsilon$.

It follows from the third and the fourth equations of system (2) that, for $t > T_1 + \tau$,

$$\frac{dY_1}{dt} \leq \frac{pb(1-m)\left(M_1^I + \varepsilon\right)Y_2(t-\tau)}{aY_2(t-\tau) + (1-m)\left(M_1^I + \varepsilon\right)} - \left(r_1 + d_1\right)Y_1(t),$$
$$\frac{dY_2}{dt} = r_1 Y_1(t) - d_2 Y_2(t). \tag{50}$$

Consider the following auxiliary equations:

$$\frac{du_1}{dt} = \frac{pb(1-m)\left(M_1^I + \varepsilon\right)u_2(t-\tau)}{au_2(t-\tau) + (1-m)\left(M_1^I + \varepsilon\right)} - \left(r_1 + d_1\right)u_1(t),$$
$$\frac{du_2}{dt} = r_1 u_1(t) - d_2 u_2(t). \tag{51}$$

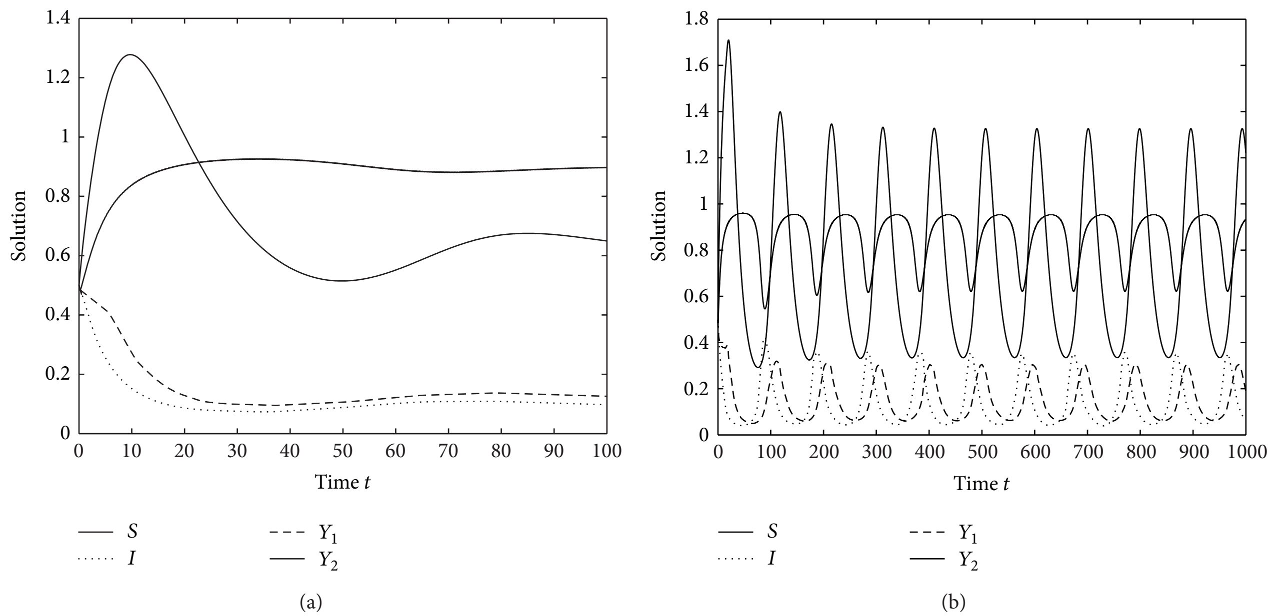

Figure 2: The temporal solution found by numerical integration of system (2) with (a) $\tau = 1$ and (b) $\tau = 15$.

If $r_1pb > d_2(r_1 + d_1)$, then by Lemma 2.4 in [9], it follows from (51) that

$$\lim_{t\to+\infty} u_1(t) = \frac{(1-m)\left(M_1^I+\varepsilon\right)\left[r_1pb - d_2(r_1+d_1)\right]}{ar_1(r_1+d_1)},$$
$$\lim_{t\to+\infty} u_2(t) = \frac{(1-m)\left(M_1^I+\varepsilon\right)\left[r_1pb - d_2(r_1+d_1)\right]}{ad_2(r_1+d_1)}. \quad (52)$$

By comparison, we obtain that

$$U_{Y_1} = \limsup_{t\to+\infty} Y_1(t) \le \frac{(1-m)\left[r_1pb - d_2(r_1+d_1)\right]}{ar_1(r_1+d_1)} M_1^I := M_1^{Y_1},$$
$$U_{Y_2} = \limsup_{t\to+\infty} Y_2(t) = \frac{(1-m)\left[r_1pb - d_2(r_1+d_1)\right]}{ad_2(r_1+d_1)} M_1^I := M_1^{Y_2}. \quad (53)$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1 + \tau$ such that if $t > T_2$, then $Y_2(t) \le M_1^{Y_2} + \varepsilon$.

We derive from the first and the second equations of system (2) that

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \beta SI,$$
$$\frac{dI}{dt} \ge \beta SI - dI - \frac{b}{a}(1-m)I. \quad (54)$$

Since $(1-m) < (a/b)(K\beta - d)$ holds, by Theorem 3.1 in [4], it follows from (54) and comparison argument that

$$L_S = \liminf_{t\to+\infty} \ge \frac{d}{\beta} + \frac{b(1-m)}{a\beta} := N_1^S,$$
$$L_I = \liminf_{t\to+\infty} \ge \frac{r}{r+\beta K}\left(K - \frac{ad + b(1-m)}{a\beta}\right) := N_1^I. \quad (55)$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_3 > T_2$ such that if $t > T_3$, then $I(t) \ge N_1^I - \varepsilon$. We derive from the third and the fourth equations of system (2) that, for $t > T_3 + \tau$,

$$\frac{dY_1}{dt} \ge \frac{pb(1-m)\left(N_1^I - \varepsilon\right)Y_2(t-\tau)}{(1-m)\left(N_1^I-\varepsilon\right) + aY_2(t-\tau)} - (d_1 + r_1)Y_1(t)$$
$$\frac{dY_2}{dt} = r_1Y_1(t) - d_2Y_2(t). \quad (56)$$

Since $r_1pb > d_2(r_1 + d_1)$ holds, by Lemma 2.4 of [9], it follows from (56) and comparison argument that

$$L_{Y_1} = \liminf_{t\to+\infty} Y_1(t) \ge \frac{(1-m)\left[r_1pb - d_2(r_1+d_1)\right]}{ar_1(r_1+d_1)}\left(N_1^I - \varepsilon\right)$$
$$L_{Y_2} = \liminf_{t\to+\infty} Y_2(t) \ge \frac{(1-m)\left[r_1pb - d_2(r_1+d_1)\right]}{ad_2(r_1+d_1)}\left(N_1^I - \varepsilon\right). \quad (57)$$

Since these two inequalities hold, for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $L_{Y_1} \geq N_1^{Y_1}$, $L_{Y_2} \geq N_1^{Y_2}$, where

$$N_1^{Y_1} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)} N_1^I,$$
$$N_1^{Y_2} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)} N_1^I. \quad (58)$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_4 \geq T_3 + \tau$, such that if $t > T_4$, $Y_2(t) \geq N_1^{Y_2} - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first and the second equations of system (2) that, for $t > T_4$,

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \beta SI,$$
$$\frac{dI}{dt} \leq \beta SI - dI - \frac{b(1-m)\left(N_1^{Y_2} - \varepsilon\right)}{a\left(N_1^{Y_2} - \varepsilon\right) + (1-m)\left(M_1^I + \varepsilon\right)} I. \quad (59)$$

By comparison and Theorem 3.1 in [4], it follows that

$$U_S = \limsup_{t\to+\infty} S(t) \leq \frac{d}{\beta} + \frac{b(1-m)\left(N_1^{Y_2} - \varepsilon\right)}{\beta\left[a\left(N_1^{Y_2} - \varepsilon\right) + (1-m)\left(M_1^I + \varepsilon\right)\right]} := \overline{M}_2^S,$$
$$U_I = \limsup_{t\to+\infty} I(t) \leq \frac{r}{r + K\beta}\left(K - \overline{M}_2^S\right). \quad (60)$$

Since these two inequalities hold, for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_S \leq M_2^S$, $U_I \leq M_2^I$, where

$$M_2^S = \frac{d}{\beta} + \frac{b(1-m)N_1^{Y_2}}{\beta\left[aN_1^{Y_2} + (1-m)M_1^I\right]},$$
$$M_2^I = \frac{r}{r + K\beta}\left(K - M_2^S\right). \quad (61)$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_5 \geq T_4$ such that if $t > T_5$, $I(t) \leq M_2^I + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the third and the fourth equations of system (2) that, for $t > T_5 + \tau$,

$$\frac{dY_1}{dt} \leq \frac{pb(1-m)\left(M_2^I + \varepsilon\right)Y_2(t-\tau)}{(1-m)\left(M_2^I + \varepsilon\right) + aY_2(t-\tau)} - \left(d_1 + r_1\right)Y_1(t),$$
$$\frac{dY_2}{dt} = r_1 Y_1(t) - d_2 Y_2(t). \quad (62)$$

Since $pbr_1 > d_2(r_1 + d_1)$ holds, by Lemma 2.4 of [9], it follows from (62) that

$$U_{Y_1} = \limsup_{t\to+\infty} Y_1(t) \leq \frac{(1-m)\left[r_1 pb - d_2\left(d_1 + r_1\right)\right]}{ar_1\left(d_1 + r_1\right)}\left(M_2^I + \varepsilon\right),$$
$$U_{Y_2} = \limsup_{t\to+\infty} Y_2(t) \leq \frac{(1-m)\left[r_1 pb - d_2\left(d_1 + r_1\right)\right]}{ad_2\left(d_1 + r_1\right)}\left(M_2^I + \varepsilon\right). \quad (63)$$

Since these two inequalities hold, for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_{Y_1} \leq M_2^{Y_1}$, $U_{Y_2} \leq M_2^{Y_2}$, where

$$M_2^{Y_1} = \frac{(1-m)\left[r_1 pb - d_2\left(d_1 + r_1\right)\right]}{ar_1\left(d_1 + r_1\right)} M_2^I,$$
$$M_2^{Y_2} = \frac{(1-m)\left[r_1 pb - d_2\left(d_1 + r_1\right)\right]}{ad_2\left(d_1 + r_1\right)} M_2^I. \quad (64)$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \geq T_5 + \tau$ such that if $t > T_6$, $y_2(t) \leq M_2^{Y_2} + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first and the second equations of system (2) that, for $t > T_6$,

$$\frac{dS}{dt} = rS\left(1 - \frac{S+I}{K}\right) - \beta SI,$$
$$\frac{dI}{dt} \geq \beta SI - dI - \frac{b(1-m)\left(M_2^{Y_2} + \varepsilon\right)}{a\left(M_2^{Y_2} + \varepsilon\right) + (1-m)\left(N_1^I - \varepsilon\right)} I(t). \quad (65)$$

By Theorem 3.1 in [4] and comparison argument, we can obtain

$$L_S = \liminf_{t\to+\infty} S(t) \geq \frac{d}{\beta} + \frac{b(1-m)\left(M_2^{Y_2} + \varepsilon\right)}{\beta\left[a\left(M_2^{Y_2} + \varepsilon\right) + (1-m)\left(N_1^I - \varepsilon\right)\right]},$$
$$L_I = \liminf_{t\to+\infty} I(t) \geq \frac{r}{r + \beta K}\left[K - \frac{d}{\beta} - \frac{b(1-m)\left(M_2^{Y_2} + \varepsilon\right)}{\beta\left[a\left(M_2^{Y_2} + \varepsilon\right) + (1-m)\left(N_1^I - \varepsilon\right)\right]}\right]. \quad (66)$$

Since these two inequalities hold, for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $L_S \geq N_2^S$, $L_I \geq N_2^I$, where

$$N_2^S = \frac{d}{\beta} + \frac{b(1-m)M_2^{Y_2}}{\beta\left[aM_2^{Y_2} + (1-m)N_1^I\right]},$$
$$N_2^I = \frac{r}{r + \beta K}\left(K - N_2^S\right). \quad (67)$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_7 \geq T_6$ such that if $t > T_7$, $I(t) \geq N_2^I - \varepsilon$. We therefore obtain from the third and the fourth equations of system (2) that, for $t > T_7 + \tau$,

$$\frac{dY_1}{dt} \geq \frac{pb(1-m)\left(N_2^I - \varepsilon\right)Y_2(t-\tau)}{(1-m)\left(N_2^I - \varepsilon\right) + aY_2(t-\tau)} - \left(d_1 + r_1\right)Y_1(t),$$
$$\frac{dY_2}{dt} = r_1 Y_1(t) - d_2 Y_2(t). \tag{68}$$

Since $pbr_1 > d_2(r_1 + d_1)$ holds, by Lemma 2.4 in [9] and comparison argument, we derive that

$$L_{Y_1} = \liminf_{t\to+\infty} Y_1(t) \geq \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)}\left(N_2^I - \varepsilon\right),$$
$$L_{Y_2} = \liminf_{t\to+\infty} Y_2(t) \geq \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)}\left(N_2^I - \varepsilon\right). \tag{69}$$

Since these inequalities hold for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $L_{Y_1} \geq N_2^{Y_1}$, $L_{Y_2} \geq N_2^{Y_2}$, where

$$N_2^{Y_1} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)}N_2^I,$$
$$N_2^{Y_2} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)}N_2^I. \tag{70}$$

Continuing this process, we derive eight sequences M_k^S, M_k^I, $M_k^{Y_1}$, $M_k^{Y_2}$, N_k^S, N_k^I, $N_k^{Y_1}$, and $N_k^{Y_2}$ $(k = 1, 2, \ldots)$ such that, for $k \geq 2$,

$$M_k^S = \frac{1}{\beta}\left[d + \frac{b(1-m)N_{k-1}^{Y_2}}{aN_{k-1}^{Y_2} + (1-m)M_{k-1}^I}\right],$$
$$M_k^I = \frac{r}{r + K\beta}\left(K - M_k^S\right),$$
$$M_k^{Y_1} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)}M_k^I,$$
$$M_k^{Y_2} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)}M_k^I,$$
$$N_k^S = \frac{1}{\beta}\left[d + \frac{b(1-m)M_k^{Y_2}}{aM_k^{Y_2} + (1-m)N_{k-1}^I}\right],$$
$$N_k^I = \frac{r}{r + K\beta}\left(K - N_k^S\right),$$
$$N_k^{Y_1} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)}N_k^I,$$
$$N_k^{Y_2} = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)}N_k^I. \tag{71}$$

It is readily seen that

$$N_k^S \leq L_S \leq U_S \leq M_k^S, \qquad N_k^I \leq L_I \leq U_I \leq M_k^I,$$
$$N_k^{Y_i} \leq L_{Y_i} \leq U_{Y_i} \leq M_k^{Y_i}, \quad (i = 1, 2). \tag{72}$$

Noting that the sequences M_k^S, M_k^I, $M_k^{Y_1}$, $M_k^{Y_2}$ are nonincreasing, and the sequences N_k^S, N_k^I, $N_k^{Y_1}$, $N_k^{Y_2}$ are nondecreasing. Hence, the limit of each sequence in M_k^S, M_k^I, $M_k^{Y_1}$, $M_k^{Y_2}$, N_k^S, N_k^I, $N_k^{Y_1}$, and $N_k^{Y_2}$ exists. Denote

$$\lim_{k\to+\infty} M_k^S = \overline{S}, \qquad \lim_{k\to+\infty} M_k^I = \overline{I},$$
$$\lim_{k\to+\infty} M_k^{Y_i} = \overline{Y}_i, \quad (i = 1, 2),$$
$$\lim_{k\to+\infty} N_k^S = \underline{S}, \qquad \lim_{k\to+\infty} N_k^I = \underline{I},$$
$$\lim_{k\to+\infty} N_k^{Y_i} = \underline{Y}_i, \quad (i = 1, 2). \tag{73}$$

From (71), we can obtain

$$\overline{S} = \frac{1}{\beta}\left[d + \frac{b(1-m)\underline{Y}_2}{a\underline{Y}_2 + (1-m)\overline{I}}\right], \qquad \overline{I} = \frac{r}{r + K\beta}\left(K - \overline{S}\right),$$
$$\overline{Y}_1 = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)}\overline{I},$$
$$\overline{Y}_2 = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)}\overline{I},$$
$$\underline{S} = \frac{1}{\beta}\left[d + \frac{b(1-m)\overline{Y}_2}{a\overline{Y}_2 + (1-m)\underline{I}}\right], \qquad \underline{I} = \frac{r}{r + K\beta}\left(K - \underline{S}\right),$$
$$\underline{Y}_1 = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ar_1\left(r_1 + d_1\right)}\underline{I},$$
$$\underline{Y}_2 = \frac{(1-m)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]}{ad_2\left(r_1 + d_1\right)}\underline{I}. \tag{74}$$

It follows from (74) that

$$\alpha\beta(r + K\beta)d_2\left(r_1 + d_1\right)\left(\overline{I}\right)^2 + \alpha\beta(r + K\beta)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]\overline{I}\underline{I}$$
$$= r(K\beta - d)ad_2\left(r_1 + d_1\right)\overline{I} + r\left[a(K\beta - d) - b(1-m)\right]\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]\underline{I}, \tag{75}$$

$$\alpha\beta(r + K\beta)d_2\left(r_1 + d_1\right)\left(\underline{I}\right)^2 + \alpha\beta(r + K\beta)\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]\overline{I}\underline{I}$$
$$= r(K\beta - d)ad_2\left(r_1 + d_1\right)\underline{I} + r\left[a(K\beta - d) - b(1-m)\right]\left[r_1 pb - d_2\left(r_1 + d_1\right)\right]\overline{I}. \tag{76}$$

(75) minus (76),

$$\begin{aligned} &\alpha\beta(K\beta+r)d_2(r_1+d_1)\left[(\bar{I})^2-(\underline{I})^2\right] \\ &\quad = r(K\beta-d)ad_2(r_1+d_1)(\bar{I}-\underline{I}) \\ &\qquad - r\left[a(K\beta-d)-b(1-m)\right] \\ &\qquad\quad \times\left[r_1pb-d_2(r_1+d_1)\right](\bar{I}-\underline{I}). \end{aligned} \tag{77}$$

Assume that $\bar{I}\neq\underline{I}$. Then we derive from (77) that

$$\begin{aligned} &\alpha\beta(K\beta+r)d_2(r_1+d_1)(\bar{I}+\underline{I}) \\ &\quad = r(K\beta-d)ad_2(r_1+d_1) \\ &\qquad - r\left[a(K\beta-d)-b(1-m)\right]\left[r_1pb-d_2(r_1+d_1)\right]. \end{aligned} \tag{78}$$

(75) plus (76),

$$\begin{aligned} &\alpha\beta(K\beta+r)d_2(r_1+d_1)(\bar{I}+\underline{I})^2 \\ &\qquad + 2\alpha\beta(K\beta+r)\left[r_1pb-2d_2(r_1+d_1)\right]\bar{I}\underline{I} \\ &\quad = \left[r(K\beta-d)ad_2(r_1+d_1)\right. \\ &\qquad + r\left(a(K\beta-d)-b(1-m)\right) \\ &\qquad\quad \left.\times\left(r_1pb-d_2(r_1+d_1)\right)\right](\bar{I}+\underline{I}). \end{aligned} \tag{79}$$

On substituting (78) into (79), it follows that

$$\begin{aligned} &\alpha\beta(K\beta+r)\left[r_1pb-2d_2(r_1+d_1)\right]\bar{I}\underline{I} \\ &\quad = r\left[a(K\beta-d)-b(1-m)\right] \\ &\qquad \times\left[r_1pb-d_2(r_1+d_1)\right](\bar{I}+\underline{I}). \end{aligned} \tag{80}$$

Note that $\bar{I}>0$ and $\underline{I}>0$. If $d_2(r_1+d_1)<r_1pb<2d_2(r_1+d_1)$, we derive that $1-m>a(K\beta-d)/b$. This is a contradiction. Hence, we have $\bar{S}=\underline{S}$. It therefore follows from (74) that $\bar{I}=\underline{I}$, $\bar{Y}_1=\underline{Y}_1$, and $\bar{Y}_2=\underline{Y}_2$. We therefore conclude that E^* is globally attractive. The proof is complete. □

5. Conclusion

In this paper, we have incorporated a prey refuge, stage structure for the predator and time delay due to the gestation of the predator into a predator-prey system. Incorporating a refuge into system (1) provides a more realistic model. A refuge can be important for the biological control of a pest; however, increasing the amount of refuge can increase prey densities and lead to population outbreaks. By using the iteration technique and comparison arguments, respectively, we have established sufficient conditions for the global stability of the predator-extinction equilibrium and the globally attractivity for the coexistence equilibrium. As a result, we have shown the threshold for the permanence and extinction of the system. By Theorem 3, we see that the predator population go to extinction if $0<(1-m)<(a/b)(K\beta-d)$ and $k\beta>d$, $d_2(r_1+d_1)>pbr_1$. By Theorem 7, we see that if $0<(1-m)<(a/b)(K\beta-d)$ and $d_2(r_1+d_1)<r_1pb<2d_2(r_1+d_1)$, then both the prey and predator species of system (2) are permanent.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (11101117).

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