

Research Article

Existence of Positive Solutions for an Elastic Beam Equation with Nonlinear Boundary Conditions

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We study the existence and nonexistence of positive solutions for the following fourth-order two-point boundary value problem subject to nonlinear boundary conditions $u''''(t) = \lambda f(t, u(t))$, $t \in (0, 1)$, $u(0) = 0$, $u'(0) = \mu h(u(0))$, $u''(1) = 0$, $u'''(1) = \mu g(u(1))$, where $\lambda > 0$, $\mu \geq 0$ are parameters, and $f : [0, 1] \times [0, +\infty) \rightarrow (0, +\infty)$, $h : [0, +\infty) \rightarrow [0, +\infty)$, and $g : [0, +\infty) \rightarrow (-\infty, 0]$ are continuous. By using the fixed-point index theory, we prove that the problem has at least one positive solution for λ , μ sufficiently small and has no positive solution for λ large enough.

1. Introduction

In this paper, we study the existence and nonexistence of positive solutions for the following fourth-order boundary value problem with nonlinear boundary conditions

$$\begin{aligned} u''''(t) &= \lambda f(t, u(t)), \quad t \in (0, 1), \\ u(0) &= 0, \quad u'(0) = \mu h(u(0)), \\ u''(1) &= 0, \quad u'''(1) = \mu g(u(1)), \end{aligned} \quad (1)$$

where $\lambda > 0$, $\mu \geq 0$ are parameters and $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $h, g \in C(\mathbb{R}, \mathbb{R})$ are real functions. If $\lambda = 1$, $\mu = 0$, in mechanics, problem (1) is called cantilever beam equation [1, 2]. The equation describes the deflection of an elastic beam fixed at the left and freed at the right. There are some papers discussing the existence of solutions of the equation by using various methods, such as the lower and upper solution method, the Leray-Schauder continuation method, fixed-point theory, and the monotone iterative method; see [3–11]. If $\mu \neq 0$, problem (1) has also been studied; see [12–17].

In the case $\mu = 1$, $h(u(0)) = 0$, Yang et al. [12] obtained sufficient conditions of the existence of two solutions of problem (1) by using variational technique and a three-critical-point theorem. Recently, Li and Zhang [13] are concerned with the existence and uniqueness of monotone positive

solution of problem (1) with $\lambda = \mu = 1$, $h(u(0)) = 0$ by using a new fixed-point theorem of generalized concave operators, but some monotone assumptions on f and g are needed.

In 2013, by using a three-critical-point theorem, Cabada and Tersian [14] studied the existence and multiplicity of solutions of problem (1) with $\lambda = \mu$, $h(u(0)) = 0$.

A natural question is what would happen if $\lambda \neq \mu$ and $h(u(0)) \neq 0$?

Motivated by the above papers, we will prove the existence and nonexistence of positive solution for problem (1) by using the fixed-point index theory with the nonlinearity f satisfying superlinear growth condition at infinity.

We make the following assumptions.

- (A1) $f : [0, 1] \times [0, +\infty) \rightarrow (0, +\infty)$ is continuous.
- (A2) $h : [0, +\infty) \rightarrow [0, +\infty)$ is continuous and $g : [0, +\infty) \rightarrow (-\infty, 0]$ is continuous.
- (A3) For any $t \in [0, 1]$, $f_\infty = \lim_{u \rightarrow \infty} (f(t, u)/u) = +\infty$.

The main results of the present paper are summarized as follows.

Theorem 1. Assume that (A1)–(A3) hold. Then problem (1) has at least one positive solution for λ , μ sufficiently small.

Theorem 2. Assume that (A1)–(A3) hold. Then problem (1) has no positive solution for λ large enough.

Remark 3. The results obtained in this paper are not a consequence of the previous theorem in the previous literature. Clearly, the boundary condition of (1) is more general than the above pieces of literature and problem (1) considers two different parameters which is more extensive.

Remark 4. It is pointed out that we do not need any monotone assumption on f , g , and h , which is weaker than the corresponding assumptions on f , g in [13]. References [12, 14] are only considered with the existence of solution for problem (1) with nonlinear boundary condition; however, we study the existence and nonexistence of positive solution of (1).

The remainder of this paper is arranged as follows. Section 2 presents some preliminaries. The proofs of Theorems 1 and 2 are given in Section 3. Finally, we give a simple example to illustrate our main results.

2. Preliminaries

In this section we collect some preliminary results that will be used in subsequent sections. Let $X = C[0, 1]$; then X is a Banach space under the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$. Define $P = \{u \in X \mid u(t) \geq 0, t \in [0, 1]\}$; then $P \subset X$ is a nonnegative cone.

Lemma 5 (see [11, Lemma 2.2]). *Let $h \in C[0, 1]$; then the solution $u(t)$ of the problem*

$$\begin{aligned} u^{(4)}(t) &= h(t), \quad t \in (0, 1), \\ u(0) &= u'(0) = u''(1) = u'''(1) = 0 \end{aligned} \tag{2}$$

is

$$u(t) = \int_0^1 G(t, s) h(s) ds, \tag{3}$$

where

$$G(t, s) = \begin{cases} \frac{1}{6}t^2(3s - t), & 0 \leq t \leq s \leq 1, \\ \frac{1}{6}s^2(3t - s), & 0 \leq s \leq t \leq 1. \end{cases} \tag{4}$$

Now, let us set

$$\phi(t) = \frac{t^2}{2} - \frac{t^3}{6}, \quad \Phi(t) = h(u(0))t - g(u(1))\phi(t), \tag{5}$$

$$\forall t \in [0, 1].$$

It follows from (3) and (5) with simple computation that problem (1) is equivalent to integral equation

$$u(t) = \lambda \int_0^1 G(t, s) f(s, u(s)) ds + \mu \Phi(t) := Tu(t). \tag{6}$$

Note that for any $u \in X$ the function $(Tu)(t)$ satisfies the boundary conditions in (1) by the definition of Green's function $G(t, s)$. In view of Lemma 5, it is easy to see that $u \in X$ is a fixed point of the operator T if and only if $u \in X$ is a solution of problem (1).

Lemma 6 (see [6, Lemma 2.2], [13, Lemma 2.1]). *$G(t, s)$, $\phi(t)$ have the following properties:*

- (i) $G(t, s) > 0, \forall 0 < t, s < 1$;
- (ii) $\max_{t \in [0,1]} G(t, s) \leq G(1, s) = (1/6)s^2(3-s), \forall 0 \leq s \leq 1$;
- (iii) $G(t, s) \geq t^2G(1, s) \geq (t^2/3)G(1, s), \forall 0 \leq t, s \leq 1$;
- (iv) $t^2/3 \leq \phi(t) \leq t^2/2, \forall 0 \leq t \leq 1$.

Lemma 7. *$\Phi(t)$ has the following properties:*

- (i) $\Phi(t) \leq \Phi(1) \leq h(u(0)) - g(u(1)), \forall 0 < t < 1$;
- (ii) $\Phi(t) \geq \Phi(t^2/3) \geq 0, \forall 0 < t < 1$.

Proof. First, we will prove (i). For any $t \in [0, 1]$, it follows that

$$\Phi'(t) = h(u(0)) - g(u(1)) \left(t - \frac{1}{2}t^2 \right). \tag{7}$$

From (A2), $\Phi'(t) \geq 0$. So $\Phi(t)$ is nondecreasing for $t \in [0, 1]$. Moreover,

$$\Phi(t) \leq \Phi(1) = h(u(0)) - \frac{1}{3}g(u(1)) \leq h(u(0)) - g(u(1)), \tag{8}$$

so (i) holds.

Now, we prove (ii). For any $t \in [0, 1]$, obviously, $t \geq t^2/3 \geq 0$.

From (i), $\Phi(t)$ is nondecreasing for $t \in [0, 1]$; then $\Phi(t) \geq \Phi(t^2/3) \geq \Phi(0) = 0$. \square

For any fixed $\tau \in (0, 1)$, define the cone

$$K = \left\{ u \in P \mid \min_{t \in [\tau, 1]} u(t) \geq \frac{\tau^2}{3} \|u\| \right\}. \tag{9}$$

Then $K \subset X$ is a positive cone; letting $K_r = \{u \in K \mid \|u\| < r\}$, $\partial K_r = \{u \in K \mid \|u\| = r\}$ with $r > 0$ is a constant.

Lemma 8. *Assume that (A1) and (A2) hold; then $T : K \rightarrow K$ is completely continuous.*

Proof. Obviously, for any $t \in [0, 1]$, $u \in K$, it follows that $Tu(t) \geq 0$.

For any $u \in K$, from (6) and Lemma 7(i), we have

$$\begin{aligned} \max_{t \in [0,1]} T(u(t)) &= \max_{t \in [0,1]} \left(\lambda \int_0^1 G(t, s) f(s, u(s)) ds + \mu \Phi(t) \right) \\ &\leq \lambda \int_0^1 G(1, s) f(s, u(s)) ds \\ &\quad + \mu (h(u(0)) - g(u(1))). \end{aligned} \tag{10}$$

Hence

$$\|Tu\| \leq \lambda \int_0^1 G(1, s) f(s, u(s)) ds + \mu (h(u(0)) - g(u(1))). \tag{11}$$

For any fixed $\tau \in (0, 1)$, (11) together with Lemma 7(ii), we get

$$\begin{aligned} \min_{\tau \leq t \leq 1} T(u(t)) &= \min_{\tau \leq t \leq 1} \left(\lambda \int_0^1 G(t, s) f(s, u(s)) ds \right. \\ &\quad \left. + \mu (h(u(0))t - g(u(1))\phi(t)) \right) \\ &\geq \frac{\tau^2}{3} \left(\lambda \int_0^1 G(1, s) f(s, u(s)) ds \right. \\ &\quad \left. + \mu (h(u(0)) - g(u(1))) \right) \\ &\geq \frac{\tau^2}{3} \|Tu\|. \end{aligned} \tag{12}$$

So, $T(K) \subset K$.

According to (A1), (A2), and Arzela-Ascoli theorem, it is not difficult to verify that $T : K \rightarrow K$ is completely continuous. \square

The following well-known fixed-point index theorem in cones is crucial to our arguments.

Lemma 9 (see [18]). *Let E be a Banach space and let $K \subset E$ be a cone. For $r > 0$, define $K_r = \{u \in K \mid \|u\| < r\}$. Assume that $T : \bar{K}_r \rightarrow K$ is compact map such that $Tu \neq u$ for $u \in \partial K_r = \{u \in K \mid \|u\| = r\}$.*

- (i) If $\|Tu\| \geq \|u\|$ for $u \in \partial K_r$, then $i(T, K_r, K) = 0$.
- (ii) If $\|Tu\| \leq \|u\|$ for $u \in \partial K_r$, then $i(T, K_r, K) = 1$.

3. Proof of Main Results

In this section, we will prove our main results.

Proof of Theorem 1. Let

$$M = \max_{(t,s) \in [0,1] \times [0,1]} G(t, s) > 0. \tag{13}$$

Let $q > 0$; set

$$I(q) = M \max_{u \in K, \|u\|=q} \left(\int_0^1 f(s, u(s)) ds \right) > 0. \tag{14}$$

For any number $r_1 > 0$, let $K_{r_1} = \{u \in K \mid \|u\| \leq r_1\}$. Choose $\sigma > 0$ so that

$$\sigma \leq \frac{r_1}{2I(r_1)}, \quad \forall u \in \partial K_{r_1}, \tag{15}$$

$$\sigma (h(u(0)) - g(u(1))) \leq \frac{r_1}{2}, \quad \forall u \in \partial K_{r_1}.$$

Then, for $\lambda \leq \sigma, \mu \leq \sigma$ and $u \in \partial K_{r_1}$, combining (6) with Lemma 7, we have

$$\begin{aligned} Tu(t) &\leq \lambda M \int_0^1 f(s, u(s)) ds \\ &\quad + \mu (h(u(0)) - g(u(1))) \\ &\leq \sigma M \int_0^1 f(s, u(s)) ds + \sigma (h(u(0)) - g(u(1))) \\ &\leq \sigma I(r_1) + \frac{r_1}{2} \leq r_1, \end{aligned} \tag{16}$$

which implies that

$$\|Tu\| \leq r_1 = \|u\|, \quad u \in \partial K_{r_1}. \tag{17}$$

Thus, Lemma 9 implies that

$$i(T, K_{r_1}, K) = 1. \tag{18}$$

According to (A3), for any $t \in [0, 1]$, there exists a $p > 0$ such that

$$f(t, u) \geq \eta u, \quad \forall u \geq p, \tag{19}$$

where η is chosen so that

$$\lambda \left(\frac{\tau^2}{3} \right)^2 \eta \int_\tau^1 G(1, s) ds \geq 1, \quad \forall \tau \in (0, 1). \tag{20}$$

For any $\tau \in (0, 1)$, choose $r_2 > \max\{(3/\tau^2)p, 2r_1\}$, and set $K_{r_2} = \{u \in K \mid \|u\| \leq r_2\}$. If $u \in \partial K_{r_2}$, then

$$\min_{\tau \leq t \leq 1} u(t) \geq \frac{\tau^2}{3} r_2 > p. \tag{21}$$

It follows from (A2) and Lemma 6 that

$$\begin{aligned} \min_{t \in [\tau, 1]} T(u(t)) &\geq \min_{t \in [\tau, 1]} \left(\lambda \int_0^1 G(t, s) f(s, u(s)) ds \right) \\ &\geq \lambda \frac{\tau^2}{3} \int_0^1 G(1, s) f(s, u(s)) ds \\ &\geq \lambda \frac{\tau^2}{3} \int_0^1 G(1, s) \eta u(s) ds \\ &\geq \lambda \frac{\tau^2}{3} \eta \int_\tau^1 G(1, s) u(s) ds \\ &\geq \lambda \left(\frac{\tau^2}{3} \right)^2 \eta \int_\tau^1 G(1, s) ds \|u\| \\ &\geq \|u\|, \end{aligned} \tag{22}$$

which implies that

$$\|T(u)\| \geq \|u\|, \quad u \in \partial K_{r_2}. \tag{23}$$

Hence, from Lemma 9, we get

$$i(T, K_{r_2}, K) = 0. \tag{24}$$

From (18) and (24), it follows that

$$i(T, K_{r_2} \setminus \bar{K}_{r_1}, K) = i(T, K_{r_2}, K) - i(T, K_{r_1}, K) = -1. \tag{25}$$

Therefore, T has a fixed point in $K_{r_2} \setminus \bar{K}_{r_1}$; that is, problem (1) has at least one positive solution. \square

Proof of Theorem 2. From (A1) and (A3), for any $t \in [0, 1]$, there is a constant c such that

$$f(t, u) \geq cu, \quad \forall u \geq 0. \tag{26}$$

Let $u \in X$ be a positive solution of (1); then $u \in K$ satisfies (6). Choose λ large enough such that

$$\lambda \left(\frac{\tau^2}{3} \right)^2 c \int_{\tau}^1 G(1, s) ds > 1, \quad \forall \tau \in (0, 1). \tag{27}$$

For any $\tau \in (0, 1)$, we have

$$\begin{aligned} \min_{t \in [\tau, 1]} u(t) &\geq \min_{t \in [\tau, 1]} \left(\lambda \int_0^1 G(t, s) f(s, u(s)) ds \right) \\ &\geq \lambda \frac{\tau^2}{3} \int_{\tau}^1 G(1, s) cu(s) ds \\ &\geq \lambda \left(\frac{\tau^2}{3} \right)^2 c \int_{\tau}^1 G(1, s) ds \|u\| \\ &> \|u\|, \end{aligned} \tag{28}$$

which is a contradiction. Therefore, problem (1) has no positive solution for λ large enough. \square

Finally, we give an example to illustrate our main result.

Example 10. Consider the following fourth-order two-point problem with nonlinear boundary conditions:

$$\begin{aligned} u''''(t) &= \lambda(u^3(t)(3 + \sin(u(t))) + t^2 + 1), \quad t \in (0, 1), \\ u(0) &= 0, \quad u'(0) = \mu(2[u(0)]^2 + 1), \\ u''(1) &= 0, \quad u'''(1) = \mu([-u(1)])^{1/2} - 1. \end{aligned} \tag{29}$$

Clearly, the nonlinearity

$$\begin{aligned} f(t, u) &= u^3(t)(3 + \sin(u(t))) + t^2 + 1, \\ (t, u) &\in [0, 1] \times [0, \infty), \end{aligned} \tag{30}$$

$$h(s) = 2s^2 + 1, \quad g(s) = -s^{1/2} - 1, \quad \forall s \in [0, +\infty).$$

It is easy to check that (A1)–(A3) are satisfied. By simple computation, we have $M = 1/2$. Set $q = 2$; then $I(q) = 17$.

If $r_1 = 4$ (where r_1 can be any real number greater than 0), $u \in \partial K_{r_1}$, then $r_1/2I(q) = 2/17$, $\Phi(t) \leq 36$, and take $\sigma = 1/20$; then $\|T(u)\| \leq \|u\|$ obviously holds for $\lambda, \mu < 1/20$.

Choosing $r_2 = 9$, it is easy to verify that

$$\|T(u)\| \geq \|u\|, \quad \forall u \in \partial K_{r_2}. \tag{31}$$

From Theorem 1, T has a fixed point in $K_{r_2} \setminus \bar{K}_{r_1}$; that is, problem (29) has at least one positive solution.

For any $t \in [0, 1]$, there is a constant $c = 3$, so that

$$f(t, u) \geq 3u, \quad \forall u \geq 0. \tag{32}$$

If $\tau = 1/2$, then problem (29) has no positive solution for $\lambda > 450$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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