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Research Article

Global Exponential Stability of Pseudo Almost Periodic Solutions for SICNNs with Time-Varying Leakage Delays

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This paper is concerned with the shunting inhibitory cellular neural networks (SICNNs) with time-varying delays in the leakage (or forgetting) terms. Under proper conditions, we employ a novel argument to establish a criterion on the global exponential stability of pseudo almost periodic solutions by using Lyapunov functional method and differential inequality techniques. We also provide numerical simulations to support the theoretical result.

1. Introduction

In the last three decades, shunting inhibitory cellular neural networks (SICNNs) have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of the equilibrium point and periodic and almost periodic solutions of SICNNs with time-varying delays in the literature. We refer the reader to [1–7] and the references cited therein.

It is well known that SICNNs have been introduced as new cellular neural networks (CNNs) in Bouzerdoum et al. in [1, 8, 9], which can be described by

$$\begin{aligned} x'_{ij}(t) = {} & -a_{ij}(t)\, x_{ij}(t) \\ & - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\, f\left(x_{kl}\left(t-\tau_{kl}(t)\right)\right) x_{ij}(t) \\ & - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\ & \cdot \int_0^\infty K_{ij}(u)\, g\left(x_{kl}(t-u)\right) du\, x_{ij}(t) + L_{ij}(t), \\ & \qquad i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n, \end{aligned} \tag{1}$$

where C_{ij} denotes the cell at the (i, j) position of the lattice. The r-neighborhood $N_r(i, j)$ of C_{ij} is given as

$$\begin{aligned} N_r(i,j) = \{ & C_{kl} : \max\left(|k-i|, |l-j|\right) \le r, \\ & 1 \le k \le m, 1 \le l \le n \}, \end{aligned} \tag{2}$$

where $N_q(i, j)$ is similarly specified, x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, the function $a_{ij}(t) > 0$ represents the passive decay rate of the cell activity, $C_{ij}^{kl}(t)$ and $B_{ij}^{kl}(t)$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity functions $f(\cdot)$ and $g(\cdot)$ are continuous functions representing the output or firing rate of the cell C_{kl}, and $\tau_{kl}(t) \ge 0$ corresponds to the transmission delay.

Obviously, the first term in each of the right side of (1) corresponds to stabilizing negative feedback of the system which acts instantaneously without time delay; these terms are variously known as "forgettin" or leakage terms (see, for instance, Kosko [10], Haykin [11]). It is known from the literature on population dynamics and neural networks dynamics (see Gopalsamy [12]) that time delays in the stabilizing negative feedback terms will have a tendency to destabilize a system. Therefore, the authors of [13–19] dealt with the existence and stability of equilibrium and periodic solutions for neuron networks model involving

leakage delays. Recently, Liu and Shao [20] considered the following SICNNs with time-varying leakage delays:

$$\begin{aligned} x'_{ij}(t) = &-a_{ij}(t)\, x_{ij}\left(t-\eta_{ij}(t)\right) \\ &- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\, f\left(x_{kl}\left(t-\tau_{kl}(t)\right)\right) x_{ij}(t) \\ &- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\ &\times \int_0^{\infty} K_{ij}(u)\, g\left(x_{kl}(t-u)\right) du\, x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{3}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $\eta_{ij}: \mathbb{R} \to [0 + \infty)$ denotes the leakage delay. By using Lyapunov functional method and differential inequality techniques, in [20], some sufficient conditions have been established to guarantee that all solutions of (1) converge exponentially to the almost periodic solution. Moreover, it is well known that the global exponential convergence behavior of solutions plays a key role in characterizing the behavior of dynamical system since the exponential convergent rate can be unveiled (see [21–24]). However, to the best of our knowledge, few authors have considered the exponential convergence on the pseudo almost periodic solution for (1). Motivated by the above discussions, in this paper, we will establish the existence and uniqueness of pseudo almost periodic solution of (1) by using the exponential dichotomy theory and contraction mapping fixed point theorem. Meanwhile, we also will give the conditions to guarantee that all solutions and their derivatives of solutions for (1) converge exponentially to the pseudo almost periodic solution and its derivative, respectively.

For convenience, we denote by $\mathbb{R}^p (\mathbb{R} = \mathbb{R}^1)$ the set of all p-dimensional real vectors (real numbers). We will use

$$\begin{aligned} \left\{x_{ij}(t)\right\} = &\left(x_{11}(t), \ldots, x_{1n}(t), \ldots, x_{i1}(t), \ldots,\right. \\ &\left. x_{in}(t), \ldots, x_{m1}(t), \ldots, x_{mn}(t)\right) \in \mathbb{R}^{m\times n}. \end{aligned} \tag{4}$$

For any $x(t) = \{x_{ij}(t)\} \in \mathbb{R}^{m\times n}$, we let $|x|$ denote the absolute-value vector given by $|x| = \{|x_{ij}|\}$ and define $\|x(t)\| = \max_{(i,j)}\{|x_{ij}(t)|\}$. A matrix or vector $A \geq 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A_1 and A_2, $A_1 \geq A_2$ (resp. $A_1 > A_2$) means that $A_1 - A_2 \geq 0$ (resp. $A_1 - A_2 > 0$). For the convenience, we will introduce the notations:

$$h^+ = \sup_{t\in\mathbb{R}} |h(t)|, \qquad h^- = \inf_{t\in\mathbb{R}} |h(t)|, \tag{5}$$

where $h(t)$ is a bounded continuous function.

The initial conditions associated with system (3) are of the form:

$$\begin{aligned} &x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0], \\ &ij \in J := \{11, \ldots, 1n, 21, \ldots, 2n, \ldots, m1, \ldots, mn\}, \end{aligned} \tag{6}$$

where $\varphi_{ij}(\cdot)$ and $\varphi'_{ij}(\cdot)$ are real-valued bounded continuous functions defined on $(-\infty, 0]$.

The paper is organized as follows. Section 2 includes some lemmas and definitions, which can be used to check the existence of almost periodic solutions of (3). In Section 3, we present some new sufficient conditions for the existence of the continuously differentiable pseudo almost periodic solution of (3). In Section 4, we establish sufficient conditions on the global exponential stability of pseudo almost periodic solutions of (3). At last, an example and its numerical simulation are given to illustrate the effectiveness of the obtained results.

2. Preliminary Results

In this section, we will first recall some basic definitions and lemmas which are used in what follows.

In this paper, $\mathrm{BC}(\mathbb{R}, \mathbb{R}^p)$ denotes the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}^p$. Note that $(\mathrm{BC}(\mathbb{R}, \mathbb{R}^p), \|\cdot\|_\infty)$ is a Banach space where $\|\cdot\|_\infty$ denotes the sup norm $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$.

Definition 1 (see [25, 26]). Let $u(t) \in \mathrm{BC}(\mathbb{R}, \mathbb{R}^p)$. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : \|u(t+\delta) - u(t)\| < \varepsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense; that is, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$; for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $\|u(t+\delta) - u(t)\| < \varepsilon$, for all $t \in \mathbb{R}$.

We denote by $\mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$. Besides, the concept of pseudo almost periodicity (pap) was introduced by Zhang in the early nineties. It is a natural generalization of the classical almost periodicity. Precisely, define the class of functions $\mathrm{PAP}_0(\mathbb{R}, \mathbb{R})$ as follows:

$$\left\{f \in \mathrm{BC}\left(\mathbb{R}, \mathbb{R}^n\right) \mid \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|\, dt = 0\right\}. \tag{7}$$

A function $f \in \mathrm{BC}(\mathbb{R}, \mathbb{R}^n)$ is called pseudo almost periodic if it can be expressed as

$$f = h + \varphi, \tag{8}$$

where $h \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$ and $\varphi \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}^n)$. The collection of such functions will be denoted by $\mathrm{PAP}(\mathbb{R}, \mathbb{R}^n)$. The functions h and φ in the above definition are, respectively, called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. The decomposition given in definition above is unique. Observe that $(\mathrm{PAP}(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space and $\mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$ is a proper subspace of $\mathrm{PAP}(\mathbb{R}, \mathbb{R}^n)$ since the function $\phi(t) = \cos \pi t + \cos t + e^{-t^4 \sin^2 t}$ is pseudo almost periodic function but not almost periodic. It should be mentioned that pseudo almost periodic functions possess many interesting properties; we shall need only a few of them and for the proofs we shall refer to [25].

Lemma 2 (see [25, page 57]). *If $f \in PAP(\mathbb{R}, \mathbb{R})$ and g is its almost periodic component, then we have*

$$g(\mathbb{R}) \subset \overline{f(\mathbb{R})}. \tag{9}$$

Therefore $\|f\|_\infty \geq \|g\|_\infty \geq \inf_{x\in\mathbb{R}} |g(x)| \geq \inf_{x\in\mathbb{R}} |f(x)|$.

Lemma 3 (see [25, page 140]). *Suppose that both functions f and its derivative f' are in $PAP(\mathbb{R},\mathbb{R})$. That is, $f = g + \varphi$ and $f' = \alpha + \beta$, where $g, \alpha \in AP(\mathbb{R},\mathbb{R})$ and $\varphi, \beta \in PAP_0(\mathbb{R},\mathbb{R})$. Then the functions g and φ are continuous differentiable so that*

$$g' = \alpha, \qquad \varphi' = \beta. \tag{10}$$

Lemma 4. *Let $B^* = \{f \mid f, f' \in PAP(\mathbb{R},\mathbb{R})\}$ equipped with the induced norm defined by $\|f\|_{B^*} = \max\{\|f\|_\infty, \|f'\|_\infty\} = \max\{\sup_{t\in\mathbb{R}}|f(t)|, \sup_{t\in\mathbb{R}}|f'(t)|\}$, and then B^* is a Banach space.*

Proof. Suppose that $\{f_p\}_{p=1}^{+\infty}$ is a Cauchy sequence in B^*, and then for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$, such that

$$\begin{aligned}&\left\|f_p - f_q\right\|_{B^*}\\ &= \max\left\{\sup_{t\in\mathbb{R}}\left|f_p(t) - f_q(t)\right|, \sup_{t\in\mathbb{R}}\left|f_{p'}(t) - f_{q'}(t)\right|\right\} < \varepsilon,\\ &\qquad \forall p, q \geq N(\varepsilon).\end{aligned} \tag{11}$$

By the definition of pseudo almost periodic function, let

$$\begin{aligned}f_p &= g_p + \varphi_p, \quad \text{where } g_p \in \mathrm{AP}(\mathbb{R},\mathbb{R}),\\ &\varphi_p \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R}),\ p = 1, 2, \ldots.\end{aligned} \tag{12}$$

From Lemma 3, we obtain

$$\begin{aligned}f_p' &= g_p' + \varphi_p', \quad \text{where } g_p' \in \mathrm{AP}(\mathbb{R},\mathbb{R}),\\ &\varphi_p' \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R}),\ p = 1, 2, \ldots.\end{aligned} \tag{13}$$

On combining (11) with Lemma 2, we deduce that, $\{g_p\}_{p=1}^{+\infty}, \{g_p'\}_{p=1}^{+\infty} \subset \mathrm{AP}(\mathbb{R},\mathbb{R})$ are Cauchy sequence, so that $\{\varphi_p\}_{p=1}^{+\infty}, \{\varphi_p'\}_{p=1}^{+\infty} \subset \mathrm{PAP}_0(\mathbb{R},\mathbb{R})$ are also Cauchy sequence.

Firstly, we show that there exists $g \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ such that g_p uniformly converges to g, as $p \to +\infty$.

Note that $\{g_p\}$ is Cauchy sequence in $\mathrm{AP}(\mathbb{R},\mathbb{R})$. for all $\varepsilon > 0$, $\exists N(\varepsilon)$, such that for all $p, q \geq N(\varepsilon)$

$$\left|g_p(t) - g_q(t)\right| < \varepsilon, \quad \forall t \in \mathbb{R}. \tag{14}$$

So for fixed $t \in \mathbb{R}$, it is easy to see $\{g_p(t)\}_{p=1}^{+\infty}$ is Cauchy number sequence. Thus, the limits of $g_p(t)$ exist as $p \to +\infty$ and let $g(t) = \lim_{p\to+\infty} g_p(t)$. In (14), let $q \to +\infty$, and we have

$$\left|g(t) - g_p(t)\right| \leq \varepsilon, \quad \forall t \in \mathbb{R},\ p \geq N(\varepsilon). \tag{15}$$

Thus, g_n uniformly converges to g, as $p \to +\infty$. Moreover, from the Theorem 1.9 [26, page 5], we obtain $g \in \mathrm{AP}(\mathbb{R},\mathbb{R})$. Similarly, we also obtain that there exist $g^* \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $\varphi, \varphi^* \in \mathrm{BC}(\mathbb{R},\mathbb{R})$, such that

$$\begin{aligned}&\left|g^*(t) - g_p'(t)\right| \leq \varepsilon,\\ &\left|\varphi(t) - \varphi_p(t)\right| \leq \varepsilon,\\ &\left|\varphi^*(t) - \varphi_p'(t)\right| \leq \varepsilon,\\ &\forall t \in \mathbb{R},\ p \geq N(\varepsilon),\end{aligned} \tag{16}$$

which lead to

$$g_p' \Longrightarrow g^*, \qquad \varphi_p \Longrightarrow \varphi, \qquad \varphi_p' \Longrightarrow \varphi^*, \tag{17}$$

where $p \to +\infty$ and "$\Rightarrow$" means uniform convergence.

Next, we claim that $\varphi, \varphi^* \in \mathrm{PAP}_0(\mathbb{R})$. Together with (16) and the facts that

$$\begin{aligned}&\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\left|\varphi_p(s)\right|ds = 0, \qquad \lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\left|\varphi_p'(s)\right|ds = 0,\\ &\qquad p = 1, 2, \ldots,\\ &\frac{1}{2r}\int_{-r}^{r}\left|\varphi(s)\right|ds \leq \frac{1}{2r}\int_{-r}^{r}\left|\varphi(s) - \varphi_p(s)\right|ds\\ &\qquad + \frac{1}{2r}\int_{-r}^{r}\left|\varphi_p(s)\right|ds, \quad r > 0,\ n = 1, 2, \ldots,\\ &\frac{1}{2r}\int_{-r}^{r}\left|\varphi^*(s)\right|ds \leq \frac{1}{2r}\int_{-r}^{r}\left|\varphi^*(s) - \varphi_p'(s)\right|ds\\ &\qquad + \frac{1}{2r}\int_{-r}^{r}\left|\varphi_p'(s)\right|ds, \quad r > 0,\ p = 1, 2, \ldots,\end{aligned} \tag{18}$$

we have

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\left|\varphi(s)\right|ds = 0, \qquad \lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\left|\varphi^*(s)\right|ds = 0. \tag{19}$$

Hence $\varphi, \varphi^* \in \mathrm{PAP}_0(\mathbb{R})$. Let $f = \mathrm{g} + \varphi$, $f^* = g^* + \varphi^*$, then $f = g + \varphi \in \mathrm{PAP}(\mathbb{R})$, $f^* = g^* + \varphi^* \in \mathrm{PAP}(\mathbb{R})$ and $f_p \Rightarrow f$, $f_p' \Rightarrow f^*$ as $p \to +\infty$.

Finally, we reveal $f' = f^*$. For $t, \Delta t \in \mathbb{R}$, it follows that

$$f_p(t + \Delta t) - f_p(t) = \int_t^{t+\Delta t} f_p'(s)\, ds. \tag{20}$$

In view of the uniform convergence of f_p and f_p', let $p \to +\infty$ for (20), and we get

$$f(t + \Delta t) - f(t) = \int_t^{t+\Delta t} f^*(s)\, ds, \tag{21}$$

which implies that

$$\begin{aligned}f^*(t) &= \lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t} f^*(s)\, ds}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{f(t + \Delta t) - f(t)}{\Delta t} = f'(t).\end{aligned} \tag{22}$$

In summary, in view of (15), (16), and (22), we obtain that the Cauchy sequence $\{f_p\}_{p=1}^{+\infty} \subset B^*$ satisfies

$$\left\|f_p - f\right\|_{B^*} \longrightarrow 0\,(p \longrightarrow +\infty), \tag{23}$$

and $f \in B^*$. This yields that B^* is a Banach space. The proof is completed. □

Remark 5. Let $B = \{f \mid f, f' \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^{n\times m})\}$ equipped with the induced norm defined by $\|f\|_B = \max\{\|f\|_\infty, \|f'\|_\infty\} = \max\{\sup_{t\in\mathbb{R}}\|f(t)\|, \sup_{t\in\mathbb{R}}\|f'(t)\|\}$. It follows from Lemma 4 that B is a Banach space.

Definition 6 (see [19, 20]). Let $x \in \mathbb{R}^p$ and $Q(t)$ be a $p \times p$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = Q(t)\,x(t) \tag{24}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α, and projection P and the fundamental solution matrix $X(t)$ of (24) satisfying

$$\begin{aligned} &\left\|X(t)\,PX^{-1}(s)\right\| \le ke^{-\alpha(t-s)}, \quad \text{for } t \ge s,\\ &\left\|X(t)\,(I-P)\,X^{-1}(s)\right\| \le ke^{-\alpha(s-t)}, \quad \text{for } t \le s. \end{aligned} \tag{25}$$

Lemma 7 (see [19]). *Assume that $Q(t)$ is an almost periodic matrix function and $g(t) \in PAP(\mathbb{R}, \mathbb{R}^p)$. If the linear system* (24) *admits an exponential dichotomy, then pseudo almost periodic system*

$$x'(t) = Q(t)\,x(t) + g(t) \tag{26}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$\begin{aligned} x(t) = &\int_{-\infty}^{t} X(t)\,PX^{-1}(s)\,g(s)\,ds\\ &-\int_{t}^{+\infty} X(t)\,(I-P)\,X^{-1}(s)\,g(s)\,ds. \end{aligned} \tag{27}$$

Lemma 8 (see [19, 20]). *Let $c_i(t)$ be an almost periodic function on $\mathbb{R}$ and*

$$M[c_i] = \lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T} c_i(s)\,ds > 0, \quad i = 1, 2, \ldots, p. \tag{28}$$

Then the linear system

$$x'(t) = \mathrm{diag}\left(-c_1(t), -c_2(t), \ldots, -c_p(t)\right)x(t) \tag{29}$$

admits an exponential dichotomy on $\mathbb{R}$.

3. Existence of Pseudo Almost Periodic Solutions

In this section, we establish sufficient conditions on the existence of pseudo almost periodic solutions of (3).

For $ij, kl \in J$, $a_{ij} : \mathbb{R} \to (0, +\infty)$ is an almost periodic function, $\eta_{ij}, \tau_{kl} : \mathbb{R} \to [0, +\infty)$, and $L_{ij}, C_{ij}^{kl}, B_{ij}^{kl} : \mathbb{R} \to \mathbb{R}$ are pseudo almost periodic functions. We also make the following assumptions which will be used later.

We also make the following assumptions.

(S1) There exist constants M_f, M_g, L_f, and L_g such that

$$\begin{aligned} &\left|f(u) - f(v)\right| \le L_f|u-v|, \quad \left|f(u)\right| \le M_f,\\ &\left|g(u) - g(v)\right| \le L_g|u-v|, \quad \left|g(u)\right| \le M_g, \qquad \forall u, v \in \mathbb{R}. \end{aligned} \tag{30}$$

(S2) For $ij \in J$, the delay kernels $K_{ij} : [0, \infty) \to \mathbb{R}$ are continuous, and $|K_{ij}(t)|e^{\beta t}$ are integrable on $[0, \infty)$ for a certain positive constant β.

(S3) Let

$$\begin{aligned} L = \max\Bigg\{&\max_{(i,j)}\left\{\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du}L_{ij}(s)\,ds\right|\right\},\\ &\max_{(i,j)}\left\{\sup_{t\in\mathbb{R}}\left|L_{ij}(t) - a_{ij}(t)\right.\right.\\ &\left.\left.\times\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du}L_{ij}(s)\,ds\right|\right\}\Bigg\} > 0. \end{aligned} \tag{31}$$

Moreover, there exists a constant κ such that

$$\begin{aligned} &0 < \kappa \le L, \qquad \max_{(i,j)}\left\{\frac{1}{a_{ij}^-}E_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right)E_{ij}\right\} \le \kappa,\\ &\max_{(i,j)}\left\{\frac{1}{a_{ij}^-}F_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right)F_{ij}\right\} < 1, \end{aligned} \tag{32}$$

where

$$\begin{aligned} E_{ij} = \Bigg[&a_{ij}^+\eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+}\left(L^f(\kappa + L) + \left|f(0)\right|\right)\\ &+ \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+}\\ &\times\int_0^\infty \left|K_{ij}(u)\right|du\left(L^g(\kappa + L) + \left|g(0)\right|\right)\Bigg](\kappa + L), \qquad ij \in J, \end{aligned}$$

$$\begin{aligned} F_{ij} = \Bigg[&a_{ij}^+\eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+}\left(M_f + L^f(\kappa + L)\right)\\ &+ \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+}\left(\int_0^\infty \left|K_{ij}(u)\right|duM_g\right.\\ &\left.+\int_0^\infty \left|K_{ij}(u)\right|duL^g(\kappa + L)\right)\Bigg], \qquad ij \in J. \end{aligned} \tag{33}$$

Lemma 9. *Assume that assumptions (S_1) and (S_2) hold. Then, for $\varphi(\cdot) \in PAP(\mathbb{R}, \mathbb{R})$, the function $\int_0^\infty K_{ij}(u)g(\varphi(t-u))\,du$ belongs to $PAP(\mathbb{R}, \mathbb{R})$, where $ij \in J$.*

Proof. Let $\varphi \in \mathrm{PAP}(\mathbb{R},\mathbb{R})$. Obviously, (S_1) implies that g is a uniformly continuous function on $\mathbb{R}$. By using Corollary 5.4 in [25, page 58], we immediately obtain the following:

$$g(\varphi(t)) = \chi_1(t) + \chi_2(t) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \tag{34}$$

where $\chi_1 \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $\chi_2 \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R})$. Then, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$; for any interval with length l, there exists a number $\tau = \tau(\varepsilon)$ in this interval such that

$$\left|\chi_1(t+\tau) - \chi_1(t)\right| < \frac{\varepsilon}{1 + \int_0^\infty \left|K_{ij}(u)\right| du}, \quad \forall t \in \mathbb{R},\ ij \in J,$$
$$\lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} \left|\chi_2(v)\right| dv = 0. \tag{35}$$

It follows that

$$\begin{aligned}
&\left|\int_0^\infty K_{ij}(u)\,\chi_1(t+\tau-u)\,du - \int_0^\infty K_{ij}(u)\,\chi_1(t-u)\,du\right| \\
&\quad\le \int_0^\infty \left|K_{ij}(u)\right| \left|\chi_1(t+\tau-u) - \chi_1(t-u)\right| du \\
&\quad< \int_0^\infty \left|K_{ij}(u)\right| du \frac{\varepsilon}{1 + \int_0^\infty \left|K_{ij}(u)\right| du} \\
&\quad< \varepsilon, \quad \forall t \in \mathbb{R},\ ij \in J, \\
&\lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} \left|\int_0^\infty K_{ij}(u)\,\chi_2(v-u)\,du\right| dv \\
&\quad\le \lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} \int_0^\infty \left|K_{ij}(u)\right| \left|\chi_2(v-u)\right| du\,dv \\
&\quad= \lim_{r\to+\infty} \frac{1}{2r} \int_0^\infty \left|K_{ij}(u)\right| \int_{-r}^{r} \left|\chi_2(v-u)\right| dv\,du \\
&\quad= \lim_{r\to+\infty} \frac{1}{2r} \int_0^\infty \left|K_{ij}(u)\right| \int_{-r-u}^{r-u} \left|\chi_2(z)\right| dz\,du \\
&\quad\le \lim_{r\to+\infty} \frac{1}{2r} \int_0^\infty \left|K_{ij}(u)\right| \int_{-r-u}^{r+u} \left|\chi_2(z)\right| dz\,du \\
&\quad\le \lim_{r\to+\infty} \int_0^\infty \left|K_{ij}(u)\right| \left(1 + \frac{1}{r}u\right) \frac{1}{2(r+u)} \\
&\qquad\times \int_{-r-u}^{r+u} \left|\chi_2(z)\right| dz\,du \\
&\quad\le \lim_{r\to+\infty} \int_0^\infty \left|K_{ij}(u)\right| e^{(1/r)u} \frac{1}{2(r+u)} \int_{-r-u}^{r+u} \left|\chi_2(z)\right| dz\,du \\
&\quad\le \lim_{r\to+\infty} \int_0^\infty \left|K_{ij}(u)\right| e^{\beta u} \frac{1}{2(r+u)} \int_{-r-u}^{r+u} \left|\chi_2(z)\right| dz\,du \\
&\quad= 0, \qquad \text{where } r > \frac{1}{\beta},\ ij \in J.
\end{aligned} \tag{36}$$

Thus,

$$\begin{aligned}
&\int_0^\infty K_{ij}(u)\,\chi_1(t-u)\,du \in \mathrm{AP}(\mathbb{R},\mathbb{R}), \\
&\int_0^\infty K_{ij}(u)\,\chi_2(t-u)\,du \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R}),
\end{aligned} \tag{37}$$

which yield

$$\begin{aligned}
&\int_0^\infty K_{ij}(u)\,g_j(\varphi(t-u))\,du \\
&\quad= \int_0^\infty K_{ij}(u)\,\chi_1(t-u)\,du \\
&\qquad+ \int_0^\infty K_{ij}(u)\,\chi_2(t-u)\,du \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \quad ij \in J.
\end{aligned} \tag{38}$$

The proof of Lemma 9 is completed. □

Theorem 10. *Let (S_1), (S_2), and (S_3) hold. Then, there exists at least one continuously differentiable pseudo almost periodic solution of system* (3).

Proof. Let $\varphi \in B$. Obviously, the boundedness of φ' and (S_1) imply that f and φ_{ij} are uniformly continuous functions on $\mathbb{R}$ for $ij \in J$. Set $\widetilde{f}(t,z) = \varphi_{ij}(t-z)$ $(ij \in J)$. By Theorem 5.3 in [25, page 58] and Definition 5.7 in [25, page 59], we can obtain that $\widetilde{f} \in \mathrm{PAP}(\mathbb{R}\times\Omega)$ and $\widetilde{f}$ is continuous in $z \in K$ and uniformly in $t \in \mathbb{R}$ for all compact subset K of Ω. This, together with $\tau_{ij}, \eta_{ij} \in \mathrm{PAP}(\mathbb{R},\mathbb{R})$ and Theorem 5.11 in [25, page 60], implies that

$$\begin{aligned}
&\varphi_{ij}\left(t - \tau_{ij}(t)\right) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \\
&\varphi_{ij}\left(t - \eta_{ij}(t)\right) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \\
&\qquad ij \in J.
\end{aligned} \tag{39}$$

Again from Corollary 5.4 in [25, page 58], we have

$$f\left(\varphi_{ij}\left(t - \tau_{ij}(t)\right)\right) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \quad ij \in J, \tag{40}$$

which, together with Lemma 9, implies

$$\begin{aligned}
&a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} \varphi_{ij}'(s)\,ds \\
&\quad= a_{ij}(t)\,\varphi_{ij}(t) - a_{ij}(t)\,\varphi_{ij}\left(t - \eta_{ij}(t)\right) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \\
&\qquad ij \in J, \\
&- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\,f\left(\varphi_{kl}\left(t - \tau_{kl}(t)\right)\right)\varphi_{ij}(t) \\
&- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\
&\times \int_0^\infty K_{ij}(u)\,g\left(\varphi_{kl}(t-u)\right) du\,\varphi_{ij}(t) + L_{ij}(t) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}), \\
&\qquad ij \in J.
\end{aligned} \tag{41}$$

For any $\varphi \in B$, we consider the pseudo almost periodic solution $x^{\varphi}(t)$ of nonlinear pseudo almost periodic differential equations

$$\begin{aligned} x'_{ij}(t) = {} & -a_{ij}(t)\, x_{ij}(t) + a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\, ds \\ & - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\, f\left(\varphi_{kl}\left(t - \tau_{kl}(t)\right)\right) \varphi_{ij}(t) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \\ & \times \int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(t-u)\right) du \varphi_{ij}(t) + L_{ij}(t), \\ & ij \in J. \end{aligned} \tag{42}$$

Then, notice that $M[a_{ij}] > 0$, $ij \in J$, and it follows from Lemma 8 that the linear system,

$$x'_{ij}(t) = -a_{ij}(t)\, x_{ij}(t), \quad ij \in J, \tag{43}$$

admits an exponential dichotomy on $\mathbb{R}$. Thus, by Lemma 7, we obtain that the system (42) has exactly one pseudo almost periodic solution:

$$\begin{aligned} x^{\varphi}(t) = {} & \left\{x_{ij}^{\varphi}(t)\right\} \\ = {} & \left\{ \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \right. \\ & \times \left[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} \varphi'_{ij}(u)\, du \right. \\ & - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \\ & \times f\left(\varphi_{kl}\left(s - \tau_{kl}(s)\right)\right) \varphi_{ij}(s) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) \\ & \times \int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(s-u)\right) du \varphi_{ij}(s) \\ & \left. \left. + L_{ij}(s) \right] ds \right\}. \end{aligned} \tag{44}$$

From (S_1), (S_2), and the Corollary 5.6 in [25, page 59], we get

$$\begin{aligned} \left(x^{\varphi}(t)\right)' = {} & \left\{x_{ij}^{\varphi\prime}(t)\right\} \\ = {} & \left\{ \left[a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\, ds \right. \right. \\ & - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\, f\left(\varphi_{kl}\left(t - \tau_{kl}(t)\right)\right) \varphi_{ij}(t) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \\ & \times \int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(t-u)\right) du \varphi_{ij}(t) + L_{ij}(t) \Big] \\ & - a_{ij}(t) \\ & \times \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \\ & \times \left[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} \varphi'_{ij}(u)\, du \right. \\ & - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \\ & \times f\left(\varphi_{kl}\left(s - \tau_{kl}(s)\right)\right) \varphi_{ij}(s) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) \\ & \times \int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(s-u)\right) du \varphi_{ij}(s) \\ & \left. \left. \left. + L_{ij}(s) \right] ds \right\} \end{aligned} \tag{45}$$

which is a pseudo almost periodic function. Therefore, $x^{\varphi} \in B$. Let $\varphi^0(t) = x^0(t)$. Then,

$$\begin{aligned} \varphi^0(t) = \left\{\varphi_{ij}^0(t)\right\} = \left\{ \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} L_{ij}(s)\, ds \right\} \in B, \\ L = \left\|\varphi^0\right\|_B. \end{aligned} \tag{46}$$

Set

$$B^{**} = \left\{\varphi \mid \varphi \in B, \left\|\varphi - \varphi^0\right\|_B \le \kappa\right\}. \tag{47}$$

If $\varphi \in B^{**}$, then

$$\left\|\varphi\right\|_B \le \left\|\varphi - \varphi^0\right\|_B + \left\|\varphi^0\right\|_B \le \kappa + L. \tag{48}$$

Now, we define a mapping $T : B^{**} \to B^{**}$ by setting

$$T(\varphi)(t) = x^{\varphi}(t), \quad \forall \varphi \in B^{**}. \tag{49}$$

We next prove that the mapping T is a contraction mapping of the B^{**}.

First we show that, for any $\varphi \in B^{**}$, $T(\varphi) = x^{\varphi} \in B^{**}$.

Note that

$$
\begin{aligned}
&\left|T(\varphi)(t)-\varphi^{0}(t)\right| \\
&= \left\{\left|\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{ij}(u)du}\right.\right. \\
&\qquad \times \left[a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s}\varphi_{ij}'(u)\,du\right. \\
&\qquad\quad -\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\, f\left(\varphi_{kl}(s-\tau_{kl}(s))\right)\varphi_{ij}(s) \\
&\qquad\quad -\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s) \\
&\qquad\quad \left.\left.\times \int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(s-u)\right)du\varphi_{ij}(s)\right]ds\right| \Bigg\} \\
&\le \left\{\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{ij}^{-}du}\right. \\
&\qquad \times \left[a_{ij}^{+}\eta_{ij}^{+}\|\varphi\|_{B}\right. \\
&\qquad\quad +\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}}\left(\left|f\left(\varphi_{kl}(s-\tau_{kl}(s))\right)\right.\right. \\
&\qquad\qquad \left.\left.-f(0)\right|+\left|f(0)\right|\right)\|\varphi\|_{B} \\
&\qquad\quad +\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^{+}} \\
&\qquad\quad \times\int_{0}^{\infty}\left|K_{ij}(u)\right| \\
&\qquad\qquad \times\left(\left|g\left(\varphi_{kl}(s-u)\right)-g(0)\right|+\left|g(0)\right|\right)du \\
&\qquad\qquad \left.\times\ \|\varphi\|_{B}\right]ds\Bigg\} \\
&\le \left\{\frac{1}{a_{ij}^{-}}\left[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}}\left(L^{f}\|\varphi\|_{B}+\left|f(0)\right|\right)\right.\right. \\
&\qquad\quad \left.\left.+\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^{+}}\int_{0}^{\infty}\left|K_{ij}(u)\right|du\left(L^{g}\|\varphi\|_{B}+\left|g(0)\right|\right)\right] \right. \\
&\qquad \left.\times\ \|\varphi\|_{B}\right\} \\
&\le \left\{\frac{1}{a_{ij}^{-}}\left[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}}\left(L^{f}(\kappa+L)+\left|f(0)\right|\right)\right.\right. \\
&\qquad\quad \left.\left.+\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^{+}}\right.\right. \\
&\qquad\quad \left.\left.\times\int_{0}^{\infty}\left|K_{ij}(u)\right|du\left(L^{g}(\kappa+L)+\left|g(0)\right|\right)\right](\kappa+L)\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\left(T(\varphi)(t)-\varphi^{0}(t)\right)'\right| \\
&= \left\{\left|\left[a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t}\varphi_{ij}'(s)\,ds\right.\right.\right. \\
&\qquad -\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\, f\left(\varphi_{kl}(t-\tau_{kl}(t))\right)\varphi_{ij}(t) \\
&\qquad -\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\
&\qquad \left.\times\int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(t-u)\right)du\varphi_{ij}(t)\right] \\
&\qquad -a_{ij}(t)\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{ij}(u)du} \\
&\qquad \times\left[a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s}\varphi_{ij}'(u)\,du\right. \\
&\qquad\quad -\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\, f\left(\varphi_{kl}(s-\tau_{kl}(s))\right)\varphi_{ij}(s) \\
&\qquad\quad -\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s) \\
&\qquad\quad \left.\left.\times\int_{0}^{\infty} K_{ij}(u)\, g\left(\varphi_{kl}(s-u)\right)du\varphi_{ij}(s)\right]ds\right|\Bigg\} \\
&\le \left\{\left(1+\frac{a_{ij}^{+}}{a_{ij}^{-}}\right)\right. \\
&\qquad \times\left[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}}\left(L^{f}\|\varphi\|_{B}+\left|f(0)\right|\right)\right. \\
&\qquad\quad +\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^{+}} \\
&\qquad\quad \left.\left.\times\int_{0}^{\infty}\left|K_{ij}(u)\right|du\left(L^{g}\|\varphi\|_{B}+\left|g(0)\right|\right)\right]\|\varphi\|_{B}\right\} \\
&\le \left\{\left(1+\frac{a_{ij}^{+}}{a_{ij}^{-}}\right)\right. \\
&\qquad \times\left[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}}\left(L^{f}(\kappa+L)+\left|f(0)\right|\right)\right. \\
&\qquad\quad \left.+\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^{+}}\right.
\end{aligned}
$$

$$\times \int_0^\infty \left|K_{ij}(u)\right| du \left(L^g(\kappa + L) + \left|g(0)\right|\right) \Bigg] \times (\kappa + L) \Bigg\}. \tag{50}$$

It follows that

$$\left\|T(\varphi) - \varphi^0\right\|_B \le \max_{(i,j)} \left\{ \frac{1}{a_{ij}^-} E_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right) E_{ij} \right\} \le \kappa; \tag{51}$$

that is, $T(\varphi) = x^\varphi \in B^{**}$.

Second, we show that T is a contract operator.

In fact, in view of (44), (48), (S_1), (S_2), and (S_3), for $\varphi, \psi \in B^{**}$, we have

$$\begin{aligned}
&\left|T(\varphi(t)) - T(\psi(t))\right| \\
&= \left\{\left|\left(T(\varphi(t)) - T(\psi(t))\right)_{ij}\right|\right\} \\
&= \left\{\left| \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} \right.\right. \\
&\qquad \times \Bigg[a_{ij}(s) \int_{s-\eta_{ij}(s)}^s \left(\varphi_{ij}'(u) - \psi_{ij}'(u)\right) du \\
&\qquad\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \\
&\qquad\quad \times \Big(f\left(\varphi_{kl}(s - \tau_{kl}(s))\right) \varphi_{ij}(s) \\
&\qquad\qquad - f\left(\psi_{kl}(s - \tau_{kl}(s))\right) \psi_{ij}(s) \Big) \\
&\qquad\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) \\
&\qquad\quad \times \Bigg(\int_0^\infty K_{ij}(u)\, g\left(\varphi_{kl}(s-u)\right) du \varphi_{ij}(s) \\
&\qquad\qquad - \int_0^\infty K_{ij}(u)\, g \\
&\qquad\qquad\quad \times \left(\psi_{kl}(s-u)\right) du \psi_{ij}(s) \Bigg) \Bigg] ds \Bigg| \Bigg\} \\
&\le \left\{ \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} \right. \\
&\qquad \times \Bigg[a_{ij}^+ \eta_{ij}^+ \left\|\varphi - \psi\right\|_B \\
&\qquad\quad + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \left(\left|f\left(\varphi_{kl}(s - \tau_{kl}(s))\right)\right| \right. \\
&\qquad\qquad \left. \times \left|\varphi_{ij}(s) - \psi_{ij}(s)\right| \right. \\
&\qquad\qquad + \left|f\left(\varphi_{kl}(s - \tau_{kl}(s))\right)\right. \\
&\qquad\qquad \left. - f\left(\psi_{kl}(s - \tau_{kl}(s))\right)\right| \left|\psi_{ij}(s)\right| \Big) \\
&\qquad\quad + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+} \Bigg(\int_0^\infty \left|K_{ij}(u)\right| \\
&\qquad\qquad \times \left|g\left(\varphi_{kl}(s-u)\right)\right| du \left|\varphi_{ij}(s) - \psi_{ij}(s)\right| \\
&\qquad\qquad + \int_0^\infty \left|K_{ij}(u)\right| \\
&\qquad\qquad \times \left|g\left(\varphi_{kl}(s-u)\right) - g\left(\psi_{kl}(s-u)\right)\right| du \\
&\qquad\qquad \left. \times \left|\psi_{ij}(s)\right| \Bigg) \Bigg] ds \Bigg\} \\
&\le \Bigg\{ \frac{1}{a_{ij}^-} \Bigg[a_{ij}^+ \eta_{ij}^+ \left\|\varphi - \psi\right\|_B \\
&\qquad\quad + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \left(M_f \left\|\varphi - \psi\right\|_B \right. \\
&\qquad\qquad \left. + L^f \left\|\varphi - \psi\right\|_B \left\|\psi\right\|_B \right) \\
&\qquad\quad + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+} \\
&\qquad\quad \times \Bigg(\int_0^\infty \left|K_{ij}(u)\right| du M_g \left\|\varphi - \psi\right\|_B \\
&\qquad\qquad + \int_0^\infty \left|K_{ij}(u)\right| du L^g \left\|\varphi - \psi\right\|_B \left\|\psi\right\|_B \Bigg) \Bigg] \Bigg\} \\
&= \Bigg\{ \frac{1}{a_{ij}^-} \Bigg[a_{ij}^+ \eta_{ij}^+ \\
&\qquad\quad + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \left(M_f + L^f \left\|\psi\right\|_B \right) \\
&\qquad\quad + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+} \\
&\qquad\quad \times \Bigg(\int_0^\infty \left|K_{ij}(u)\right| du M_g \\
&\qquad\qquad + \int_0^\infty \left|K_{ij}(u)\right| du L^g \left\|\psi\right\|_B \Bigg) \Bigg] \left\|\varphi - \psi\right\|_B \Bigg\} \\
&\le \Bigg\{ \frac{1}{a_{ij}^-} \Bigg[a_{ij}^+ \eta_{ij}^+ \\
&\qquad\quad + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \left(M_f + L^f (\kappa + L) \right) \\
&\qquad\quad + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+}
\end{aligned}$$

$$\times \left(\int_0^\infty \left|K_{ij}(u)\right| du M_g + \int_0^\infty \left|K_{ij}(u)\right| du L^g(\kappa + L) \right) \Bigg] \times \|\varphi - \psi\|_B \Bigg\},$$

$$\begin{aligned} &\left|\left(T(\varphi(t)) - T(\psi(t))\right)'\right| \\ &= \left\{\left|\left(T'(\varphi(t)) - T'(\psi(t))\right)_{ij}\right|\right\} \\ &= \Bigg\{\Bigg|\Bigg[a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} \left(\varphi'_{ij}(s) - \psi'_{ij}(s)\right) ds \\ &\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) \\ &\quad \times \left(f\left(\varphi_{kl}(t - \tau_{kl}(t))\right) \varphi_{ij}(t) \right. \\ &\quad \left. - f\left(\psi_{kl}(t - \tau_{kl}(t))\right) \psi_{ij}(t) \right) \\ &\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\ &\quad \times \left(\int_0^\infty K_{ij}(u)\, g\left(\varphi_{kl} \right.\right. \\ &\quad \left. \times (t-u)\right) du \varphi_{ij}(t) \\ &\quad - \int_0^\infty K_{ij}(u)\, g\left(\psi_{kl}\ (t-u)\right) du \\ &\quad \left. \times\ \psi_{ij}(t) \right) \Bigg] \\ &\quad - a_{ij}(t) \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \\ &\quad \times \Bigg[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} \left(\varphi'_{ij}(u) - \psi'_{ij}(u)\right) du \\ &\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \\ &\quad \times \left(f\left(\varphi_{kl}(s - \tau_{kl}(s))\right) \varphi_{ij}(s) \right. \\ &\quad - f\left(\psi_{kl}(s - \tau_{kl}(s))\right) \\ &\quad \left. \times\ \psi_{ij}(s) \right) \\ &\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s) \\ &\quad \times \left(\int_0^\infty K_{ij}(u)\, g\left(\varphi_{kl}(s-u)\right) du \varphi_{ij}(s) \right. \\ &\quad - \int_0^\infty K_{ij}(u)\, g \\ &\quad \left. \times\ \left(\psi_{kl}(s-u)\right) du \psi_{ij}(s) \right) \Bigg] ds \Bigg| \Bigg\} \\ &\le \Bigg\{ \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right) \Bigg[a_{ij}^+ \eta_{ij}^+ \\ &\quad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl+} \left(M_f + L^f(\kappa + L)\right) \\ &\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl+} \left(\int_0^\infty \left|K_{ij}(u)\right| du M_g \right. \\ &\quad \left. + \int_0^\infty \left|K_{ij}(u)\right| du \right. \\ &\quad \left. \times\ L^g(\kappa + L) \right) \Bigg] \\ &\quad \times \|\varphi - \psi\|_B \Bigg\}, \end{aligned} \tag{52}$$

which yields

$$\|T(\varphi) - T(\psi)\|_B \le \max_{(i,j)} \left\{ \frac{1}{a_{ij}^-} F_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right) F_{ij} \right\} \|\varphi - \psi\|_B, \tag{53}$$

which implies that the mapping $T : B^{**} \to B^{**}$ is a contraction mapping. Therefore, using Theorem 0.3.1 of [27], we obtain that the mapping T possesses a unique fixed point

$$x^* = \left\{x_{ij}^*(t)\right\} \in B^{**}, \qquad Tx^* = x^*. \tag{54}$$

By (42) and (44), x^* satisfies (42). So (3) has at least one continuously differentiable pseudo almost periodic solution x^*. The proof of Theorem 10 is now completed. □

4. Exponential Stability of the Pseudo Almost Periodic Solution

In this section, we will discuss the exponential stability of the pseudo almost periodic solution of system (3).

Definition 11. Let $x^*(t) = \{x_{ij}^*(t)\}$ be the pseudo almost periodic solution of system (3). If there exist constants $\alpha > 0$ and $M > 1$ such that, for every solution $x(t) = \{x_{ij}(t)\}$ of system (3) with any initial value $\varphi(t) = \{\varphi_{ij}(t)\}$ satisfying (6),

$$\begin{aligned} &\|x(t) - x^*(t)\|_1 \\ &\quad = \max_{(i,j)} \left\{ \max\left\{ \left|x_{ij}(t) - x_{ij}^*(t)\right|, \left|x'_{ij}(t) - x_{ij}^{*\prime}(t)\right| \right\} \right\} \\ &\quad \le M \|\varphi - x^*\|_0 e^{-\alpha t}, \quad \forall t > 0, \end{aligned} \tag{55}$$

where $\|\varphi - x^*\|_0 = \max\{\sup_{t\le 0}\max_{(i,j)}|\varphi_{ij}(t) - x_{ij}^*(t)|, \sup_{t\le 0}\max_{(i,j)}|\varphi'_{ij}(t) - x_{ij}^{*\prime}(t)|\}$. Then $x^*(t)$ is said to be globally exponentially stable.

Theorem 12. *Suppose that all conditions in Theorem 10 are satisfied. Then system* (3) *has at least one pseudo almost periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable.*

Proof. By Theorem 10, (3) has at least one continuously differentiable pseudo almost periodic solution $x^*(t) = \{x_{ij}^*(t)\}$ such that

$$\left\|x^*\right\|_B \le \kappa + L. \tag{56}$$

Suppose that $x(t) = \{x_{ij}(t)\}$ is an arbitrary solution of (1) associated with initial value $\varphi(t) = \{\varphi_{ij}(t)\}$ satisfying (6). Let $y(t) = \{y_{ij}(t)\} = \{x_{ij}(t) - x_{ij}^*(t)\}$. Then

$$\begin{aligned}
y_{ij}'(t) &= -a_{ij}(t)\, y_{ij}\left(t - \eta_{ij}(t)\right) \\
&\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) \\
&\quad \times \left[f\left(x_{kl}\left(t - \tau_{kl}(t)\right)\right) x_{ij}(t) \right. \\
&\qquad \left. - f\left(x_{kl}^*\left(t - \tau_{kl}(t)\right)\right) x_{ij}^*(t) \right] \\
&\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\
&\qquad \times \left[\int_0^\infty K_{ij}(u)\, g\left(x_{kl}(t-u)\right) du x_{ij}(t) \right. \\
&\qquad \left. - \int_0^\infty K_{ij}(u)\, g\left(x_{kl}^*(t-u)\right) du x_{ij}^*(t) \right] \\
&= -a_{ij}(t)\, y_{ij}(t) \\
&\quad + a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} y_{ij}'(u)\, du \\
&\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) \left[f\left(x_{kl}\left(t - \tau_{kl}(t)\right)\right) x_{ij}(t) \right. \\
&\qquad \left. - f\left(x_{kl}^*\left(t - \tau_{kl}(t)\right)\right) \times x_{ij}^*(t) \right] \\
&\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t) \\
&\qquad \times \left[\int_0^\infty K_{ij}(u)\, g\left(x_{kl}(t-u)\right) du x_{ij}(t) \right. \\
&\qquad \left. - \int_0^\infty K_{ij}(u)\, g\left(x_{kl}^*(t-u)\right) du x_{ij}^*(t) \right].
\end{aligned} \tag{57}$$

Define continuous functions $\Gamma_i(\omega)$ and $\Pi_i(\omega)$ by setting

$$\begin{aligned}
\Gamma_{ij}(\omega) &= -a_{ij}^- + \omega + a_{ij}^+ \eta_{ij}^+ e^{\omega\eta_{ij}^+} \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+} \left(M^f + L^f e^{\omega\tau_{kl}^+} (\kappa + L) \right) \\
&\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+} \\
&\quad \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g \right. \\
&\qquad \left. + \int_0^\infty \left|K_{ij}(u)\right| L^g e^{\omega u} du\, (\kappa + L) \right), \\
\Pi_{ij}(\omega) &= \left(1 + \frac{a_{ij}^+}{a_{ij}^- - \omega} \right) \\
&\quad \times \left[a_{ij}^+ \eta_{ij}^+ e^{\omega\eta_{ij}^+} \right. \\
&\qquad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+} \left(M^f + L^f e^{\omega\tau_{kl}^+} (\kappa + L) \right) \\
&\qquad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+} \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g \right. \\
&\qquad\qquad \left. \left. + \int_0^\infty \left|K_{ij}(u)\right| L^g e^{\omega u} du \right. \right. \\
&\qquad\qquad \left. \left. \times (\kappa + L) \right) \right],
\end{aligned} \tag{58}$$

where $t > 0$, $\omega \in [0, \beta]$, $ij \in J$. Then, from (S_3), we have

$$\begin{aligned}
\Gamma_{ij}(0) &= -a_{ij}^- + a_{ij}^+ \eta_{ij}^+ \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+} \left(M^f + L^f (\kappa + L) \right) \\
&\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+} \\
&\quad \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g \right. \\
&\qquad \left. + \int_0^\infty \left|K_{ij}(u)\right| L^g du\, (\kappa + L) \right) \\
&= -a_{ij}^- \left(1 - \frac{1}{a_{ij}^-} F_{ij} \right) < 0, \quad ij \in J, \\
\Pi_{ij}(0) &= \left(1 + \frac{a_{ij}^+}{a_{ij}^-} \right)
\end{aligned}$$

$$\times \left[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+}\left(M^f + L^f(\kappa+L)\right) + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+} \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g + \int_0^\infty \left|K_{ij}(u)\right| L^g du(\kappa+L)\right)\right] = \left(1+\frac{a_{ij}^{+}}{a_{ij}^{-}}\right) F_{ij} < 1, \quad ij \in J, \tag{59}$$

which, together with the continuity of $\Gamma_{ij}(\omega)$ and $\Pi_{ij}(\omega)$, implies that we can choose a constant $\lambda \in (0, \min\{\beta, \min_{(i,j)} a_{ij}^{-}\})$ such that

$$\begin{aligned}\Gamma_{ij}(\lambda) &= -a_{ij}^{-} + \lambda + a_{ij}^{+}\eta_{ij}^{+}e^{\lambda\eta_{ij}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+}\left(M^f + L^f e^{\lambda\tau_{kl}^{+}}(\kappa+L)\right) + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+} \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g + \int_0^\infty \left|K_{ij}(u)\right| L^g e^{\lambda u} du(\kappa+L)\right) \\ &= \left(a_{ij}^{-}-\lambda\right)\left(\frac{\beta_{ij}}{a_{ij}^{-}-\lambda} - 1\right) < 0,\end{aligned} \tag{60}$$

$$\begin{aligned}\Pi_{ij}(\lambda) &= \left(1+\frac{a_{ij}^{+}}{a_{ij}^{-}-\lambda}\right) \times \left[a_{ij}^{+}\eta_{ij}^{+}e^{\lambda\eta_{ij}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+}\left(M^f + L^f e^{\lambda\tau_{kl}^{+}}(\kappa+L)\right) + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+}\left(\int_0^\infty \left|K_{ij}(u)\right| du M^g + \int_0^\infty \left|K_{ij}(u)\right| L^g e^{\lambda u} du \times (\kappa+L)\right)\right] \\ &= \left(1+\frac{a_{ij}^{+}}{a_{ij}^{-}-\lambda}\right)\beta_{ij} < 1,\end{aligned} \tag{61}$$

where

$$\begin{aligned}\beta_{ij} &= a_{ij}^{+}\eta_{ij}^{+}e^{\lambda\eta_{ij}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^+}\left(M^f + L^f e^{\lambda\tau_{kl}^{+}}(\kappa+L)\right) + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl^+} \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g + \int_0^\infty \left|K_{ij}(u)\right| L^g e^{\lambda u} du(\kappa+L)\right), \quad ij \in J.\end{aligned} \tag{62}$$

Let M be a constant such that

$$M > \frac{a_{ij}^{-}-\lambda}{\beta_{ij}} > 1, \quad \forall ij \in J, \tag{63}$$

which, together with (60), yields

$$\frac{1}{M} - \frac{\beta_{ij}}{a_{ij}^{-}-\lambda} < 0, \qquad \frac{\beta_{ij}}{a_{ij}^{-}-\lambda} < 1, \quad \forall ij \in J. \tag{64}$$

Consequently, for any $\varepsilon > 0$, it is obvious that

$$\left\|y(t)\right\|_1 < \left(\left\|\varphi - x^*\right\|_0 + \varepsilon\right)e^{-\lambda t} < M\left(\left\|\varphi - x^*\right\|_0 + \varepsilon\right)e^{-\lambda t} \quad t \in (-\infty, 0]. \tag{65}$$

In the following, we will show that

$$\left\|y(t)\right\|_1 < M\left(\left\|\varphi - x^*\right\|_0 + \varepsilon\right)e^{-\lambda t}, \quad \forall t > 0. \tag{66}$$

Otherwise, there must exist $ij \in J$ and $\theta > 0$ such that

$$\begin{aligned}\left\|y(\theta)\right\|_1 &= \max\left\{\left|y_{ij}(\theta)\right|, \left|y_{ij}'(\theta)\right|\right\} = M\left(\left\|\varphi - x^*\right\|_0 + \varepsilon\right)e^{-\lambda\theta}, \\ \left\|y(t)\right\|_1 &< M\left(\left\|\varphi - x^*\right\|_0 + \varepsilon\right)e^{-\lambda t}, \quad \forall t \in (-\infty, \theta).\end{aligned} \tag{67}$$

Note that

$$\begin{aligned}y_{ij}'(s) + a_{ij}(s)\,y_{ij}(s) &= a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s} y_{ij}'(u)\,du - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \times \left[f\left(x_{kl}(s-\tau_{kl}(s))\right)x_{ij}(s) - f\left(x_{kl}^*(s-\tau_{kl}(s))\right)x_{ij}^*(s)\right] - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s) \\ &\quad \times \left[\int_0^\infty K_{ij}(u)\,g\left(x_{kl}(s-u)\right)du x_{ij}(s)\right.\end{aligned}$$

$$- \int_0^\infty K_{ij}(u)\, g\left(x_{kl}^*(s-u)\right) du x_{ij}^*(s)\Bigg],$$
$$s \in [0,t],\ t \in [0,\theta]. \tag{68}$$

Multiplying both sides of (68) by $e^{\int_0^s a_{ij}(u)du}$ and integrating on $[0,t]$, we get

$$\begin{aligned}
y_{ij}(t) &= y_{ij}(0)\, e^{-\int_0^t a_{ij}(u)du} \\
&\quad + \int_0^t e^{-\int_s^t a_{ij}(u)du} \\
&\quad \times \Bigg[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} y_{ij}'(u)\, du \\
&\qquad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \\
&\qquad \times \left(f\left(x_{kl}(s-\tau_{kl}(s))\right) x_{ij}(s) \right. \\
&\qquad\quad \left. - f\left(x_{kl}^*(s-\tau_{kl}(s))\right) x_{ij}^*(s) \right) \\
&\qquad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s) \\
&\qquad \times \left(\int_0^\infty K_{ij}(u)\, g\left(x_{kl}(s-u)\right) du x_{ij}(s) \right. \\
&\qquad\quad \left. - \int_0^\infty K_{ij}(u)\, g\left(x_{kl}^*(s-u)\right) du x_{ij}^*(s) \right) \Bigg] ds, \\
&\qquad t \in [0,\theta].
\end{aligned} \tag{69}$$

Thus, with the help of (67), we have

$$\begin{aligned}
&\left|y_{ij}(\theta)\right| \\
&= \Bigg| y_{ij}(0)\, e^{-\int_0^\theta a_{ij}(u)du} \\
&\quad + \int_0^\theta e^{-\int_s^\theta a_{ij}(u)du} \\
&\quad \times \Bigg[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} y_{ij}'(u)\, du \\
&\qquad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \left(f\left(x_{kl}(s-\tau_{kl}(s))\right) x_{ij}(s) \right. \\
&\qquad\qquad \left. - f\left(x_{kl}^*(s-\tau_{kl}(s))\right) x_{ij}^*(s) \right) \\
&\qquad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s) \\
&\qquad \times \left(\int_0^\infty K_{ij}(u)\, g\left(x_{kl}(s-u)\right) du x_{ij}(s) \right. \\
&\qquad\quad \left. - \int_0^\infty K_{ij}(u)\, g\left(x_{kl}^*(s-u)\right) du x_{ij}^*(s) \right) \Bigg] ds \Bigg| \\
&\le \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-a_{ij}^- \theta} + \int_0^\theta e^{-\int_s^\theta a_{ij}(u)du} \\
&\quad \times \Bigg[a_{ij}^+ \eta_{ij}^+ M \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-\lambda(s-\eta_{ij}(s))} \\
&\qquad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl+} \left(\left|f\left(x_{kl}(s-\tau_{kl}(s))\right)\right| \left|x_{ij}(s) - x_{ij}^*(s)\right| \right. \\
&\qquad\quad + \left| f\left(x_{kl}(s-\tau_{kl}(s))\right) \right. \\
&\qquad\qquad \left. - f\left(x_{kl}^*(s-\tau_{kl}(s))\right) \right| \\
&\qquad\quad \left. \times \left|x_{ij}^*(s)\right| \right) + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl+} \\
&\qquad \times \left(\int_0^\infty \left|K_{ij}(u)\right| \left|g\left(x_{kl}(s-u)\right)\right| du \left|x_{ij}(s) - x_{ij}^*(s)\right| \right. \\
&\qquad\quad + \int_0^\infty \left|K_{ij}(u)\right| \left| g\left(x_{kl}(s-u)\right) \right. \\
&\qquad\qquad \left. - g\left(x_{kl}^*(s-u)\right) \right| du \\
&\qquad\quad \left. \times \left|x_{ij}^*(s)\right| \right) \Bigg] ds \\
&\le \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-a_{ij}^- \theta} + \int_0^\theta e^{-\int_s^\theta a_{ij}(u)du} \\
&\quad \times \Bigg[a_{ij}^+ \eta_{ij}^+ M \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-\lambda(s-\eta_{ij}(s))} \\
&\qquad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl+} \left(M^f \left|y_{ij}(s)\right| \right. \\
&\qquad\quad \left. + L^f \left|y_{kl}\left(s-\tau_{kl}(s)\right)\right| \left|x_{ij}^*(s)\right| \right) \\
&\qquad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl+} \\
&\qquad \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g \left|y_{ij}(s)\right| \right. \\
&\qquad\quad + \int_0^\infty \left|K_{ij}(u)\right| L^g \left|y_{kl}(s-u)\right| du \\
&\qquad\quad \left. \times \left|x_{ij}^*(s)\right| \right) \Bigg] ds \\
&\le \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-a_{ij}^- \theta} + \int_0^\theta e^{-\int_s^\theta a_{ij}(u)du} \\
&\quad \times \Bigg[a_{ij}^+ \eta_{ij}^+ M \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-\lambda(s-\eta_{ij}(s))} \\
&\qquad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl+} \\
&\qquad \times \left(M^f M \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-\lambda s} \right. \\
&\qquad\quad \left. + L^f M \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-\lambda(s-\tau_{kl}(s))} \left|x_{ij}^*(s)\right| \right) \\
&\qquad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl+} \\
&\qquad \times \left(\int_0^\infty \left|K_{ij}(u)\right| du M^g M \left(\|\varphi - x^*\|_0 + \varepsilon\right) e^{-\lambda s} \right. \\
&\qquad\quad + \int_0^\infty \left|K_{ij}(u)\right| L^g M \left(\|\varphi - x^*\|_0 + \varepsilon\right) \\
&\qquad\quad \left. \times\ e^{-\lambda(s-u)} du \left|x_{ij}^*(s)\right| \right) \Bigg] ds \\
&\le M \left(\|\varphi - x^*\|_0 + \varepsilon\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \frac{e^{-a_{ij}^{-}\theta}}{M} + \int_0^\theta e^{-\int_s^\theta a_{ij}(u)du} e^{-\lambda s} \right. \\
&\quad \times \left[a_{ij}^{+} \eta_{ij}^{+} e^{\lambda \eta_{ij}^{+}} \right. \\
&\qquad + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \left(M^f + L^f e^{\lambda \tau_{kl}^{+}} (\kappa + L) \right) \\
&\qquad + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+} \\
&\qquad \times \left(\int_0^\infty \left| K_{ij}(u) \right| du M^g \right. \\
&\qquad\quad \left. \left. + \int_0^\infty \left| K_{ij}(u) \right| L^g e^{\lambda u} du (\kappa + L) \right) \right] ds \Bigg\} \\
&\le M \left(\left\| \varphi - x^* \right\|_0 + \varepsilon \right) \\
&\quad \times \left\{ \frac{e^{-a_{ij}^{-}\theta}}{M} + e^{-a_{ij}^{-}\theta} \int_0^\theta e^{(a_{ij}^{-} - \lambda)s} ds \beta_{ij} \right\} \\
&\le M \left(\left\| \varphi - x^* \right\|_0 + \varepsilon \right) e^{-\lambda\theta} \\
&\quad \times \left[\frac{e^{(\lambda - a_{ij}^{-})\theta}}{M} + \frac{\beta_{ij}}{a_{ij}^{-} - \lambda} \left(1 - e^{(\lambda - a_{ij}^{-})\theta} \right) \right] \\
&= M \left(\left\| \varphi - x^* \right\|_0 + \varepsilon \right) e^{-\lambda\theta} \\
&\quad \times \left[\left(\frac{1}{M} - \frac{\beta_{ij}}{a_{ij}^{-} - \lambda} \right) e^{(\lambda - a_{ij}^{-})\theta} + \frac{\beta_{ij}}{a_{ij}^{-} - \lambda} \right],
\end{aligned}
\tag{70}
$$

which, together with (64) and (67), implies that

$$
\left| y_{ij}(\theta) \right| < M \left(\| \varphi - x^* \|_0 + \varepsilon \right) e^{-\lambda\theta}, \tag{71}
$$

$$
\begin{aligned}
\left\| y(\theta) \right\|_1 &= \max \left\{ \left| y_{ij}(\theta) \right|, \left| y_{ij}'(\theta) \right| \right\} \\
&= \left| y_{ij}'(\theta) \right| = M \left(\left\| \varphi - x^* \right\|_\xi + \varepsilon \right) e^{-\lambda\theta}.
\end{aligned}
\tag{72}
$$

From (60), (61) and (67)–(72) yield

$$
\begin{aligned}
&\left| y_{ij}'(\theta) \right| \\
&\quad \le a_{ij}(\theta) \left| y_{ij}(\theta) \right| \\
&\qquad + \left| a_{ij}(\theta) \int_{\theta - \eta_{ij}(\theta)}^{\theta} y_{ij}'(u) du \right. \\
&\qquad\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\theta) \\
&\qquad\quad \times \left[f\left(x_{kl}(\theta - \tau_{kl}(\theta)) \right) x_{ij}(\theta) \right. \\
&\qquad\qquad \left. - f\left(x_{kl}^*(\theta - \tau_{kl}(\theta)) \right) x_{ij}^*(\theta) \right] \\
&\qquad\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\theta) \\
&\qquad\quad \times \left[\int_0^\infty K_{ij}(u)\, g\left(x_{kl}(\theta - u) \right) du x_{ij}(\theta) \right. \\
&\qquad\qquad \left. - \int_0^\infty K_{ij}(u)\, g\left(x_{kl}^*(\theta - u) \right) du x_{ij}^*(\theta) \right] \Bigg| \\
&\quad \le a_{ij}^{+} \left| y_{ij}(\theta) \right| \\
&\qquad + \left[a_{ij}^{+} \eta_{ij}^{+} M \left(\left\| \varphi - x^* \right\|_\xi + \varepsilon \right) e^{-\lambda(\theta - \eta_{ij}(\theta))} \right. \\
&\qquad\quad + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \\
&\qquad\quad \times \left(\left| f\left(x_{kl}(\theta - \tau_{kl}(\theta)) \right) \right| \left| x_{ij}(\theta) - x_{ij}^*(\theta) \right| \right. \\
&\qquad\qquad + \left| f\left(x_{kl}(\theta - \tau_{kl}(\theta)) \right) - f\left(x_{kl}^*(\theta - \tau_{kl}(\theta)) \right) \right| \\
&\qquad\qquad \left. \times \left| x_{ij}^*(\theta) \right| \right) \\
&\qquad\quad + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+} \left(\int_0^\infty \left| K_{ij}(u) \right| \left| g\left(x_{kl}(\theta - u) \right) \right| du \right. \\
&\qquad\qquad\qquad \times \left| x_{ij}(\theta) - x_{ij}^*(\theta) \right| \\
&\qquad\qquad\qquad + \int_0^\infty \left| K_{ij}(u) \right| \\
&\qquad\qquad\qquad\quad \times \left| g\left(x_{kl}(\theta - u) \right) \right. \\
&\qquad\qquad\qquad\qquad \left. - g\left(x_{kl}^*(\theta - u) \right) \right| du \\
&\qquad\qquad\qquad\quad \left. \times \left| x_{ij}^*(\theta) \right| \right) \\
&\quad \le \left\{ a_{ij}^{+} \left[\left(\frac{1}{M} - \frac{\beta_{ij}}{a_{ij}^{-} - \lambda} \right) e^{(\lambda - a_{ij}^{-})\theta} + \frac{\beta_{ij}}{a_{ij}^{-} - \lambda} \right] \right. \\
&\qquad + a_{ij}^{+} \eta_{ij}^{+} e^{\lambda \eta_{ij}^{+}} + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl^+} \\
&\qquad\quad \times \left(M^f + L^f e^{\lambda \tau_{kl}^{+}} (\kappa + L) \right) + \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl^+} \\
&\qquad\quad \times \left(\int_0^\infty \left| K_{ij}(u) \right| du M^g \right. \\
&\qquad\qquad \left. \left. + \int_0^\infty \left| K_{ij}(u) \right| L^g e^{\lambda u} du (\kappa + L) \right) \right\} M \\
&\qquad\quad \times \left(\left\| \varphi - x^* \right\|_0 + \varepsilon \right) e^{-\lambda\theta} \\
&\quad \le M \left(\left\| \varphi - x^* \right\|_0 + \varepsilon \right) e^{-\lambda\theta} \\
&\qquad\quad \times \left[a_{ij}^{+} \left(\frac{1}{M} - \frac{\beta_{ij}}{a_{ij}^{-} - \lambda} \right) e^{(\lambda - a_{ij}^{-})\theta} + \beta_{ij} \left(\frac{a_{ij}^{+}}{a_{ij}^{-} - \lambda} + 1 \right) \right] \\
&\quad < M \left(\left\| \varphi - x^* \right\|_0 + \varepsilon \right) e^{-\lambda\theta},
\end{aligned}
\tag{73}
$$

which contradicts (72). Hence, (66) holds. Letting $\varepsilon \to 0^+$, we have from (66) that

$$\|y(t)\|_1 \le M\|\varphi - x^*\|_0 e^{-\lambda t}, \quad \forall t > 0, \tag{74}$$

which implies

$$\|x(t) - x^*(t)\|_1 \le M\|\varphi - x^*\|_0 e^{-\lambda t}, \quad \forall t > 0. \tag{75}$$

This completes the proof. □

5. An Example

In this section, we give an example with numerical simulation to demonstrate the results obtained in previous sections.

Example 13. Consider the following SICNNs with time-varying delays in the leakage terms:

$$\begin{aligned}\frac{dx_{ij}}{dt} &= -a_{ij}(t)\, x_{ij}\left(t - \eta_{ij}(t)\right) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\left(x_{kl}\left(t - \sin^2 t\right)\right) x_{ij}(t) \\ &\quad - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl} \int_0^\infty K_{ij}(u)\, g\left(x_{kl}(t-u)\right) du x_{ij} + L_{ij}(t), \quad i, j = 1, 2, 3,\end{aligned} \tag{76}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0 & 0.2 \\ 0.1 & 0.2 & 0.1 \end{bmatrix},$$

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} = 0.01 \begin{bmatrix} \sin^2\sqrt{3}t + \dfrac{0.1}{1+t^2} & \cos^2\sqrt{3}t + \dfrac{0.1}{1+t^2} & \sin^2 2t + \dfrac{0.1}{1+t^2} \\ \cos^2\sqrt{5}t + \dfrac{0.1}{1+t^2} & \sin^2\sqrt{5}t + \dfrac{0.1}{1+t^2} & \cos^2 2t + \dfrac{0.1}{1+t^2} \\ \sin^2 2t + \dfrac{0.1}{1+t^2} & \cos^2 3t + \dfrac{0.1}{1+t^2} & \sin^2\sqrt{2}t + \dfrac{0.1}{1+t^2} \end{bmatrix} \tag{77}$$

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 0.7 + 0.24\sin^2\sqrt{2}t - \dfrac{1}{1+t^2} & 0.41 + 0.5\cos^2 t & 1 \\ 0.61 + 0.2\cos^2 t - \dfrac{1}{1+t^2} & 0.67 + 0.2\sin^2 t & 1 \\ 0.59 + 0.4\cos^4 t - \dfrac{1}{1+t^2} & 0.5 + 0.41\sin^2 t & 1 \end{bmatrix}.$$

Set

$$\begin{aligned} &\kappa = 0.7, \qquad r = q = 1, \qquad K_{ij}(u) = |\sin u|\, e^{-u}, \\ &\qquad\qquad\qquad\qquad i = 1, 2, 3, \;\; j = 1, 2, 3, \\ &f(x) = g(x) = \frac{1}{50}\left(|x-1| - |x+1|\right), \end{aligned} \tag{78}$$

clearly,

$$\begin{aligned} &M_f = M_g = 0.04, \qquad L_f = L_g = 0.04, \\ &\sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl} = \sum_{C_{kl}\in N_1(1,1)} B_{11}^{kl} = 0.5, \\ &\sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl} = \sum_{C_{kl}\in N_1(1,2)} B_{12}^{kl} = 0.8, \\ &\sum_{C_{kl}\in N_1(1,3)} C_{13}^{kl} = \sum_{C_{kl}\in N_1(1,3)} B_{13}^{kl} = 0.5, \\ &\sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl} = \sum_{C_{kl}\in N_1(2,1)} B_{21}^{kl} = 0.8, \\ &\sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl} = \sum_{C_{kl}\in N_1(2,2)} B_{22}^{kl} = 1.2, \\ &\sum_{C_{kl}\in N_1(2,3)} C_{23}^{kl} = \sum_{C_{kl}\in N_1(2,3)} B_{23}^{kl} = 0.8, \\ &\sum_{C_{kl}\in N_1(3,1)} C_{31}^{kl} = \sum_{C_{kl}\in N_1(3,1)} B_{31}^{kl} = 0.5, \end{aligned}$$

$$\sum_{C_{kl}\in N_1(3,2)} C_{32}^{kl} = \sum_{C_{kl}\in N_1(3,2)} B_{32}^{kl} = 0.8,$$
$$\sum_{C_{kl}\in N_1(3,3)} C_{33}^{kl} = \sum_{C_{kl}\in N_1(3,3)} B_{33}^{kl} = 0.5,$$
(79)

where $ij \in J = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$. Then,

$$L = \max\left\{\max_{(i,j)}\left\{\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} L_{ij}(s)\,ds\right|\right\},\ \max_{(i,j)}\left\{\sup_{t\in\mathbb{R}}\left|L_{ij}(t) - a_{ij}(t) \times \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} L_{ij}(s)\,ds\right|\right\}\right\}$$
$$= 1 > 0,$$
$$0.7 = \kappa \le L = 1,$$
$$\max_{(i,j)}\left\{\frac{1}{a_{ij}^-}E_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right)E_{ij}\right\} = 0.6603 \le \kappa,$$
$$\max_{(i,j)}\left\{\frac{1}{a_{ij}^-}F_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-}\right)F_{ij}\right\} = 0.5804 < 1.$$
(80)

It follows that system (56) satisfies all the conditions in Theorems 10 and 12. Hence, system (76) has exactly one pseudo almost periodic solution. Moreover, the pseudo almost periodic solution is globally exponentially stable. The fact is verified by the numerical simulation in Figures 1, 2, and 3 and there are three different initial values which are $\varphi_{11} \equiv 1$, $\varphi_{12} \equiv -3$, $\varphi_{13} \equiv 4$, $\varphi_{21} \equiv 2$, $\varphi_{22} \equiv 5$, $\varphi_{23} \equiv 3$, $\varphi_{33} \equiv -1$, $\varphi_{32} \equiv -2$, $\varphi_{33} \equiv -5$; $\varphi_{11} \equiv 2$, $\varphi_{12} \equiv -1$, $\varphi_{13} \equiv 5$, $\varphi_{21} \equiv 4$, $\varphi_{22} \equiv 2$, $\varphi_{23} \equiv 1$, $\varphi_{33} \equiv -3$, $\varphi_{32} \equiv -4$, $\varphi_{33} \equiv 3$ and $\varphi_{11} \equiv -2$, $\varphi_{12} \equiv 1$, $\varphi_{13} \equiv -5$, $\varphi_{21} \equiv -4$, $\varphi_{22} \equiv -2$, $\varphi_{23} \equiv -1$, $\varphi_{33} \equiv 3$, $\varphi_{32} \equiv 4$, $\varphi_{33} \equiv -3$, respectively.

Remark 14. By using the inequality analysis technique, in [19, 20], the authors obtained the existence of almost periodic solution of SICNNs with leakage delays, but they did not give the existence and global exponential convergence for the pseudo almost periodic solution. Since [1–9] only dealt with SICNNs without leakage delays, [14–18, 21–24] give no opinions about the problem of pseudo almost periodic solutions for SICNNs with leakage delays. One can observe that all the results in these literatures and the references therein cannot be applicable to prove the existence and exponential stability of pseudo almost periodic solutions for SICNNs (56).

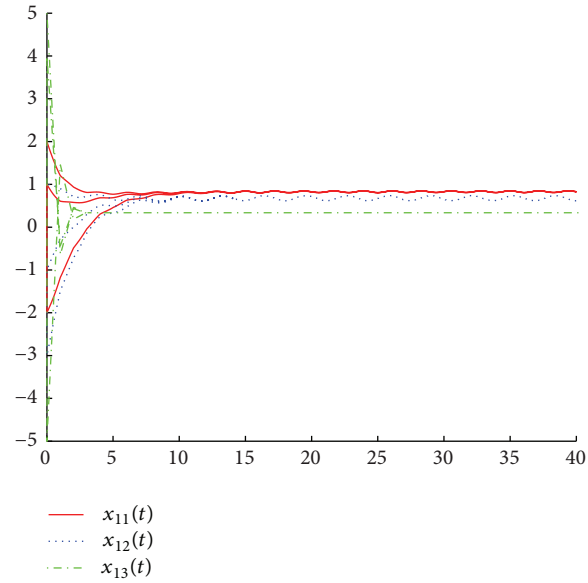

Figure 1: Numerical solutions of system (76) for different initial values.

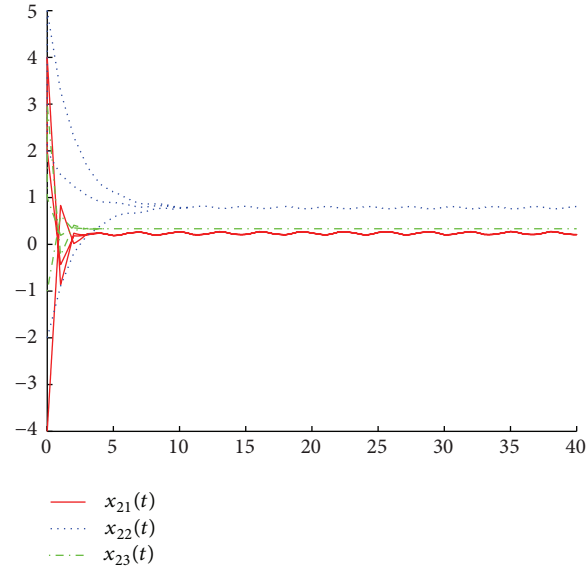

Figure 2: Numerical solutions of system (76) for different initial values.

Conflict of Interests

The authors declare no conflict of interests. They also declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work; there is no professional or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this present paper.

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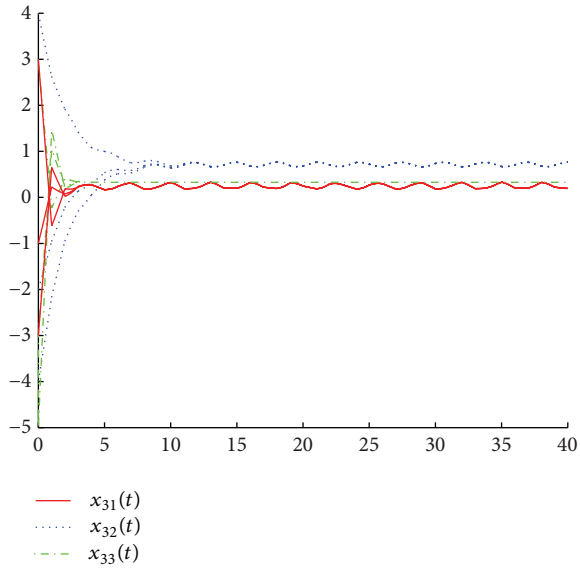

FIGURE 3: Numerical solutions of system (76) for different initial values.

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