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Research Article

Orbital Stability of Solitary Waves for Generalized Symmetric Regularized-Long-Wave Equations with Two Nonlinear Terms

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This paper investigates the orbital stability of solitary waves for the generalized symmetric regularized-long-wave equations with two nonlinear terms and analyzes the influence of the interaction between two nonlinear terms on the orbital stability. Since J is not onto, Grillakis-Shatah-Strauss theory cannot be applied on the system directly. We overcome this difficulty and obtain the general conclusion on orbital stability of solitary waves in this paper. Then, according to two exact solitary waves of the equations, we deduce the explicit expression of discrimination $d''(c)$ and give several sufficient conditions which can be used to judge the orbital stability and instability for the two solitary waves. Furthermore, we analyze the influence of the interaction between two nonlinear terms of the equations on the wave speed interval which makes the solitary waves stable.

1. Introduction

Symmetric regularized-long-wave equations (SRLWE)

$$u_{xxt} - u_t = \left(v + \frac{1}{2}u^2\right)_x, \qquad v_t + u_x = 0, \tag{1}$$

which are the mathematical models describing the propagation of weakly nonlinear ion acoustic waves [1] and the typical equations in the field of nonlinear science, arise in many other areas of nonlinear mathematical physics [2]. References [1, 2] studied the solitary wave solutions, conservation laws, and interaction among the solitary wave solutions of (1). Moreover, [3–5] discussed the global solution and numerical solution of (1).

Many authors have studied some extended forms of (1).

Guo [3] studied the periodic initial value problem for generalized nonlinear wave equations including (1)

$$u_t - u_{xxt} + \rho_x + f(u)_x = g(u, \rho, u_x), \qquad \rho_t + u_x = h(\rho), \tag{2}$$

by spectral method, then proved the existence and uniqueness of the global generalized solution and classical solution, and gave the convergence and error estimates for the approximate solution in 1987. Zhang [6] obtained the exact solitary wave solutions for a class of the generalized SRLWE with high-order nonlinear terms in 2003.

In terms of the orbital stability of solitary wave solutions, Chen [7] studied it in 1998 for the following generalized SRLWE:

$$u_{tt} - u_{xx} + f(u)_{xt} - u_{xxtt} = 0, \tag{3}$$

where $f(u)$ is a C^1 function, satisfying $f(s) > 0$ if $s > 0$ and $|f(s)| = o(|s|^p)$; $|f'(s)| = o(|s|^{p-1})$ as $s \to 0$ for $p > 1$. In particular, $\varphi_c > 0$ in the solitary wave solution $(\varphi_c, \psi_c)^T$ (T represents transposition) in Assumption 1 of [7]. Moreover, H_c only has a simple negative eigenvalue, whose kernel is spanned by φ_c'. In addition, the rest of its eigenvalues are positive and bounded away from zero.

In this paper, we will consider the orbital stability and instability of solitary wave solutions for the following generalized SRLWE with two nonlinear terms:

$$u_{xxt} - u_t = \left(v + b_2 u^2 + b_3 u^3\right)_x,$$
$$b_i = \text{constant}, \quad b_3 \geq 0,\ i = 2, 3, \tag{4}$$
$$v_t + u_x = 0.$$

Our purpose is to investigate the influence of the interaction of the nonlinear terms on the orbital stability.

Equation (4) is the generalization of (1). If (4) is converted into (3), then $f(u) = b_2 u^2 + b_3 u^3$, where $f(u)$ has two nonlinear terms and the symbols of b_2, b_3 are unfixed. Indeed, $f(u)$ is not always positive when $u > 0$, so the problem studied in this paper is not included by [7]. In the other hand, according to Theorem 1 in this paper, (4) has two bell-profile solitary wave solutions $(\varphi_i, \psi_i)^T$, $i = 1, 2$, where $\varphi_1(\xi) > 0$ and $\varphi_2(\xi) < 0$. But the orbital stability of the solitary wave solution $(\varphi_2(\xi), \psi_2(\xi))^T$ is not considered in [7]. In this paper, we will consider it as well. So the content of this study is new. More significantly, we will study the influence of the interaction between nonlinear terms $b_2 u^2$ and $b_3 u^3$ on the orbital stability. It is meaningful for the stability in the application of the practical problems and the selection of the models.

The paper is organized as follows. In Section 2, we will present two exact bell-profile solitary wave solutions of (4) and local existence for the solution of Cauchy problem. In Section 3, we will verify that (4) and its solitary wave solutions meet the requirements of the orbital stability theory of Grillakis-Shatah-Strauss and give the general conclusion. In Section 4, according to two exact solitary waves of the equations obtained in Section 2, we deduce the explicit expression of discrimination $d''(c)$ and give several sufficient conditions which can be used to judge the orbital stability and instability for the two solitary waves. Moreover, we will analyze the influence of two nonlinear terms on the orbital stability. In Section 5, we will focus on studying the orbital instability of solitary wave solutions for (4). Since the skew symmetric operator J is not onto, we will define a new conservational functional $I(\vec{u}) = \int_R \vec{u}\,dx$ and estimate solutions of the initial value problem. We will construct a formal Lyapunov function and present the sufficient condition on orbital instability of solitary wave solutions.

2. The Bell-Profile Solitary Wave Solutions and Local Existence for the Solution of Cauchy Problem

According to [6], the solitary wave solution of (4) satisfies

$$u(x,t) = u(\xi) = u(x - ct)$$
$$v(x,t) = v(\xi) = v(x - ct)$$

$$-cu'''(\xi) + cu'(\xi) = v'(\xi) + \left(b_2 u^2(\xi) + b_3 u^3(\xi)\right)_\xi,$$
$$v'(\xi) = \frac{1}{c} u'(\xi), \tag{5}$$

where $u'(\xi), u''(\xi) \to 0, |\xi| \to \infty$. Their exact expressions are given by the following theorem.

Theorem 1. *Suppose that $c^2 - 1 > 0$.*

(1) *If $b_3 c > 0$, or $b_3 = 0$ and $b_2 c > 0$, then (4) has a bell-profile solitary wave solution*

$$u(x,t) = \varphi_1(\xi) = \varphi_1(x - ct),$$
$$v(x,t) = \psi_1(\xi) = \frac{1}{c}\varphi_1(x - ct), \tag{6a}$$

where

$$\varphi_1(\xi) = \frac{A_1 \operatorname{sech}^2(\alpha_1/2)\xi}{2 + B_1 \operatorname{sech}^2(\alpha_1/2)\xi}, \tag{6b}$$

$$\alpha_1 = \frac{\sqrt{c^2 - 1}}{c}, \qquad A_1 = \frac{3\sqrt{2}\left(c^2 - 1\right)}{\sqrt{c\left[2b_2^2 c + 9b_3\left(c^2 - 1\right)\right]}},$$
$$B_1 = -1 + \frac{b_2 A_1}{3\alpha_1^2 c}. \tag{6c}$$

(2) *If $b_3 c > 0$, or $b_3 = 0$ and $b_2 c < 0$, then (4) has another bell-profile solitary wave solution*

$$u(x,t) = \varphi_2(\xi) = \varphi_2(x - ct),$$
$$v(x,t) = \psi_2(\xi) = \frac{1}{c}\varphi_2(x - ct), \tag{7a}$$

where

$$\varphi_2(\xi) = \frac{A_2 \operatorname{sech}^2(\alpha_2/2)\xi}{2 + B_2 \operatorname{sech}^2(\alpha_2/2)\xi}, \tag{7b}$$

$$\alpha_2 = \alpha_1, \qquad A_2 = -A_1, \qquad B_2 = -1 + \frac{b_2 A_2}{3\alpha_2^2 c} = -2 - B_1. \tag{7c}$$

Next, we study the local existence for the solution of Cauchy problem for (4) by semigroup theory. Firstly, we give two lemmas (see [8, 9]).

Lemma 2 (Hille-Yosida-Phillips). *A linear unbounded operator A is the infinitesimal generator of a C_0 semigroup of $\{T(t) : t \geq 0\}$ if and only if A is a closed operator with dense domain and there exist real numbers M and ω, such that when $\lambda > \omega$, one has*

(1) $\lambda \in \rho(A)$,

(2) $\|R(\lambda; A)^n\| \le M/(\lambda-\omega)^n$, $n = 1, 2, \ldots$,

where $\rho(A)$ is the resolvent set and $R(\lambda; A)^n$ is the resolvent of A.

Lemma 3. *Consider the Cauchy problem of nonlinear equation*

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), \quad t > 0, \qquad u(0) = u_0, \quad u_0 \in X. \tag{8}$$

If the following two conditions hold:

(1) *A is the infinitesimal generator of a C_0 semigroup $T(t)$ in X;*

(2) *$f \in C(R^+ \times X, X)$ satisfies the Lipschitz manner, which means for any $T > 0$,*

there exists $K = K(t)$, such that $\|f(t,u) - f(t,v)\| \le K(t)\|u - v\|$, for all $u, v \in X, t \in [0, T]$, then the initial value problem (8) has a unique solution

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s, u(s))\, ds \tag{9}$$

in R^+.

From Lemmas 2 and 3, we can prove the following Lemma 4, which describes the local existence for the solution of Cauchy problem for (4).

Lemma 4. *For any $\vec{u}_0 \in X(H^1(R) \times L^2(R))$, there exists $t_0 > 0$, which only depends on $\|\vec{u}_0\|_X$, such that (4) has a unique solution $\vec{u} \in C([0, t_0); H^1(R) \times L^2(R))$ with $\vec{u}(0) = \vec{u}_0$.*

Proof. Firstly, (4) can be written as

$$u_t = (\Delta - 1)^{-1}\left(v + b_2u^2 + b_3u^3\right)_x, \qquad b_i = \text{constant}, \quad b_3 \ge 0, \qquad v_t = -u_x, \tag{10}$$

where $\Delta = \partial^2/\partial x^2$. $(\Delta - 1)^{-1}$ is the pseudodifferential operator. The initial value problem of (10) is equal to

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) + f(t, \vec{u}(t)), \quad t > 0, \qquad \vec{u}(0) = \vec{u}_0, \tag{11}$$

where

$$\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \qquad \vec{u}_0 = \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & (\Delta - 1)^{-1}\partial_x \\ -\partial_x & 0 \end{pmatrix}, \tag{12}$$

$$f(t, \vec{u}(t)) = \begin{pmatrix} (\Delta - 1)^{-1}\left(b_2u^2 + b_3u^3\right)_x \\ 0 \end{pmatrix}.$$

Since for any $T > 0$, there exists $K = K(t)$, such that for any $\vec{u}_1, \vec{u}_2 \in X$, $t \in [0, T]$, we have

$$\begin{aligned} \|f(t, \vec{u}_1) - f(t, \vec{u}_2)\| &= \left\| \begin{pmatrix} (\Delta - 1)^{-1}\left(b_2u_1^2 + b_3u_1^3\right)_x \\ 0 \end{pmatrix} - \begin{pmatrix} (\Delta - 1)^{-1}\left(b_2u_2^2 + b_3u_2^3\right)_x \\ 0 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} (\Delta - 1)^{-1}\left[b_2\left(u_1^2 - u_2^2\right) + b_3\left(u_1^3 - u_2^3\right)\right]_x \\ 0 \end{pmatrix} \right\| \\ &\le K(t)\|\vec{u}_1 - \vec{u}_2\|. \end{aligned} \tag{13}$$

Therefore, $f(t, \vec{u}(t))$ satisfies the local Lipschitz manner.

Now we want to verify that A is the infinitesimal generator of a C_0 semigroup in X and $D(A) = H^1 \times L^2$.

According to Lemma 2, we only need to prove that there exists ω, such that

$$\left\|(\lambda I - A)^{-1}\right\| \le \frac{1}{\lambda - \omega} \tag{14}$$

if $\lambda > \omega$ and $\lambda \in \rho(A)$.

Indeed, since $\lambda \in \rho(A)$, for any $\vec{v} = \binom{\vec{v}_1}{\vec{v}_2} \in X$, we have $\vec{u} = \binom{\vec{u}_1}{\vec{u}_2} \in D(A)$ and $(\lambda I - A)\vec{u} = \vec{v}$. Thus $\vec{u} = (\lambda I - A)^{-1}\vec{v}$. Taking the Fourier transform yields

$$\widehat{u}_1 = \frac{\lambda}{\lambda^2 + \xi^2/(\xi^2+1)}\widehat{v}_1 - \frac{i\xi/\left(1+\xi^2\right)}{\lambda^2 + \xi^2/\left(1+\xi^2\right)}\widehat{v}_2, \tag{15}$$

$$\widehat{u}_2 = \frac{\lambda}{\lambda^2 + \xi^2/(\xi^2+1)}\widehat{v}_2 - \frac{i\xi}{\lambda^2 + \xi^2/\left(1+\xi^2\right)}\widehat{v}_1. \tag{16}$$

By (15), we have

$$\begin{aligned} \left|(\lambda - \omega)\widehat{u}_1\right| &\le \left|\frac{\lambda(\lambda - \omega)}{\lambda^2 + \xi^2/(\xi^2+1)}\widehat{v}_1\right| + \left|\frac{(\lambda - \omega)\xi}{\lambda^2(\xi^2+1) + \xi^2}\widehat{v}_2\right| \\ &\le \left|\widehat{v}_1\right| + \left|\frac{(\lambda - \omega)\xi}{\lambda^2(\xi^2+1) + \xi^2}\widehat{v}_2\right|. \end{aligned} \tag{17}$$

Since $(\lambda - \omega)\xi/(\lambda^2(\xi^2+1)+\xi^2) \to 0$ as $|\xi| \to \infty$, there exists positive real number N_1, when $|\xi| \ge N_1$, such that $|(\lambda - \omega)\xi/(\lambda^2(\xi^2+1)+\xi^2)| \le 1$; that is, $|(\lambda - \omega)\widehat{u}_1| \le |\widehat{v}_1| + |\widehat{v}_2|$.

Solving the inequality $|(\lambda - \omega)\xi/(\lambda^2(\xi^2+1)+\xi^2)| \le |(\lambda - \omega)\xi/\lambda^2| \le 1$ when $|\xi| \le N_1$, we can obtain that when $\omega \ge N_1/4$,

$$\left|(\lambda - \omega)\widehat{u}_1\right| \le \left|\widehat{v}_1\right| + \left|\widehat{v}_2\right|. \tag{18}$$

By (16), we know

$$\begin{aligned}\left|(\lambda-\omega)\widehat{u}_2\right| &\le \left|\frac{\lambda(\lambda-\omega)}{\lambda^2+\xi^2/(\xi^2+1)}\widehat{v}_2\right| \\ &\quad+\left|\frac{(\lambda-\omega)\xi}{\lambda^2+\xi^2/(\xi^2+1)}\widehat{v}_1\right| \\ &\le \left|\widehat{v}_2\right|+\left|\frac{(\lambda-\omega)\xi}{\lambda^2}\widehat{v}_1\right|.\end{aligned} \tag{19}$$

Since $(\lambda-\omega)|\xi|/\lambda^2 \to 0$ as $|\xi| \to 0$, there exists a positive real number N_2, when $|\xi| \le N_2$, such that $|(\lambda-\omega)\xi/\lambda^2| \le 1$; that is, $|(\lambda-\omega)\widehat{u}_2| \le |\widehat{v}_2| + |\widehat{v}_1|$.

Solving the inequality $(\lambda-\omega)|\xi|/\lambda^2 \le 1$ when $|\xi| \ge N_2$, we can obtain that when $\omega \ge N_2/4$,

$$\left|(\lambda-\omega)\widehat{u}_2\right| \le \left|\widehat{v}_2\right| + \left|\widehat{v}_1\right|. \tag{20}$$

Combining (18) and (20) and choosing $\omega = \mathrm{Max}\{N_1/4, N_2/4\}$, then we get (14) due to the definition of the operator norm.

In conclusion, we can obtain Lemma 4 from Lemmas 2 and 3. □

3. General Results for the Orbital Stability of Solitary Wave Solutions

Equation (4) can be written in a Hamiltonian form

$$\frac{d\vec{u}}{dt} = JE'(\vec{u}), \tag{21}$$

where

$$J = \frac{\partial}{\partial x}\begin{pmatrix}(1-\Delta)^{-1} & 0\\ 0 & 1\end{pmatrix}, \qquad \Delta = \frac{\partial^2}{\partial x^2}, \tag{22}$$

$$E(\vec{u}) = -\int_R\left(vu + \frac{b_2}{3}u^3 + \frac{b_3}{4}u^4\right)dx, \tag{23}$$

$$E'(\vec{u}) = \begin{pmatrix}E'(u)\\ E'(v)\end{pmatrix} = \begin{pmatrix}-v - b_2u^2 - b_3u^3\\ -u\end{pmatrix}. \tag{24}$$

Let $X = H^1(R) \times L^2(R)$, whose dual space is $X^* = H^{-1}(R) \times L^2(R)$, and the inner product of X is

$$(\vec{u}_1, \vec{u}_2) = \int_R (u_1u_2 + v_1v_2 + v_{1x}v_{2x})\,dx, \qquad \forall \vec{u}_1, \vec{u}_2 \in X. \tag{25}$$

There exists a natural isomorphism $I : X \to X^*$ defined by $\langle I\vec{u}_1, \vec{u}_2\rangle = (\vec{u}_1, \vec{u}_2)$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*, and

$$\langle \vec{u}_1, \vec{u}_2\rangle = \int_R (u_1u_2 + v_1v_2)\,dx. \tag{26}$$

From (25) and (26), we know that $I = \begin{pmatrix}1 & 0\\ 0 & 1-\Delta\end{pmatrix}$. And we can verify that J is an skew symmetric operator; that is, $\langle J\vec{u}, \vec{v}\rangle = -\langle \vec{u}, J\vec{v}\rangle$.

Let T be a one-parameter group of unitary operator on X defined by $T(s)\vec{u}(\cdot) = \vec{u}(\cdot - s)$, where $\vec{u}(s) \in X$, for all $s \in R$. Obviously, $T'(0) = \begin{pmatrix}-\partial/\partial x & 0\\ 0 & -\partial/\partial x\end{pmatrix}$. Since $T'(0) = JB$, we can get $B = \begin{pmatrix}\Delta-1 & 0\\ 0 & -1\end{pmatrix}$.

Therefore, we define

$$\begin{aligned}V(\vec{u}) &= -\frac{1}{2}\langle B\vec{u}, \vec{u}\rangle = -\frac{1}{2}\left\langle\begin{pmatrix}\Delta-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}, \begin{pmatrix}u\\ v\end{pmatrix}\right\rangle \\ &= \frac{1}{2}\left\langle\begin{pmatrix}u-u_{xx}\\ v\end{pmatrix}, \begin{pmatrix}u\\ v\end{pmatrix}\right\rangle \\ &= \frac{1}{2}\int_R\left(u^2 - u_{xx}u + v^2\right)dx.\end{aligned} \tag{27}$$

Then $V'(\vec{u}) = \begin{pmatrix}V'(u)\\ V'(v)\end{pmatrix} = \begin{pmatrix}u-u_{xx}\\ v\end{pmatrix}$, $V''(\vec{u}) = \begin{pmatrix}1-\Delta & 0\\ 0 & 1\end{pmatrix}$.

The solitary waves (6a) and (7a) of (4) can be written as

$$\vec{\phi}_{ic} = \begin{pmatrix}\varphi_{ic}(x-ct)\\ \psi_{ic}(x-ct)\end{pmatrix} = T(ct)\vec{\phi}_i(x), \quad i = 1, 2, \tag{28}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are defined by (6b) and (7b), respectively. Now, we consider the orbital stability of solitary waves $T(ct)\vec{\phi}_i(x)$. Avoiding repetition, we let $\vec{\phi}_c(x)$ be one of $\vec{\phi}_1(x)$ and $\vec{\phi}_2(x)$. We will verify that (4) and the solitary wave $T(ct)\vec{\phi}_c(x)$ satisfy the three assumption conditions of the orbital stability theory presented by Grillakis et al. in [10].

Verification of Assumption 1. From Lemma 4 in Section 2, we obtain that the initial value problem of (4) has a unique solution. And it is easy to prove that $E(\vec{u})$ and $V(\vec{u})$ defined by (23) and (27) satisfy

$$\begin{aligned}E(\vec{u}(t)) &= E(\vec{u}(0)) = E(\vec{u}_0),\\ V(\vec{u}(t)) &= V(\vec{u}(0)) = V(\vec{u}_0),\end{aligned} \tag{29}$$

respectively.

This shows that (4) satisfies the Assumption 1 in [10].

Verification of Assumption 2. Firstly, we can prove the following lemma.

Lemma 5. *$\vec{\phi}_c$ is a bounded state solution of* (4), *satisfying* $E'(\vec{\phi}_c) + cV'(\vec{\phi}_c) = 0$.

Proof. Substituting the solution $\vec{\phi}_c = \begin{pmatrix}\varphi_c\\ \psi_c\end{pmatrix}$ into (4), we obtain

$$\begin{aligned}\varphi_{c\xi} &= c\psi_{c\xi},\\ -c\varphi_{c\xi\xi\xi} + c\varphi_{c\xi} &= \left(\psi_c + b_2\varphi_c^2 + b_3\varphi_c^3\right)_\xi.\end{aligned} \tag{30}$$

Integrating (30) once, we get

$$\begin{aligned}\varphi_c &= c\psi_c + a_1,\\ -c\varphi_{c\xi\xi} + c\varphi_c &= \left(\psi_c + b_2\varphi_c^2 + b_3\varphi_c^3\right) + a_2,\end{aligned} \tag{31}$$

where a_1, a_2 are integral constants.

$\varphi_c, \psi_c, \varphi_{c\xi\xi} \to 0$ as $|\xi| \to \infty$, so $a_1 = 0$ and $a_2 = 0$. Thus,

$$\begin{gathered} \varphi_c = c\psi_c, \\ -c\varphi_{c\xi\xi} + c\varphi_c = \psi_c + b_2\varphi_c^2 + b_3\varphi_c^3. \end{gathered} \tag{32}$$

Furthermore,

$$\begin{aligned} & E'\left(\vec{\phi}_c\right) + cV'\left(\vec{\phi}_c\right) \\ & \quad = \begin{pmatrix} -\psi_c - b_2\varphi_c^2 - b_3\varphi_c^3 \\ -\varphi_c \end{pmatrix} + c\begin{pmatrix} \varphi_c - \varphi_{c\xi\xi} \\ \psi_c \end{pmatrix} \\ & \quad = \begin{pmatrix} -\psi_c - b_2\varphi_c^2 - b_3\varphi_c^3 - c\varphi_{c\xi\xi} + c\varphi_c \\ -\varphi_c + c\psi_c \end{pmatrix}. \end{aligned} \tag{33}$$

Due to (32), we have $E'(\vec{\phi}_c) + cV'(\vec{\phi}_c) = 0$. □

The above Lemma 5 shows that (4) has the bounded state solutions, and the two solitary waves $\vec{\phi}_{1c}$ and $\vec{\phi}_{2c}$ given in Theorem 1 both are the bounded state solutions of the equation.

Verification of Assumption 3. We consider spectrum analysis of the operator H_c.

Now we define the operator $H_c : X \to X^*$ as $H_c = E''(\vec{\phi}_c) + cV''(\vec{\phi}_c)$, where

$$E''(\vec{u}) = \begin{pmatrix} -2b_2u - 3b_3u^2 & -1 \\ -1 & 0 \end{pmatrix}, \qquad V''(\vec{u}) = \begin{pmatrix} 1-\Delta & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

Therefore,

$$\begin{aligned} H_c &= E''\left(\vec{\phi}_c\right) + cV''\left(\vec{\phi}_c\right) \\ &= \begin{pmatrix} -2b_2\varphi_c - 3b_3\varphi_c^2 & -1 \\ -1 & 0 \end{pmatrix} + c\begin{pmatrix} 1-\Delta & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2b_2\varphi_c - 3b_3\varphi_c^2 - c\dfrac{\partial^2}{\partial x^2} + c & -1 \\ -1 & c \end{pmatrix}. \end{aligned} \tag{35}$$

For any $\vec{u}_1, \vec{u}_2 \in H^1(R) \times L^2(R)$, we have $\langle H_c\vec{u}_1, \vec{u}_2\rangle = \langle H_c\vec{u}_2, \vec{u}_1\rangle$. This means that H_c is a self-conjugate operator, that is, $H_c = H_c^*$, and that $I^{-1}H_c$ is a bounded self-conjugate operator on X. The eigenvalues of H_c consist of the real numbers λ which ensure that $H_c - \lambda I$ are irreversible.

From (30), we have $-c\varphi_{c\xi\xi\xi} + c\varphi_{c\xi} = ((1/c)\varphi_c + b_2\varphi_c^2 + b_3\varphi_c^3)_\xi$. Namely,

$$\left[-2b_2\varphi_c - 3b_3\varphi_c^2 - c\frac{\partial^2}{\partial x^2} + \left(c - \frac{1}{c}\right)\right]\varphi_{cx} = 0. \tag{36}$$

Let $L = -2b_2\varphi_c - 3b_3\varphi_c^2 - c(\partial^2/\partial x^2) + (c - (1/c))$. Since the existence of solitary wave solution $\vec{\phi}_c = \binom{\varphi_c}{\psi_c}$ of (4) is based on the condition that $c^2 - 1 > 0$, $c - 1/c > 0$ as $c > 1$, and $-2b_2\varphi_c - 3b_3\varphi_c^2 \to 0$ as $|\xi| \to \infty$, it is easy to know that $\sigma_{ess}(L) = [c - 1/c, +\infty)$ by Weyl's essential spectrum theorem.

Moreover, from (36), we have $L\varphi_{cx} = 0$, where $x = 0$ is a unique zero point of φ_{cx}. By Sturm-Liouville theorem we know that zero is the second eigenvalue of L. Thus L only has one strictly negative eigenvalue $-\sigma^2$ in the case of $c > 1$, whose corresponding eigenfunction is denoted by χ; that is, $L\chi = -\sigma^2\chi$.

Therefore, H_c has a unique simple negative eigenvalue, and zero is its eigenvalue and the rest of its spectrums are bounded away from zero. So, H_c satisfies the Assumption 3 in [10].

According to [10, 11], we can get the following lemma.

Lemma 6. *For any real function $y \in H^1(R)$ with $\langle y, \chi\rangle = \langle y, \varphi_{cx}\rangle = 0$, there exists $\delta > 0$ such that $\langle Ly, y\rangle \geq \delta\|y\|^2_{H^2(R)}$.*

Let $\vec{\chi}_c = \binom{\chi}{(1/c)\chi}$. We have

$$\begin{aligned} & \langle H_c\vec{\chi}_c, \vec{\chi}_c\rangle \\ & \quad = \int_R \left[-2b_2\varphi_c\chi^2 - 3b_3\varphi_c^2\chi^2 - c\chi_{xx}\chi + \left(c - \frac{1}{c}\right)\chi^2\right] dx \\ & \quad = \langle L\chi, \chi\rangle \\ & \quad = -\sigma^2 \langle \chi, \chi\rangle < 0. \end{aligned} \tag{37}$$

Let $\vec{\varphi}_{cx} = \binom{\varphi_{cx}}{(1/c)\varphi_{cx}}$, and then

$$\begin{aligned} & \langle H_c\vec{\varphi}_{cx}, \vec{\varphi}_{cx}\rangle \\ & = \int_R \left[-2b_2\varphi_c\varphi_{cx}^2 - 3b_3\varphi_c^2\varphi_{cx}^2 - c\varphi_{cxxx}\varphi_{cx} + \left(c - \frac{1}{c}\right)\varphi_{cx}^2\right] dx \\ & = \langle L\varphi_{cx}, \varphi_{cx}\rangle = 0. \end{aligned} \tag{38}$$

Le

$$\begin{aligned} P = \Big\{ & \vec{p}_c \in X \mid \vec{p}_c = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \ \langle p_1, \chi\rangle = \langle p_1, \varphi_{cx}\rangle = 0, \\ & \left\langle p_2, \frac{1}{c}\chi\right\rangle = \left\langle p_2, \frac{1}{c}\varphi_{cx}\right\rangle = 0 \Big\}. \end{aligned} \tag{39}$$

We have

$$\begin{aligned} & \langle H_c\vec{p}_c, \vec{p}_c\rangle \\ & \quad = \int_R \Big[-2b_2\varphi_c p_1^2 - 3b_3\varphi_c^2 p_1^2 \\ & \qquad - cp_{1xx}p_1 + \left(c - \frac{1}{c}\right)p_1^2 + \frac{1}{c}(p_1 - cp_2)^2\Big] dx. \end{aligned} \tag{40}$$

Thus $\langle H_c\vec{p}_c, \vec{p}_c\rangle \geq \langle Lp_1, p_1\rangle \geq \delta\|\vec{p}_c\|^2 > 0$ for any $\vec{p}_c \in P$ when $c > 1$.

According to the above analysis, when $c > 1$, we can make spectrum decomposition for H_c. Let

$$N = \{k_1\vec{\chi}_c \mid k_1 \in R\},$$
$$Z = \{k_2\vec{\varphi}_{cx} \mid k_2 \in R\},$$
$$P = \left\{\vec{p}_c \in X \mid \vec{p}_c = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},\ \langle p_1, \chi\rangle = \langle p_1, \varphi_{cx}\rangle = 0, \right. \quad (41)$$
$$\left. \left\langle p_2, \frac{1}{c}\chi \right\rangle = \left\langle p_2, \frac{1}{c}\varphi_{cx} \right\rangle = 0 \right\}.$$

For any $\vec{u} \in N$, $\vec{u} \neq 0$, $\langle H_c\vec{u}, \vec{u}\rangle < 0$ due to $\langle H_c k_1\vec{\chi}_c, k_1\vec{\chi}_c\rangle = k_1^2\langle H_c\vec{\chi}_c, \vec{\chi}_c\rangle = -k_1^2\sigma^2\langle \chi, \chi\rangle < 0$.

For any $\vec{z} \in Z$, $\vec{z} \neq 0$, $\langle H_c\vec{z}, \vec{z}\rangle = 0$ due to $\langle H_c k_2\vec{\varphi}_{cx}, k_2\vec{\varphi}_{cx}\rangle = 0$.

For any $\vec{p}_c \in P$, $\vec{p}_c \neq 0$, $\langle H_c\vec{p}_c, \vec{p}_c\rangle > \delta\|\vec{p}_c\|^2_{H^2(R)}$.

Thus the space X can be decomposed as a direct sum $X = N + Z + P$, where Z is the kernel space of H_c, N is a finite-dimensional subspace and P is a closed subspace.

We now define $d(c) : R \to R$ as $d(c) = E(\vec{\phi}_c) + cV(\vec{\phi}_c)$, and then

$$d'(c) = \left\langle E'\left(\vec{\phi}_c\right), \vec{\phi}_c'\right\rangle + c\left\langle V'\left(\vec{\phi}_c\right), \vec{\phi}_c'\right\rangle + V\left(\vec{\phi}_c\right) = V\left(\vec{\phi}_c\right). \quad (42)$$

According to the above verification of Assumptions 1–3, (4) *and its solitary wave solutions satisfy the three assumptions of Theorem 2 in [10], so we can obtain the following general conclusion on orbital stability of solitary waves for* (4).

Theorem 7. *Suppose that $b_2b_3 \neq 0$, $b_3 \geq 0$, $c > 1$, and $\vec{\phi}_c = T(ct)\vec{\phi}(x)$ is the solitary wave solution of* (4). *Then,*

(1) *$\vec{\phi}_c$ is orbitally stable as $d''(c) > 0$;*

(2) *$\vec{\phi}_c$ is orbitally unstable as $d''(c) < 0$.*

Remark 8. The proof of the conclusion (2) in Theorem 7 will be given by Theorem 26 in Section 5.

4. Orbital Stability and Influence of the Interaction between Nonlinear Terms on It

In this section, by using two exact solitary waves (6a), (6b), and (6c) and (7a), (7b), and (7c) given in Theorem 1, we will give the explicit expressions for the discrimination $d_i''(c)$. Then with the analysis method, we will give several sufficient conditions to judge the orbital stability and instability of the solitary waves. Furthermore, we will also analyze the influence of the interaction between two nonlinear terms on the orbital stability. We assume that $b_3 > 0$ and $c > 1$ in this section.

4.1. Discrimination $d_i''(c)$. In view of (42), we have

$$d_i'(c) = V\left(\vec{\phi}_{ic}\right) = \frac{1}{2}\int_R \left(\varphi_{ic}^2 - \varphi_{icxx}\varphi_{ic} + \psi_{ic}^2\right) dx, \quad i = 1, 2. \quad (43)$$

Next, we simplify (43). According to (6a) and (7a) in Theorem 1, we have $\psi_i = (1/c)\varphi_i$. Substituting it into (43), and letting $z = e^{\alpha_i\xi}(\alpha_i > 0)$, we obtain

$$\begin{aligned} d_i'(c) &= \frac{1}{2}\int_R \left(\varphi_{ic}^2 - \varphi_{icxx}\varphi_{ic} + \psi_{ic}^2\right) dx \\ &= \frac{1}{2}\int_R \left[\left(1 + \frac{1}{c^2}\right)\varphi_{ic}^2 + \varphi_{icx}^2\right] dx \\ &= \frac{1}{2}\int_R \left[\left(1 + \frac{1}{c^2}\right)\varphi_{ic}^2 + \varphi_{ic\xi}^2\right] d\xi \\ &= \frac{1}{2\alpha_i}\int_0^{+\infty} \frac{1}{z}\left[\frac{2A_iz}{(1+z)^2 + 2B_iz}\right]^2 \\ &\quad \times \left\{1 + \frac{1}{c^2} + \left[\frac{\alpha_i\left(z^2 - 1\right)}{(1+z)^2 + 2B_iz}\right]^2\right\} dz \\ &= \left(1 + \frac{1}{c^2}\right)\frac{2A_i^2}{\alpha_i}\int_0^{+\infty} \frac{z}{\left[(1+z)^2 + 2B_iz\right]^2} dz \\ &\quad + 2A_i^2\alpha_i\int_0^{+\infty} \frac{z\left(z^2 - 1\right)^2}{\left[(1+z)^2 + 2B_iz\right]^4} dz, \end{aligned} \quad (44)$$

where α_i, A_i, B_i are given by (6c) and (7c).

Since $-2 < B_i < 0$, we can solve above two integrations. Then,

$$\begin{aligned} d_i'(c) &= \left(1 + \frac{1}{c^2}\right) \\ &\quad \times A_i^2\left\{\frac{B_i + 1}{\sqrt{-B_i\left(B_i + 2\right)}}\right. \\ &\quad \left.\times\left[\pi - 2\arctan\left(\frac{B_i + 1}{\sqrt{-B_i\left(B_i + 2\right)}}\right)\right] - 2\right\} \\ &\quad \times \left(2\alpha_iB_i\left(B_i + 2\right)\right)^{-1} \end{aligned}$$

$$+ A_i^2\alpha_i \left\{ 2\left(3 + 2B_i + B_i^2\right) - 3\frac{B_i + 1}{\sqrt{-B_i\left(B_i + 2\right)}} \times \left[\pi - 2\arctan\left(\frac{B_i + 1}{\sqrt{-B_i\left(B_i + 2\right)}}\right)\right]\right\} \times \left(12B_i^2\left(B_i + 2\right)^2\right)^{-1}. \quad (45)$$

If

$$B_i = B_1 = -1 + \frac{\sqrt{2}b_2 c}{\sqrt{c\left[2b_2^2 c + 9b_3\left(c^2 - 1\right)\right]}}, \quad (46)$$

then (45) can be simplified into the following form:

$$d_1'(c) = \frac{1}{3\sqrt{b_3 c}}\left\{6\alpha\sqrt{b_3 c} - \sqrt{2}b_2 \times \left[\pi - 2\arctan\left(\frac{\sqrt{2}b_2}{3}\sqrt{\frac{c}{b_3\left(c^2 - 1\right)}}\right)\right]\right\} \cdot \left(\frac{1}{2b_3 c} + \frac{3c}{2b_3} + \frac{b_2^2}{9b_3^2}\right) - \frac{c\alpha^3}{3b_3}. \quad (47)$$

If

$$B_i = B_2 = -1 - \frac{\sqrt{2}b_2 c}{\sqrt{c\left[2b_2^2 c + 9b_3\left(c^2 - 1\right)\right]}}, \quad (48)$$

then (45) can be simplified into the following form:

$$d_2'(c) = \frac{1}{3\sqrt{b_3 c}}\left\{6\alpha\sqrt{b_3 c} + \sqrt{2}b_2 \times \left[\pi + 2\arctan\left(\frac{\sqrt{2}b_2}{3}\sqrt{\frac{c}{b_3\left(c^2 - 1\right)}}\right)\right]\right\} \cdot \left(\frac{1}{2b_3 c} + \frac{3c}{2b_3} + \frac{b_2^2}{9b_3^2}\right) - \frac{c\alpha^3}{3b_3}. \quad (49)$$

By calculating, we have

$$d_1''(c) = -\frac{k\left(3c - 3/c - k^2\right)D_1}{4b_3 c^{3/2}} + \frac{\sqrt{c^2 - 1}\left(16c^4 + 7k^2c^3 - \left(8 + 3k^4\right)c^2 - 7k^2c + 16\right)}{6b_3c^3\left(c^2 - 1 + k^2c\right)}, \quad (50)$$

$$d_2''(c) = \frac{k\left(3c - 3/c - k^2\right)D_2}{4b_3 c^{3/2}} + \frac{\sqrt{c^2 - 1}\left(16c^4 + 7k^2c^3 - \left(8 + 3k^4\right)c^2 - 7k^2c + 16\right)}{6b_3c^3\left(c^2 - 1 + k^2c\right)}, \quad (51)$$

where

$$k = \frac{\sqrt{2}b_2}{3\sqrt{b_3}}, \qquad D_1 = \pi - 2\arctan\left(k\sqrt{\frac{c}{c^2 - 1}}\right), \qquad D_2 = \pi + 2\arctan\left(k\sqrt{\frac{c}{c^2 - 1}}\right). \quad (52)$$

Furthermore, suppose that

$$M_1 = k\sqrt{\frac{c}{c^2 - 1}}D_1 = k\sqrt{\frac{c}{c^2 - 1}}\left[\pi - 2\arctan\left(k\sqrt{\frac{c}{c^2 - 1}}\right)\right], \quad M_2 = k\sqrt{\frac{c}{c^2 - 1}}D_2 = k\sqrt{\frac{c}{c^2 - 1}}\left[\pi + 2\arctan\left(k\sqrt{\frac{c}{c^2 - 1}}\right)\right]. \quad (53)$$

Then (50) can be written as

$$d_1''(c) = \frac{\sqrt{c^2 - 1}}{12b_3c^2} \times \left[-3M_1\left(3c - \frac{3}{c} - k^2\right) + \frac{2\left(16c^4 + 7k^2c^3 - \left(8 + 3k^4\right)c^2 - 7k^2c + 16\right)}{c\left(c^2 - 1 + k^2c\right)}\right]. \quad (54)$$

And (51) can be written as

$$d_2''(c) = \frac{\sqrt{c^2 - 1}}{12b_3c^2} \times \left[3M_2\left(3c - \frac{3}{c} - k^2\right) + \frac{2\left(16c^4 + 7k^2c^3 - \left(8 + 3k^4\right)c^2 - 7k^2c + 16\right)}{c\left(c^2 - 1 + k^2c\right)}\right]. \quad (55)$$

Therefore, we only need to consider the conditions such that $d_i''(c) > 0$ hold in (54) and (55) to study the orbital stability of the solitary waves $\vec{\phi}_{ic}$, while needing to consider $d_i''(c) < 0$ to study instability.

4.2. Discussion on M_1 and M_2. In this section, we consider M_1 and M_2 in the case of $c > 1$ and $b_3 > 0$.

(1) For M_1. If $b_2 > 0$, then $k > 0$. Suppose that $x = k\sqrt{c/(c^2-1)} \in (0, +\infty)$, and then

$$\lim_{c\to+\infty} x = \lim_{c\to+\infty} k\sqrt{\frac{c}{c^2-1}} = 0,$$
$$\lim_{c\to 1} x = \lim_{c\to 1} k\sqrt{\frac{c}{c^2-1}} = +\infty. \tag{56}$$

Let $g(x) = x(\pi - 2\arctan x)$. We have

$$g'(x) = \pi - 2\arctan x - \frac{2x}{1+x^2},$$
$$g''(x) = -\frac{4}{(1+x^2)^2} < 0. \tag{57}$$

Moreover, $g(x)$ can obtain the local maximum at x_0, where x_0 satisfies $g'(x_0) = 0$ and

$$g(x_0) = x_0(\pi - 2\arctan x_0) = \frac{2x_0^2}{1+x_0^2} < 2. \tag{58}$$

Since

$$\lim_{x\to+\infty} M_1 = \lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} x(\pi - 2\arctan x) = 2,$$
$$M_1 \in (0,2). \tag{59}$$

When $b_2 < 0$, it is easy to know that $x \in (-\infty, 0)$, and then $M_1 \in (-\infty, 0)$.

(2) For M_2. When $b_2 > 0$, it is clear that $x \in (0, +\infty)$, and then $M_2 \in (0, +\infty)$. But if $b_2 < 0$, then $k < 0$ and $x = k\sqrt{c/(c^2-1)} \in (-\infty, 0)$. Therefore

$$\lim_{x\to-\infty} M_2 = \lim_{x\to-\infty} x(\pi + 2\arctan x) = -2. \tag{60}$$

Similarly, we can get $M_2 \in (-2, 0)$.

4.3. Orbital Stability of Solitary Waves for (4) *in the Case of* $3c - 3/c - k^2 > 0$. Based on (54), (55), and above discussion on M_1, M_2, we want to obtain much more simple conditions on the orbital stability of solitary waves $\vec{\phi}_{1c}$ and $\vec{\phi}_{2c}$.

4.3.1. Orbital Stability of $\vec{\phi}_{1c}$. (1) If $b_2 > 0$, then $M_1 \in (0,2)$. At this time, $-3M_1(3c - 3/c - k^2) < 0$. In order to find c such that $d_1''(c) > 0$, we only need to consider $M_1 = 2$ in (54). It is easy to see that $d_1''(c) > 0$ when c satisfies

$$7c^4 + k^2c^3 + 10c^2 - k^2c + 7 > 0. \tag{61}$$

Thus, $\vec{\phi}_{1c}$ is orbitally stable.

If $b_2 < 0$, $-3M_1(3c - 3/c - k^2) > 0$. In order to make $d_1''(c) > 0$, that is, $\vec{\phi}_{1c}$ is orbitally stable, only when c satisfies

$$16c^4 + 7k^2c^3 - (8 + 3k^4)c^2 - 7k^2c + 16 > 0. \tag{62}$$

(2) If $b_2 > 0$, then $M_1 \in (0,2)$. Here, $-3M_1(3c - 3/c - k^2) < 0$ in (54). In order to make $d_1''(c) < 0$, we only need to consider $M_1 = 0$ in (54). Then, it is easy to see that $d_1''(c) < 0$ when c satisfies

$$16c^4 + 7k^2c^3 - (8 + 3k^4)c^2 - 7k^2c + 16 < 0. \tag{63}$$

Thus, $\vec{\phi}_{1c}$ is orbitally unstable.

4.3.2. Orbital Stability of $\vec{\phi}_{2c}$. (1) If $b_2 > 0$, then $M_2 \in (0, +\infty)$. In order to make $d_2''(c) > 0$, it is easy to know that $\vec{\phi}_{2c}$ is orbitally stable if c satisfies (62).

If $b_2 < 0$, then $M_2 \in (-2, 0)$. In order to make $d_2''(c) > 0$, we only need to consider $M_2 = -2$ in (55). It is easy to know that $\vec{\phi}_{2c}$ is orbitally stable if c satisfies (61).

(2) If $b_2 < 0$, in order to make $d_2''(c) < 0$, we only need to consider $M_2 = 0$ in (55). Then, it is easy to know that $\vec{\phi}_{2c}$ is orbitally unstable if c satisfies (63).

In addition, we know that $3c - 3/c - k^2 > 0$ is equal to $c > (1/27b_3)[b_2^2 + \sqrt{b_2^4 + 729b_3^2}]$ or $c < (1/27b_3)[b_2^2 - \sqrt{b_2^4 + 729b_3^2}]$, but we always assume $c > 1$ through this section. So if we assume $c_0 = (1/27b_3)[b_2^2 + \sqrt{b_2^4 + 729b_3^2}]$, then $3c - 3/c - k^2 > 0$ is equal to $c > c_0$.

Summarizing above results, we have the following theorem.

Theorem 9. *Suppose that $b_3 > 0$ and $c > c_0$, where $c_0 = (b_2^2 + \sqrt{b_2^4 + 729b_3^2})/27b_3 > 1$.*

(1) *$\vec{\phi}_{1c}$ is orbitally stable if $b_2 > 0$ and the wave speed c satisfies* (61), *or $b_2 < 0$ and the wave speed c satisfies* (62); *$\vec{\phi}_{1c}$ is orbitally unstable if $b_2 > 0$ and the wave speed c satisfies* (63).

(2) *$\vec{\phi}_{2c}$ is orbitally stable if $b_2 > 0$ and the wave speed c satisfies* (62), *or $b_2 < 0$ and the wave speed c satisfies* (61); *$\vec{\phi}_{2c}$ is orbitally unstable if $b_2 < 0$ and the wave speed c satisfies* (63).

4.4. Orbital Stability of Solitary Waves for (4) *in the Case of* $3c - 3/c - k^2 < 0$

4.4.1. Orbital Stability of $\vec{\phi}_{1c}$. (1) If $b_2 > 0$, $-3M_1(3c - 3/c - k^2) > 0$. In order to make $d_1''(c) > 0$, we only need to consider $M_1 = 0$ in (54). It is easy to see that $\vec{\phi}_{1c}$ is orbitally stable if c satisfies (62).

(2) If $b_2 > 0$. In order to find c such that $d_1''(c) < 0$, we only need to consider $M_1 = 2$ in (54). It is easy to see that $d_1''(c) < 0$ when c satisfies

$$7c^4 + k^2c^3 + 10c^2 - k^2c + 7 < 0. \tag{64}$$

Thus, $\vec{\phi}_{1c}$ is orbitally unstable.

If $b_2 < 0$, $-3M_1(3c - 3/c - k^2) < 0$. In order to make $d_1''(c) < 0$, it is easy to know that $\vec{\phi}_{1c}$ is orbitally unstable if c satisfies (63).

4.4.2. Orbital Stability of $\vec{\phi}_{2c}$. (1) If $b_2 < 0$, then $M_2 \in (-2, 0)$. At this time, $3M_2(3c - 3/c - k^2) > 0$. In order to find c such that $d_2''(c) > 0$, we only need to consider $M_2 = 0$ in (55). It is easy to see that $\vec{\phi}_{2c}$ is orbitally stable if c satisfies (62).

(2) If $b_2 > 0$, then $M_2 \in (0, +\infty)$. In order to make $d_2''(c) < 0$, it is easy to know that $\vec{\phi}_{2c}$ is orbitally unstable if c satisfies (63).

If $b_2 < 0$, then $M_2 \in (-2, 0)$. Here, $3M_2(3c - 3/c - k^2) > 0$ in (55). In order to find c such that $d_2''(c) < 0$, we only need to consider $M_2 = -2$ in (55). It is easy to see that $\vec{\phi}_{2c}$ is orbitally unstable if c satisfies (64).

Summarizing above results, we have the following theorem.

Theorem 10. *Suppose that $b_3 > 0$ and $1 < c < c_0$, where $c_0 = (b_2^2 + \sqrt{b_2^4 + 729b_3^2})/27b_3 > 1$.*

(1) *$\vec{\phi}_{1c}$ is orbitally stable if $b_2 > 0$ and the wave speed c satisfies* (62). *$\vec{\phi}_{1c}$ is orbitally unstable if $b_2 > 0$ and the wave speed c satisfies* (64), *or $b_2 < 0$ and the wave speed c satisfies* (63).

(2) *$\vec{\phi}_{2c}$ is orbitally stable if $b_2 < 0$ and the wave speed c satisfies* (62). *$\vec{\phi}_{2c}$ is orbitally unstable if $b_2 > 0$ and the wave speed c satisfies* (63), *or $b_2 < 0$ and the wave speed c satisfies* (64).

4.5. Corollaries and Influences of Nonlinear Terms on Orbital Stability of the Solitary Waves for (4). In this part, we will firstly consider the orbital stability of the solitary waves for (4) with only one nonlinear term. Secondly, we will discuss the effect of nonlinear terms on orbital stability of the solitary waves for (4).

Corollary 11. *Suppose that $b_3 > 0$ and $c > 1$. If $b_2 = 0$,* (4) *has the solitary wave solutions $\vec{\phi}_{ic} = \binom{\varphi_i(\xi)}{\psi_i(\xi)}$, $i = 1, 2$, where*

$$\begin{gathered}
\varphi_1(\xi) = \frac{A_1 \operatorname{sech}^2(\alpha_1/2)\xi}{2 + B_1 \operatorname{sech}^2(\alpha_1/2)\xi}, \qquad \psi_1(\xi) = \frac{1}{c}\varphi_1(\xi), \\
\alpha_1 = \frac{\sqrt{c^2-1}}{c}, \qquad A_1 = \frac{\sqrt{2(c^2-1)}}{\sqrt{b_3 c}}, \qquad B_1 = -1, \\
\varphi_2(\xi) = \frac{A_2 \operatorname{sech}^2(\alpha_2/2)\xi}{2 + B_2 \operatorname{sech}^2(\alpha_2/2)\xi}, \qquad \psi_2(\xi) = \frac{1}{c}\varphi_2(\xi), \\
\alpha_2 = \frac{\sqrt{c^2-1}}{c}, \qquad A_2 = \frac{\sqrt{2(c^2-1)}}{\sqrt{b_3 c}}, \qquad B_2 = -1.
\end{gathered} \tag{65}$$

Under the given conditions, we can easily conclude that the solitary waves $\vec{\phi}_{ic}$, $i = 1, 2$, are both orbitally stable.

Proof. When $b_2 = 0$, the above solitary waves (65) of (4) can be deduced from Theorem 1 directly.

Actually, it is clear that $k = 0$ as $b_2 = 0$. Substituting $k = 0$ into (50) and (51), we have

$$d_1''(c) = d_2''(c) = \frac{4}{3b_3c^3}\frac{2c^4 - c^2 + 2}{\sqrt{c^2-1}}. \tag{66}$$

We know that $2c^4 - c^2 + 2 > 0$ in (66), so $d_i''(c) > 0$, $i = 1, 2$, if $c > 1$ and $b_3 > 0$. Thus, we know that the solitary waves $\vec{\phi}_{ic}$, $i = 1, 2$, of (4) are both orbitally stable according to Theorem 7. □

Corollary 12. *When $c > 1$ and $b_3 = 0$, then* (4) *has the solitary wave solution $\vec{\phi}_c = \binom{\varphi(\xi)}{\psi(\xi)}$, where*

$$\begin{gathered}
\varphi(\xi) = \frac{A \operatorname{sech}^2(\alpha/2)\xi}{2 + B \operatorname{sech}^2(\alpha/2)\xi}, \qquad \psi(\xi) = \frac{1}{c}\varphi(\xi), \\
\alpha = \frac{\sqrt{c^2-1}}{c}, \qquad A = \frac{3(c^2-1)}{b_2 c}, \qquad B = 0.
\end{gathered} \tag{67}$$

Under the given conditions, we know that the solitary wave $\vec{\phi}_c$ of (4) *is orbitally stable.*

Proof. When $b_3 = 0$, the above solitary wave (67) of (4) can be deduced from Theorem 1 directly.

Moreover, similar to deducing (50) and (51), by calculating, we can obtain

$$\begin{aligned}
d'(c) &= \frac{1}{2}\int_R \left(\varphi_c^2 - \varphi_{cxx}\varphi_c + \psi_c^2\right)dx \\
&= \frac{1}{2}\int_R \left[\left(1 + \frac{1}{c^2}\right)\varphi_c^2 + \varphi_{cx}^2\right]dx \\
&= \frac{1}{2}\int_R \left[\left(1 + \frac{1}{c^2}\right)\varphi_c^2 + \varphi_{c\xi}^2\right]d\xi \\
&= \frac{1}{2\alpha}\int_0^{+\infty}\frac{1}{z}\left[\frac{2Az}{(1+z)^2 + 2Bz}\right]^2 \\
&\quad \times \left\{1 + \frac{1}{c^2} + \left[\frac{\alpha(z^2-1)}{(1+z)^2 + 2Bz}\right]^2\right\}dz \\
&= \left(1 + \frac{1}{c^2}\right)\frac{2A^2}{\alpha}\int_0^{+\infty}\frac{z}{\left[(1+z)^2 + 2Bz\right]^2}dz \\
&\quad + 2A^2\alpha\int_0^{+\infty}\frac{z(z^2-1)^2}{\left[(1+z)^2 + 2Bz\right]^4}dz.
\end{aligned} \tag{68}$$

Substituting (67) into the above formula yields

$$d'(c) = \frac{6}{5}\frac{(c^2-1)^{3/2}(3c^2+2)}{c^3 b_2^2}. \tag{69}$$

Therefore

$$d''(c) = \frac{18}{5}\frac{\sqrt{c^2-1}\left(2c^4 + c^2 + 2\right)}{c^4 b_2^2}. \tag{70}$$

We know that $2c^4+c^2+2>0$ in (70). According to Theorem 7, $\vec{\phi}_c$ is orbitally stable if $c>1$ and $b_3=0$. Thus, Corollary 12 holds. □

According to the above Corollaries 11 and 12, we know that if (4) has only one nonlinear term $b_2(u^2)_x$ or $b_3(u^3)_x$, that is, $b_3=0$ or $b_2=0$, the solitary waves of (4) are both orbitally stable if $c>1$. That is to say the wave speed intervals which make the two solitary waves stable are both $(1,+\infty)$. But according to Theorems 9 and 10, when (4) has two nonlinear items $b_2(u^2)_x$ and $b_3(u^3)_x$, the stability of solitary waves will be affected by the interaction between them. For convenience, we call the solitary wave whose wave speed c satisfies $c>c_0$ ($c_0=(b_2^2+\sqrt{b_2^4+729b_3^2})/27b_3$) the big wave speed solitary wave, while we call the solitary wave whose wave speed c satisfies $c<c_0$ the small wave speed solitary wave. Generally, we have the results from Theorems 9 and 10 as follows.

(1) For given $b_3>0$, when $|b_2|$ is larger, the wave speed interval which makes the solitary waves stable will become smaller for the big wave speed solitary wave, but the wave speed interval which makes the solitary waves stable will become larger for the small wave speed solitary wave.

(2) For given b_2. For the big wave speed solitary wave, the wave speed interval which makes it stable will become larger if b_3 is bigger and the wave speed interval will become smaller if b_3 is smaller. For the small wave speed solitary wave, the wave speed interval which makes it stable will become smaller if b_3 is bigger and the wave speed interval will become larger if b_3 is smaller.

Summarizing the above results, it is significant to analyze the effect by multiple nonlinear terms on orbital stability of the solitary waves, at least in the application. For example, fix b_2 in (4). If we need to know the orbital stability of the small wave speed solitary wave in practical problems, since the wave speed interval which makes it stable will become larger as b_3 is smaller, and $(b_2^2+\sqrt{b_2^4+729b_3^2})/27b_3\to+\infty$ as $b_3\to 0$, it has little influence on the stability to ignore b_3u^3 in the application. But if we need to consider the orbital stability of the big wave speed solitary wave, the wave speed interval which makes it stable will become smaller as b_3 is smaller, so it is not suitable to ignore b_3u^3 in the application here.

5. Instability of the Solitary Waves

In this section, we will prove the conclusion (2) given in Theorem 7; that is, the solitary wave solution $\vec{\phi}_c$ is orbitally unstable if $d''(c)<0$.

Since J given in Section 3 is not onto, we cannot apply Grillakis-Shatah-Strauss theory on the system (4) directly. In order to prove instability, we define a new conservational functional

$$I(\vec{u})=\int_{-\infty}^{\infty}\vec{u}(x)\,dx. \tag{71}$$

We will prove that $d''(c)<0$ is the sufficient condition to judge orbital instability of solitary wave solutions by estimating to the solution of initial value problem.

5.1. Estimate to the Solution of Initial Value Problem for (4)

Lemma 13. *The unique solution $\vec{u}(t)$ of* (4) *with initial data $\vec{u}(0)=\vec{u}_0$ satisfies*

$$\begin{aligned}E(\vec{u}(t))&=E(u,v)=constant,\quad t\in R^+,\\V(\vec{u}(t))&=V(u,v)=constant,\quad t\in R^+,\end{aligned} \tag{72}$$

where $E(\vec{u})=-\int_R(vu+(b_2/3)u^3+(b_3/4)u^4)dx$ and $V(\vec{u})=(1/2)\int_R(-u_{xx}u+u^2+v^2)dx$.

From Lemma 4, (23)*, and* (27)*, we can prove Lemma 13 easily. We now prove that $I(\vec{u})=\int_{-\infty}^{\infty}\vec{u}(x)dx$ is an invariance.*

Lemma 14. *If $\int_R u_0(x)dx$ and $\int_R v_0(x)dx$ converge, then $I(u)=\int_R udx$ and $I(v)=\int_R vdx$ converge and are constants for any $t\in R^+$.*

Proof. Integrating (4) separately yields

$$\begin{aligned}&\int_a^b u(x,t)\,dx-\int_a^b u(x,0)\,dx\\&\quad=\int_0^t\int_a^b\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x^2}-1\right)^{-1}\left(v+b_2u^2+b_3u^3\right)dx\,d\tau,\\&\int_a^b v(x,t)\,dx-\int_a^b v(x,0)\,dx=-\int_0^t\int_a^b\frac{\partial}{\partial x}u(x,\tau)\,dx\,d\tau.\end{aligned} \tag{73}$$

Now we analyze the second formula and have

$$-\int_0^t\int_a^b u_x dx\,d\tau=-\int_0^t[u(b,\tau)-u(a,\tau)]\,d\tau. \tag{74}$$

For any fixed τ, $u(b,\tau)\to 0$ and $u(a,\tau)\to 0$, as $a\to-\infty$ and $b\to+\infty$,

$$-\int_a^b\int_0^t u_x d\tau\,dx\longrightarrow 0. \tag{75}$$

Thus

$$\int_a^b v(x,t)\,dx-\int_a^b v(x,0)\,dx\longrightarrow 0. \tag{76}$$

Similarly,

$$\int_a^b u(x,t)\,dx-\int_a^b u(x,0)\,dx\longrightarrow 0. \tag{77}$$

Hence $I(u)=\int_R udx$ and $I(v)=\int_R vdx$ exist and are equal to $\int_R u_0(x)dx$ and $\int_R v_0(x)dx$, respectively. This completes the proof of Lemma 14. □

The next theorem is the key step in the proof of instability, and it is the main result of this section.

Theorem 15. *Let $\Lambda^1 u_0 \in L^1$ and $v_0 \in L^1$, where $\Lambda^k = (1 - \partial^2/\partial x^2)^{k/2}$ $(k \in Z)$. Assume that $\vec{u} = \binom{u}{v}$ satisfies (4) and $u(0,x) = u_0$. Then*

$$\sup_{-\infty<x<\infty} \left| \int_{-\infty}^{x} \vec{u}(z,t)\, dz \right| \le c_0 \left(1 + t^{2/3} + t^{9/10}\right), \tag{78}$$

where the constant c_0 only depends on $\vec{u}_0$.

In order to prove Theorem 15, we need a series of lemmas. The first one is the well-known Van der Corput lemma [12]. The proofs of the following Lemmas 17 and 18 are similar to those which are given in [13], and we omit the details.

Lemma 16 (Van der Corput lemma). *Let $h(\xi)$ be either convex or concave on $[a,b]$ with $-\infty \le a < b \le \infty$. If $h''(\xi) \neq 0$ in $[a,b]$; then*

$$\left| \int_a^b e^{ih(\xi)} d\xi \right| \le 4 \left\{ \min_{[a,b]} \left| h''(\xi) \right| \right\}^{-1/2}. \tag{79}$$

Lemma 17. *Suppose $t > 0, n > 0$, one has*

$$\sup_{-\infty<\alpha<+\infty} \left| \int_{-n}^{n} e^{ith(\xi,\alpha)} d\xi \right| \le c_0 \left(t^{-1/3} + t^{-1/2} n^2\right), \tag{80}$$

where c_0 is a constant and $h(\xi,\alpha) = \xi/\sqrt{1+\xi^2} + \alpha\xi$.

Lemma 18. *For $1 \le p \le \infty$, if $u \in L^p(R)$, we have $\Lambda^{-1} u \in L^p$ and $\|\Lambda^{-1} u\|_p \le \|u\|_p$.*

The following lemma concerns the decay of the linear evolution operator.

Lemma 19. *Suppose that $S(t)$ the evolution operator of the linear equation*

$$\vec{u}_t + \begin{pmatrix} 0 & (1-\Delta)^{-1} \\ 1 & 0 \end{pmatrix} \vec{u}_x = 0, \qquad \vec{u}(0) = \vec{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{81}$$

That is to say, $S(t)\vec{u}(0) = \vec{u}(t)$. If $\Lambda^1 u_0 \in L^1$ and $v_0 \in L^1$, we have $S(t)\vec{u}_0 \in L^\infty$ and

$$\left\| S(t)\vec{u}_0 \right\|_\infty \le c_0 \left(t^{-1/3} + t^{-1/10}\right) \left(\left\|\Lambda^1 u_0\right\|_1 + \left\|v_0\right\|_1 \right), \tag{82}$$

where c_0 is a constant.

Proof. The solution of the linear equation is

$$\begin{aligned} \vec{u}(t) = S(t)\vec{u}_0(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \begin{pmatrix} \cos\left(\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) & \dfrac{1}{i\sqrt{1+\xi^2}} \sin\left(\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) \\ \dfrac{\sqrt{1+\xi^2}}{i} \sin\left(\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) & \cos\left(\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) \end{pmatrix} \cdot \widehat{\vec{u}}_0(\xi)\, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \begin{pmatrix} \cos\left(-\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) & \dfrac{i}{\sqrt{1+\xi^2}} \sin\left(-\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) \\ i\sqrt{1+\xi^2} \sin\left(-\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) & \cos\left(-\dfrac{\xi}{\sqrt{1+\xi^2}} t\right) \end{pmatrix} \cdot \widehat{\vec{u}}_0(\xi)\, d\xi, \end{aligned} \tag{83}$$

where $\widehat{\vec{u}}_0$ is the Fourier transform of $\vec{u}_0$.

According to Fubini's theorem and Lemmas 17 and 18, we have

$$\begin{aligned} &|\vec{u}(t)| \\ &= \left| S(t)\vec{u}_0(x) \right| \\ &\le \frac{1}{4\pi} \sum \left| \int_{-\infty}^{+\infty} \left(\widehat{u}_0 \pm \frac{1}{\sqrt{1+\xi^2}} \widehat{v}_0 \right) e^{it(-\xi/\sqrt{1+\xi^2} \pm (x/t)\xi)} d\xi \right| \\ &\quad + \frac{1}{4\pi} \sum \left| \int_{-\infty}^{+\infty} \left(\widehat{v}_0 \pm \sqrt{1+\xi^2}\, \widehat{u}_0 \right) e^{it(-\xi/\sqrt{1+\xi^2} \pm (x/t)\xi)} d\xi \right| \\ &\le \frac{1}{4\pi} \sum \left| \int_{-\infty}^{+\infty} \left(\widehat{u}_0 \pm \widehat{\Lambda^{-1} v_0} \right) e^{it(-\xi/\sqrt{1+\xi^2} \pm (x/t)\xi)} d\xi \right| \\ &\quad + \frac{1}{4\pi} \sum \left| \int_{-\infty}^{+\infty} \left(\widehat{v}_0 \pm \widehat{\Lambda^1 u_0} \right) e^{it(-\xi/\sqrt{1+\xi^2} \pm (x/t)\xi)} d\xi \right| \\ &\le \frac{1}{2\pi} \int_{|\xi|>n} \left(\left|\widehat{u}_0\right| + \left|\widehat{\Lambda^{-1} v_0}\right| + \left|\widehat{v}_0\right| + \left|\widehat{\Lambda^1 u_0}\right| \right) d\xi \\ &\quad + \frac{1}{4\pi} \sum \int_R \left| u_0(y) \pm \Lambda^{-1} v_0(y) \right| \\ &\qquad \times \left| \int_{-n}^{n} e^{it(-\xi/\sqrt{1+\xi^2} \pm (x/t)\xi - (y/t)\xi)} d\xi \right| dy \\ &\quad + \frac{1}{4\pi} \sum \int_R \left| v_0(y) \pm \Lambda^1 u_0(y) \right| \\ &\qquad \times \left| \int_{-n}^{n} e^{it(-\xi/\sqrt{1+\xi^2} \pm (x/t)\xi - (y/t)\xi)} d\xi \right| dy. \end{aligned} \tag{84}$$

Therefore,

$$
\begin{aligned}
|\vec{u}(t)| \le\; & c_0 \left(\left\| \Lambda^1 u_0 \right\|_{L^1} + \|v_0\|_{L^1} \right) \\
& \times \left(\int_{|\xi|>n} (1+|\xi|)^{-2} d\xi \right)^{1/2} \\
& + c_0 \left(\|\vec{u}_0\|_{L^1 \times L^1} + \left\| \Lambda^{-1} v_0 \right\|_{L^1} + \left\| \Lambda^1 u_0 \right\|_{L^1} \right) \\
& \times \sup_{-\infty<\alpha<\infty} \left| \int_{-\infty}^{\infty} e^{ith(\xi,\alpha)} d\xi \right| \\
\le\; & c_0 \left(n^{-1/2} + t^{-1/3} + t^{-1/2} n^2 \right) \left(\|v_0\|_{L^1} + \left\| \Lambda^1 u_0 \right\|_{L^1} \right).
\end{aligned} \tag{85}
$$

Choosing $n = t^{1/5}$, we have

$$
\left| S(t) \vec{u}_0 \right| \le c_0 \left(t^{-1/3} + t^{-1/10} \right) \left(\|v_0\|_{L^1} + \left\| \Lambda^1 u_0 \right\|_{L^1} \right), \tag{86}
$$

where $t > 0$. This completes the proof of Lemma 19. □

Proof of Theorem 15. Let $\vec{w}(t) = S(t)\vec{u}_0$. Then $\vec{w}(t)$ satisfies

$$
\vec{w}_t + \begin{pmatrix} 0 & \left(1 - \dfrac{\partial^2}{\partial x^2} \right)^{-1} \\ 1 & 0 \end{pmatrix} \vec{w}_x = 0, \quad \text{with } \vec{w}(0) = \vec{u}_0. \tag{87}
$$

The solution $\vec{u}(t)$ of the nonlinear (4) can be written as

$$
\vec{u}(t) = \vec{w}(t) + \frac{\partial}{\partial x} \int_0^t S(t-\tau) \begin{pmatrix} -\Lambda^{-2} \left(b_2 u^2 + b_3 u^3 \right) \\ 0 \end{pmatrix} d\tau. \tag{88}
$$

Let $\vec{U}(x,t) = \int_{-\infty}^x \vec{u}(y,t)dy$, $\vec{U}(x,0) = \int_{-\infty}^x \vec{u}(y,0)dy$, and $\vec{W}(x,t) = \int_{-\infty}^x \vec{w}(y,t)dy$. Then $\vec{U}(x,t) = \vec{W}(t) - \int_0^t S(t-\tau) \begin{pmatrix} \Lambda^{-2}(b_2u^2+b_3u^3) \\ 0 \end{pmatrix} d\tau$.

We estimate both two terms in the above formula on the right-hand side separately. Firstly, from the equation for $\vec{w}(t)$, we can obtain

$$
\vec{w}(t) = \vec{u}_0 - \partial_x \int_0^t \begin{pmatrix} 0 & (1-\Delta)^{-1} \\ 1 & 0 \end{pmatrix} \vec{w}(\tau) d\tau. \tag{89}
$$

Therefore

$$
\vec{W}(t) = \vec{U}_0 - \int_0^t \begin{pmatrix} 0 & (1-\Delta)^{-1} \\ 1 & 0 \end{pmatrix} \vec{w}(\tau) d\tau. \tag{90}
$$

Since $\vec{w}(t) = S(t)\vec{u}_0$, we obtain

$$
\begin{aligned}
& \left| \vec{W}(x,t) \right| \\
& \le |\vec{u}_0|_{L^1 \times L^1} + \left| \int_0^t \begin{pmatrix} 0 & (1-\Delta)^{-1} \\ 1 & 0 \end{pmatrix} S(\tau) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} d\tau \right| \\
& = |\vec{u}_0|_{L^1 \times L^1} + \left| \int_0^t S(\tau) \begin{pmatrix} (1-\Delta)^{-1} v_0 \\ u_0 \end{pmatrix} d\tau \right| \\
& \le |\vec{u}_0|_{L^1 \times L^1} + \int_0^t \left| S(\tau) \begin{pmatrix} (1-\Delta)^{-1} v_0 \\ u_0 \end{pmatrix} \right| d\tau.
\end{aligned} \tag{91}
$$

Using Lemma 19, substituting $\begin{pmatrix} (1-\Delta)^{-1} v_0 \\ u_0 \end{pmatrix}$ for $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, we have

$$
\begin{aligned}
\left| \vec{W}(x,t) \right| \le\; & |\vec{u}_0|_{L^1 \times L^1} \\
& + c_0 \int_0^t \left(\tau^{-1/3} + \tau^{-1/10} \right) d\tau \left(\left\| \Lambda^{-1} v_0 \right\|_{L^1} + \|u_0\|_{L^1} \right) \\
\le\; & c_0 \left(1 + t^{2/3} + t^{9/10} \right) \left(\|v_0\|_{L^1} + \|u_0\|_{L^1} \right).
\end{aligned} \tag{92}
$$

Let

$$
\vec{Y}(x,t) = \int_0^t S(t-\tau) \begin{pmatrix} \Lambda^{-2} \left(b_2 u^2 + b_3 u^3 \right) \\ 0 \end{pmatrix} d\tau. \tag{93}
$$

Using Lemma 19 again, and substituting $\begin{pmatrix} \Lambda^{-2}(b_2u^2+b_3u^3) \\ 0 \end{pmatrix}$ for $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, we obtain

$$
\begin{aligned}
& \left| \vec{Y}(x,t) \right| \\
& \le \int_0^t \left| S(t-\tau) \begin{pmatrix} \Lambda^{-2} \left(b_2 u^2 + b_3 u^3 \right) \\ 0 \end{pmatrix} \right| d\tau \\
& \le c_0 \int_0^t \left((t-\tau)^{-1/3} + (t-\tau)^{-1/10} \right) \left\| \Lambda^{-1} \left(b_2 u^2 + b_3 u^3 \right) \right\|_{L^1} d\tau.
\end{aligned} \tag{94}
$$

In view of Lemma 18, we have

$$
\left\| \Lambda^{-1} \left(b_2 u^2 + b_3 u^3 \right) \right\|_{L^1} \le \left\| b_2 u^2 + b_3 u^3 \right\|_{L^1}. \tag{95}
$$

Therefore

$$
\begin{aligned}
& \left| \vec{Y}(x,t) \right| \\
& \le c_0 \int_0^t \left((t-\tau)^{-1/3} + (t-\tau)^{-1/10} \right) \left(\left\| u^2 \right\|_{L^1} + \left\| u^3 \right\|_{L^1} \right) d\tau \\
& \le c_0 \left(t^{2/3} + t^{9/10} \right).
\end{aligned} \tag{96}
$$

Summarizing the estimate of $\vec{W}(x,t)$ and $\vec{Y}(x,t)$ above yields the result of Theorem 15; that is,

$$
\sup_{-\infty<x<\infty} \left| \int_{-\infty}^x \vec{u}(z,t) dz \right| \le c_0 \left(1 + t^{2/3} + t^{9/10} \right). \tag{97}
$$

□

5.2. Proof of Instability

Theorem 20. *Let $c \neq 0$ be fixed. If $d''(c) < 0$, then there is a curve $\omega \to \Phi_\omega$ such that $V(\vec{\Phi}_\omega) = V(\vec{\varphi}_c)$, $\vec{\Phi}_c = \vec{\varphi}_c$, and on which $E(\vec{u})$ has a strict local maximum at $\vec{u} = \vec{\varphi}_c$.*

Proof. Let χ_c be the unique negative eigenfunction of H_c, which has been proved in Section 2. Next we define

$$
\vec{\Phi}_\omega = \vec{\varphi}_\omega + s(\omega) \vec{\chi}_c, \quad \text{for } \omega \longrightarrow c, \tag{98}
$$

where $s(\omega)$ satisfies $s(c) = 0$ and $V(\vec{\Phi}_\omega) = V(\vec{\varphi}_c)$.

By the implicit function theorem, the function $s(\omega)$ can be determined. In fact,

$$\frac{\partial}{\partial s}V\left(\vec{\varphi}_\omega + s\vec{\chi}\right)\Big|_{\{s=0,\omega=c\}} = \left\langle V'\left(\vec{\varphi}_c\right), \vec{\chi}_c\right\rangle = \int_{-\infty}^{\infty}\left(\Lambda^2\varphi_c \cdot \chi_{1,c} + \psi_c \cdot \chi_{2,c}\right)dx, \tag{99}$$

where $\chi_c = \binom{\chi_{1,c}}{\chi_{2,c}}$ with $\chi_{1,c} = (c-\lambda)\chi_{2,c}$, and λ is the unique negative eigenvalue of H_c and $\varphi_c = c\psi_c$, $\varphi_c > 0$, $\chi_{1,c} > 0$, $\Lambda^2\varphi_c > 0$. Therefore

$$\begin{aligned}\frac{\partial}{\partial s}V\left(\vec{\varphi}_\omega + s\vec{\chi}\right)\Big|_{\{s=0,\omega=c\}} &= \int_{-\infty}^{+\infty}\left(\Lambda^2\varphi_c \cdot \chi_{1,c} + \psi_c \cdot \chi_{2,c}\right)dx \\ &= \int_{-\infty}^{+\infty}\left(\Lambda^2\varphi_c \cdot \chi_{1,c} + \frac{1}{c(c-\lambda)}\varphi_c \cdot \chi_{1,c}\right)dx \neq 0.\end{aligned} \tag{100}$$

It is easy to see that

$$\frac{d^2}{d\omega^2}E\left(\Phi_\omega\right)\Big|_{\omega=c} = \left\langle H_c\vec{y}, \vec{y}\right\rangle, \quad \text{where } \vec{y} = \frac{\partial\vec{\Phi}_\omega}{\partial\omega}\Big|_{\omega=c} = \frac{\partial\vec{\varphi}_c}{\partial c} + s'(c)\,\vec{\chi}_c. \tag{101}$$

So it suffices to show that $\langle H_c\vec{y}, \vec{y}\rangle < 0$. Since

$$\begin{aligned}0 = \frac{dV\left(\vec{\varphi}_c\right)}{d\omega} &= \frac{dV\left(\vec{\Phi}_\omega\right)\big|_{\omega=c}}{d\omega} \\ &= \left\langle V'\left(\vec{\varphi}_c\right), \vec{y}\right\rangle = \left\langle V'\left(\vec{\varphi}_\omega\right), \vec{y}\right\rangle\Big|_{\omega=c},\end{aligned} \tag{102}$$

then

$$\begin{aligned}d''(c) = \left\langle V'\left(\vec{\varphi}_c\right), \frac{d\vec{\varphi}_c}{dc}\right\rangle &= \left\langle V'\left(\vec{\varphi}_c\right), \vec{y} - s'(c)\,\vec{\chi}_c\right\rangle \\ &= \left\langle V'\left(\vec{\varphi}_c\right), \vec{y}\right\rangle - \left\langle V'\left(\vec{\varphi}_c\right), s'(c)\,\vec{\chi}_c\right\rangle \\ &= -s'(c)\left\langle V'\left(\vec{\varphi}_c\right), \vec{\chi}_c\right\rangle,\end{aligned} \tag{103}$$

$$\begin{aligned}H_c\vec{y} &= H_c\frac{\partial\vec{\varphi}_c}{\partial c} + H_c s'(c)\,\vec{\chi}_c \\ &= \left(E''\left(\vec{\varphi}_c\right) + cV''\left(\vec{\varphi}_c\right)\right)\frac{\partial\vec{\varphi}_c}{\partial c} + H_c s'(c)\,\vec{\chi}_c.\end{aligned}$$

Note that $E'(\vec{\varphi}_c) + cV'(\vec{\varphi}_c) = 0$. We derivate it with respect to c, and then

$$E''\left(\vec{\varphi}_c\right)\frac{\partial\vec{\varphi}_c}{\partial c} + V'\left(\vec{\varphi}_c\right) + cV''\left(\vec{\varphi}_c\right)\frac{\partial\vec{\varphi}_c}{\partial c} = 0. \tag{104}$$

Namely,

$$\left(E''\left(\vec{\varphi}_c\right) + cV''\left(\vec{\varphi}_c\right)\right)\frac{\partial\vec{\varphi}_c}{\partial c} + V'\left(\vec{\varphi}_c\right) = 0. \tag{105}$$

So

$$H_c\vec{y} = -V'\left(\vec{\varphi}_c\right) + s'(c)\,H_c\vec{\chi}_c,$$

$$\begin{aligned}\left\langle H_c\vec{y}, \vec{y}\right\rangle &= \left\langle -V'\left(\vec{\varphi}_c\right) + s'(c)\,H_c\vec{\chi}_c, \frac{\partial\vec{\varphi}_c}{\partial c} + s'(c)\,\vec{\chi}_c\right\rangle \\ &= \left\langle -V'\left(\vec{\varphi}_c\right), \frac{\partial\vec{\varphi}_c}{\partial c}\right\rangle + \left\langle -V'\left(\vec{\varphi}_c\right), s'(c)\,\vec{\chi}_c\right\rangle \\ &\quad + \left\langle s'(c)\,H_c\vec{\chi}_c, \frac{\partial\vec{\varphi}_c}{\partial c}\right\rangle + \left\langle s'(c)\,H_c\vec{\chi}_c, s'(c)\,\vec{\chi}_c\right\rangle \\ &= s'(c)\left\langle H_c\vec{\chi}_c, \frac{\partial\vec{\varphi}_c}{\partial c}\right\rangle + \left(s'(c)\right)^2\left\langle H_c\vec{\chi}_c, \vec{\chi}_c\right\rangle \\ &= s'(c)\left\langle H_c\frac{\partial\vec{\varphi}_c}{\partial c}, \vec{\chi}_c\right\rangle + \left(s'(c)\right)^2\left\langle H_c\vec{\chi}_c, \vec{\chi}_c\right\rangle \\ &= s'(c)\left\langle -V'\left(\vec{\varphi}_c\right), \vec{\chi}_c\right\rangle + \left(s'(c)\right)^2\left\langle H_c\vec{\chi}_c, \vec{\chi}_c\right\rangle \\ &= d''(c) + \left(s'(c)\right)^2\left\langle H_c\vec{\chi}_c, \vec{\chi}_c\right\rangle < 0.\end{aligned} \tag{106}$$

Hence, $(d^2/d\omega^2)E(\vec{\Phi}_\omega)|_{\omega=c} = \langle H_c\vec{y}, \vec{y}\rangle < 0$. The result in Theorem 20 holds. □

Lemma 21 (see [14]). *There exists $\varepsilon > 0$ and a unique C^1 map $\alpha : U_\varepsilon \to R$, such that for any $\vec{u} \in U_\varepsilon$ and $r \in R$,*

$$\begin{aligned}&(1)\ \left\langle \vec{u}\left(\cdot + \alpha\left(\vec{u}\right)\right), T'(0)\,\vec{\varphi}_c\right\rangle = 0, \\ &(2)\ \alpha\left(\vec{u}\left(\cdot + r\right)\right) = \alpha\left(\vec{u}\right) - r, \\ &(3)\ \alpha'\left(\vec{u}\right) = \frac{(\partial/\partial x)\,\vec{\varphi}_c\left(\cdot - \alpha\left(\vec{u}\right)\right)}{\left\langle \vec{u}, \left(\partial^2/\partial x^2\right)\vec{\varphi}_c\left(\cdot - \alpha\left(\vec{u}\right)\right)\right\rangle},\end{aligned} \tag{107}$$

where

$$U_\varepsilon = \left\{\vec{u} \in X : \inf_{s\in R}\left\|\vec{u} - \tau_s\vec{\varphi}_c\right\|_X < \varepsilon\right\}. \tag{108}$$

Next we define an auxiliary operator B which will play a critical role in the proof of instability.

Definition 22. For $\vec{u} \in U_\varepsilon$, $B(\vec{u})$ is defined by the formula

$$B\left(\vec{u}\right) = \vec{y}\left(\cdot - \alpha\left(\vec{u}\right)\right) - \left\langle I\vec{u}, \vec{y}\left(\cdot - \alpha\left(\vec{u}\right)\right)\right\rangle\frac{\partial}{\partial x}I^{-1}\alpha'\left(\vec{u}\right). \tag{109}$$

By Lemma 21, $B(\vec{u})$ can also be written as

$$\begin{aligned}B\left(\vec{u}\right) = &\ \vec{y}\left(\cdot - \alpha\left(\vec{u}\right)\right) \\ &- \frac{\left\langle I\vec{u}, \vec{y}\left(\cdot - \alpha\left(\vec{u}\right)\right)\right\rangle}{\left\langle \vec{u}, \left(\partial^2/\partial x^2\right)\vec{\varphi}_c\left(\cdot - \alpha\left(\vec{u}\right)\right)\right\rangle}I^{-1}\frac{\partial^2}{\partial x^2}\varphi_c\left(\cdot - \alpha\left(\vec{u}\right)\right),\end{aligned} \tag{110}$$

where $I = \begin{pmatrix}1-\Delta & 0\\ 0 & 1\end{pmatrix}$.

The next lemma summarizes the properties of B.

Lemma 23 (see [14]). *$B(\vec{u}) : U_\varepsilon \to X$ is a C^1 function. Moreover, B commutes with translations, $B(\vec{\varphi}_c) = \vec{y}$ and for any $\vec{u} \in U_\varepsilon$, $\langle B(\vec{u}), I\vec{u}\rangle = 0$.*

Lemma 24 (see [14]). *There exists a C^1 function*

$$\Pi : \{\vec{v} \in U_\varepsilon : V(\vec{v}) = V(\vec{\varphi}_c)\} \longrightarrow R, \tag{111}$$

which is invariant under translations, such that

$$E(\vec{\varphi}_c) < E(\vec{v}) + \Pi(\vec{v}) \left\langle E'(\vec{v}), B(\vec{v})\right\rangle, \tag{112}$$

for any $\vec{v} \in U_\varepsilon$ with $V(\vec{v}) = V(\vec{\varphi}_c)$ and $\vec{v}$ is not a translate of $\vec{\varphi}_c$.

Lemma 25 (see [14]). *According to Theorem 20, there is a curve $\omega \to \Phi_\omega$ which satisfies $E(\vec{\Phi}_\omega) < E(\vec{\varphi}_c)$ for $\omega \neq c$, $V(\vec{\Phi}_\omega) = V(\vec{\varphi}_c)$, and $\langle E'(\vec{\Phi}_\omega), B(\vec{\Phi}_\omega)\rangle$ changes sign as ω passes through c, with $c \neq 0$.*

Theorem 26. *If (4) has a bell-profile solitary wave solution $\vec{\phi}_c$, when $d''(c) < 0$, then $\vec{\phi}_c$ is orbitally unstable.*

Proof. Firstly, we consider $c \neq 0$. Let $\varepsilon > 0$, small enough, and U_ε be the tubular neighbourhood defined above. By Lemma 25 we can choose $\vec{u}_0 \in X$ which is arbitrarily close to $\vec{\varphi}_c$, such that $V(\vec{u}_0) = V(\vec{\varphi}_c)$, $E(\vec{u}_0) < E(\vec{\varphi}_c)$, and $\langle E'(\vec{u}_0), B(\vec{u}_0)\rangle > 0$. To prove the instability of $\vec{\varphi}_c$, it suffices to show that there are some elements $\vec{u}_0 \in X$ which are arbitrarily close to $\vec{\varphi}_c$, but the solution $\vec{u}(x,t)$ with the initial data $\vec{u}_0$ exits from U_ε in finite time. Let $[0, t_1)$ be the maximal interval for which $\vec{u}(x,t)$ lies continuously in U_ε, where $t_1 > 0$. Let T be the maximum existence time for the solution $\vec{u}(x,t)$ with initial data $\vec{u}_0$. If T is finite, it is easy to see that $\vec{\varphi}_c$ is orbital instability by definition, so we may assume that $T = +\infty$ and our purpose now is to show that $t_1 < +\infty$; that is to say, it is instability if it blows up at a finite time. The proof is as follows.

Firstly, in view of Lemmas 4, 13, and 14 and Theorem 15, we know that $\vec{u}$ enjoys the following properties:

$$\begin{gathered}\vec{u} \in C([0,t_1); X), \quad \vec{u}(0,x) = \vec{u}_0(x),\\ E(\vec{u}(t)), V(\vec{u}(t)) \text{ are constant}, \quad \text{for } t \in [0,t_1),\\ I(u(t)), I(v(t)) \text{ converge and are constant},\\ \text{for } t \in [0,t_1),\\ \sup_{0\le t<t_1} \|\vec{u}(x,t)\|_X \le c_1,\\ \sup_{x\in R}\left|\int_{-\infty}^{x} \vec{u}(y,t)\,dy\right| \le c_2\left(1 + t^{2/3} + t^{9/10}\right),\end{gathered} \tag{113}$$

where c_1 depends on $\vec{\varphi}_c$ and ε, c_2 depends on c_1, $\|v_0\|_{L^1}$, and $\|\vec{u}_0\|_{L^1}$.

Let $\beta(t) = \alpha(\vec{u}(t))$, where α is defined by Lemma 21 and define

$$\begin{gathered}\vec{y}(z) = \frac{d\vec{\Phi}_c}{dc} = \frac{d\vec{\varphi}_c}{dc} + s'(c)\,\vec{\chi}_c,\\ \vec{Y}(x) = \int_{-\infty}^{x} I\vec{y}(z)\,dz,\\ A(t) = \int_{-\infty}^{\infty} \vec{Y}(x - \beta(t)) \cdot \vec{u}(x,t)\,dx, \quad 0 \le t < t_1,\end{gathered} \tag{114}$$

where the function $A(t)$ serves as a Lyapunov function, and

$$\vec{\gamma} = \int_{-\infty}^{\infty} \vec{y}(x)\,dx. \tag{115}$$

Due to the assumptions above, it is observed that

$$\begin{gathered}\int_{-\infty}^{\infty} (1 + |x|)^{1/2} \frac{\partial\vec{\varphi}_c}{\partial c}dx < \infty,\\ \int_{-\infty}^{\infty} (1 + |x|)^{1/2} \vec{\chi}_c dx < \infty.\end{gathered} \tag{116}$$

Therefore $\int_{-\infty}^{\infty} (1 + |x|)^{1/2} |\vec{y}(x)| dx < \infty$, such that $|\vec{\gamma}| < +\infty$.

Indeed, if H is the Heaviside function and $\vec{W}(x) = \int_{-\infty}^{x} \vec{y}(z)dz - \vec{\gamma}H(x)$, then

$$\begin{gathered}\vec{W}(x) = \int_{-\infty}^{x} \vec{y}(z)\,dz \quad \text{for } x < 0,\\ \vec{W}(x) = -\int_{x}^{\infty} \vec{y}(z)\,dz \quad \text{for } x \ge 0.\end{gathered} \tag{117}$$

Now we can have

$$\begin{aligned}A(t) &= \int_{-\infty}^{\infty} \left(\vec{Y}(x - \beta(t)) - \vec{\gamma}H(x - \beta(t))\right) \cdot \vec{u}(x,t)\,dx\\ &\quad + \vec{\gamma}\int_{-\infty}^{\infty} \vec{H}(x - \beta(t)) \cdot \vec{u}(x,t)\,dx\\ &= \int_{-\infty}^{\infty} \left(\vec{Y}(x - \beta(t)) - \vec{\gamma}H(x - \beta(t))\right) \cdot \vec{u}(x,t)\,dx\\ &\quad + \vec{\gamma}\int_{\beta(t)}^{\infty} \vec{u}(x,t)\,dx.\end{aligned} \tag{118}$$

Hence, by (113),

$$|A(t)| \le \left\|\vec{Y} - \vec{\gamma}H\right\|_0 \|\vec{u}(t)\|_0 + c_2\left(1 + t^{2/3} + t^{9/10}\right). \tag{119}$$

It follows from Minkowski's inequality that

$$\begin{aligned}
&\left(\int_{-\infty}^{0}\left|\vec{Y}(x)-\vec{\gamma}H(x)\right|^{2}dx\right)^{1/2} \\
&\quad=\left(\int_{-\infty}^{0}\left|\vec{Y}(x)\right|^{2}dx\right)^{1/2} \\
&\quad\leq\int_{-\infty}^{0}\left(\int_{z}^{0}\left|I\vec{y}(z)\right|^{2}dx\right)^{1/2}dz \\
&\quad=\int_{-\infty}^{0}\left(\int_{z}^{0}\left|\vec{y}(z)\right|^{2}dx\right)^{1/2}dz \\
&\quad=\int_{-\infty}^{0}\left|\vec{y}(z)\right|\sqrt{-z}dz<\infty.
\end{aligned} \tag{120}$$

Similarly, $(\int_{0}^{\infty}|\vec{Y}-\vec{\gamma}H(x)|^{2}dx)^{1/2}<\infty$. Therefore

$$|A(t)|\leq c_2\left(1+t^{2/3}+t^{9/10}\right),\quad \text{for } 0\leq t<t_1. \tag{121}$$

Since $A(t)=\langle\vec{Y}(x-\beta(t)),\vec{u}(x,t)\rangle$, and $\beta(t)=\alpha(\vec{u}(t))$, now we estimate $dA(t)/dt$, by calculating, and have

$$\begin{aligned}
&\frac{dA(t)}{dt} \\
&=\left\langle\frac{d\vec{Y}(x-\beta(t))}{dt},\vec{u}(x,t)\right\rangle+\left\langle\vec{Y}(x-\beta(t)),\frac{\partial}{\partial t}\vec{u}(x,t)\right\rangle \\
&=\left\langle-\langle I\vec{y}(x-\beta),\vec{u}\rangle\alpha'(\vec{u})+\vec{Y}(x-\beta),\frac{\partial}{\partial t}\vec{u}(x,t)\right\rangle.
\end{aligned} \tag{122}$$

Since $d\vec{u}/dt=JE'(\vec{u})$, where $J=(\partial/\partial x)I^{-1}$ and $(\partial/\partial x)\vec{Y}=I\vec{y}$, it follows that

$$\begin{aligned}
\frac{dA}{dt}&=\Big\langle\langle I\vec{y}(x-\beta(t)),\vec{u}\rangle\frac{\partial}{\partial x}\alpha'(\vec{u}) \\
&\qquad -I\vec{y}(x-\beta(t)),I^{-1}E'(\vec{u})\Big\rangle \\
&=\Big\langle\langle I\vec{y}(x-\beta(t)),\vec{u}\rangle\frac{\partial}{\partial x}I^{-1}\alpha'(\vec{u}) \\
&\qquad -\vec{y}(x-\beta(t)),E'(\vec{u})\Big\rangle \\
&=-\left\langle B(\vec{u}),E'(\vec{u})\right\rangle.
\end{aligned} \tag{123}$$

Since $0<E(\vec{\varphi}_0)-E(\vec{u}_0)=E(\vec{\varphi}_c)-E(\vec{u}(t))$, from Lemma 24, we can deduce that

$$0<\Pi(\vec{u}(t))\left\langle B(\vec{u}(t)),E'(\vec{u}(t))\right\rangle. \tag{124}$$

Since $\langle E'(\vec{u}_0),B(\vec{u}_0)\rangle>0$ and it is continuous, we can obtain $\langle E'(\vec{u}(t)),B(\vec{u}(t))\rangle>0$. Therefore for all $0<t<t_1$, $\Pi(\vec{u}(t))>0$. Moreover, since $\vec{u}(t)\in U_\varepsilon$ and $\Pi(\vec{\varphi}_c)=0$, we may assume that $0<\Pi(\vec{u}(t))<1$, $0<t<t_1$, by choosing ε smaller if necessary. So for all $t\in[0,t_1)$, by Lemma 24, we have

$$\begin{aligned}
&\left\langle B(\vec{u}(t)),E'(\vec{u})\right\rangle \\
&\quad\geq\Pi(\vec{u}(t))\left\langle B(\vec{u}(t)),E'(\vec{u}(t))\right\rangle \\
&\quad>E(\vec{\varphi}_c)-E(\vec{u}(t)) \\
&\quad=E(\vec{\varphi}_c)-E(\vec{u}_0)=\varepsilon_0>0,
\end{aligned} \tag{125}$$

$$-\frac{dA}{dt}\geq E(\vec{\varphi}_c)-E(\vec{u}_0)=\varepsilon_0>0,\quad \text{for } t\in[0,t_1). \tag{126}$$

Integrating (126) on $[0,t_1)$, we have

$$A(0)-A(t_1)\geq\varepsilon_0 t_1=\left[E(\vec{\varphi}_c)-E(\vec{u}_0)\right]t_1. \tag{127}$$

And then

$$t_1\leq\frac{A(0)-A(t_1)}{E(\vec{\varphi}_c)-E(\vec{u}_0)}\leq\frac{2|A(t)|}{E(\vec{\varphi}_c)-E(\vec{u}_0)}. \tag{128}$$

Since $|A(t)|\leq c_2(1+t^{2/3}+t^{9/10})$, we can conclude that $t_1<\infty$. This completes the proof of Theorem 26. □

6. Conclusions

In this paper, we studied the orbital stability and instability of solitary waves for (4) with two nonlinear terms. By using the orbital stability theory proposed in [10, 11], we obtained a general theorem judging the orbital stability for solitary waves of (4) in Section 3 based on proof of the local existence of the solutions, existence of the bounded state solution, and the spectral analysis of operator H_c. In Section 4, we gave the explicit expressions for the discrimination $d_i''(c)$, $i=1,2$, of orbital stability in terms of the two exact solitary waves $(\varphi_i,\psi_i)^T$, $i=1,2$, of (4). Furthermore, we deduced Theorems 9 and 10 which could easily judge the orbital stability of the two solitary waves $(\varphi_i,\psi_i)^T$, $i=1,2$, and analyzed the influence of the two nonlinear terms on the orbital stability. Finally, we studied instability in Section 5. We defined a new conservational functional and estimated to the solution of initial value problem to overcome the difficulty that we could not apply Grillakis-Shatah-Strauss theory on the system directly since J is not onto. We constructed a formal Lyapunov function and proved Theorem 26.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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