Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 958931, 5 pages
http://dx.doi.org/10.1155/2014/958931

Research Article

Solving Fractional Partial Differential Equations with Corrected Fourier Series Method

Nor Hafizah Zainal[1] and Adem Kılıçman[1,2]

[1] *Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[2] *Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to Adem Kılıçman; akilic@upm.edu.my

Received 31 March 2014; Accepted 5 May 2014; Published 26 May 2014

Academic Editor: Hassan Eltayeb

Copyright © 2014 N. H. Zainal and A. Kılıçman. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The corrected Fourier series (CFS) is proposed for solving partial differential equations (PDEs) with fractional time derivative on a finite domain. In the previous work, we have been solving partial differential equations by using corrected Fourier series. The fractional derivatives are described in Riemann sense. Some numerical examples are presented to show the solutions.

1. Introduction

In recent years, differential equations of fractional orders have been appearing more and more frequently in various research and applications in the fluid mechanics, viscoelasticity, biology, physics, and engineering; see [1, 2]. There are some methods usually used in solving the fractional partial differential equations such as Laplace and Fourier transform, variational iteration method, and differential transform methods. In this study, we want to use the corrected Fourier series method in solving the problems.

In [3], corrected Fourier series method has been used in solving classical PDEs problems. The corrected Fourier series is a combination of the uniformly convergent Fourier series and the correction functions and consists of algebraic polynomials and Heaviside step function.

2. Basic Definitions

The Riemann-Liouville fractional integral is the most popular definition that we always find in the study of fractional calculus.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x)$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\, ds. \tag{1}$$

Jumarie's modified Riemann-Liouville derivative of order α is defined by the following definition.

Definition 2. Let $f : \mathfrak{R} \to \mathfrak{R}$, $x \to f(x)$ denote a continuous function. Its fractional derivative of order α is defined as follows:

$$\text{for } \alpha < 0,$$
$$D_x^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-s)^{-\alpha-1} \left(f(s) - f(0)\right) ds, \tag{2}$$
$$\text{for } \alpha > 0,$$
$$D_x^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-s)^{-\alpha} \left(f(s) - f(0)\right) ds,$$
$$\text{where } 0 < \alpha < 1,$$
$${}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\alpha-1} \left[f(s) - f(0)\right] ds, \tag{3}$$

where $n-1 < \alpha < n$ with $n \in N$.

Definition 3 (see [4–7]). Fractional derivative of compounded function is defined as

$$d^{\alpha} f \cong \Gamma(1+\alpha)\, df, \quad 0 < \alpha < 1. \tag{4}$$

Table 1

t	x	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$	
					CFS	Exact
0.2	0.25	0.248566	0.351241	0.295434	0.248567	0.262584
	0.50	0.542423	0.602263	0.548652	0.542423	0.550334
	0.75	0.832454	0.852586	0.801804	0.832454	0.863252
	1.00	1.115475	1.102422	1.054999	1.115475	1.201334

Table 2

α	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$	
x	FCFS	VIM	FCFS	VIM	FCFS	VIM
0.25	0.248566	0.263175	0.351241	0.262840	0.295434	0.262670
0.50	0.542423	0.552700	0.602263	0.551362	0.548652	0.550680
0.75	0.832454	0.868575	0.852586	0.865566	0.801804	0.864029
1.00	1.115475	1.210800	1.102422	1.205450	1.054999	1.202718

Definition 4 (see [4–7]). The integral with respect to $(dt)^{\alpha}$ is defined as the solution of the fractional differential equation

$$dx = f(x)(dt)^{\alpha}, \quad t \geq 0, \; x(0) = 0, \; 0 < \alpha < 1. \tag{5}$$

Lemma 5 (see [4–7]). *Let $f(x)$ denote a continuous function; then the solution of $y(x)$, $y(0) = 0$, (5) is defined as*

$$y = \int_0^x f(\xi)(d\xi)^{\alpha} = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi, \quad 0 < \alpha < 1. \tag{6}$$

Table 3

x	$\alpha = 2$	
	VIM	CFS
0.25	0.262584	0.245396
0.50	0.550334	0.516298
0.75	0.863252	0.787390
1.00	1.201334	1.058374

3. Corrected Fourier Series

The CFS is described in the form of

$$\begin{aligned} u(x,t) &= \sum_{|n|<\infty} \sum_{|m|<\infty} A_{nm} e^{i\alpha_n x} e^{i\beta_m t} \\ &+ \sum_{|m|<\infty} \left(a_{1m} x + a_{2m} \frac{x^2}{2!} + a_{3m} \frac{x^3}{3!} \right) e^{i\beta_m t} \\ &+ \sum_{|n|<\infty} \left(b_{1n} t + b_{2n} \frac{t^2}{2!} + b_{3n} \frac{t^3}{3!} \right) e^{i\alpha_n x} \\ &+ \sum_{l=1}^{3} \sum_{l_0=1}^{3} d_{ll_0} \frac{x^l}{l!} \frac{t^{l_0}}{l_0!}, \end{aligned} \tag{7}$$

where $\alpha_n = 2n\pi/x_0$ and $\beta_m = 2m\pi/t_0$.

Due to the periodicity of either $e^{i\alpha_n x}$ or $e^{i\beta_m t}$, we can cancel out the first three terms on the right-hand side of (7) because they are identically zero. Based on the endpoints values of $u(x,t)$ and its partial derivative, we obtain the following linear equations:

$$\begin{aligned} &\sum_{l=1}^{3} \sum_{l_0=1}^{3} d_{ll_0} \frac{x^{l-j}}{(l-j)!} \frac{t^{l_0-j_0}}{(l_0-j_0)!} H(l-j) H(l_0-j_0) \Big|_0^{x_0} \Big|_0^{t_0} \\ &\quad = \frac{\partial^{j+j_0} u(x,t)}{\partial x^j \partial t^{j_0}} \Big|_0^{x_0} \Big|_0^{t_0}, \end{aligned} \tag{8}$$

where $j = 0, 1, 2$ and $j_0 = 0, 1, 2$.

Next, we want to determine the coefficients a_{1m}, a_{2m}, and a_{3m}. With respect to x, the endpoints effect of $u(x,t)$ and its partial derivatives yields

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}\Big|_0^{x_0} &= \sum_{|m|\leq M} a_{3m} x_0 e^{i\beta_m t} + \sum_{l_0=1}^{3} d_{3l_0} x_0 \frac{t^{l_0}}{l_0!}, \\ \frac{\partial u}{\partial x}\Big|_0^{x_0} &= \sum_{|m|\leq M} \left(a_{2m} x_0 + a_{3m} \frac{x_0^2}{2!} \right) e^{i\beta_m t} \\ &\quad + \sum_{l=2}^{3} \sum_{l_0=1}^{3} d_{ll_0} \frac{x_0^{l-1}}{(l-1)!} \frac{t^{l_0}}{l_0!}, \\ u\big|_0^{x_0} &= \sum_{|m|\leq M} \left(a_{1m} x_0 + a_{2m} \frac{x_0^2}{2!} + a_{3m} \frac{x_0^3}{3!} \right) e^{i\beta_m t} \\ &\quad + \sum_{l=1}^{3} \sum_{l_0=1}^{3} d_{ll_0} \frac{x_0^l}{l!} \frac{t^{l_0}}{l_0!}. \end{aligned} \tag{9}$$

Again, due to the periodicity of $e^{i\alpha_n x}$, the first and third terms of (7) and its partial derivatives are identically zero. Then, by

Table 4

α	0.25		0.5		0.75	
x	FCFS	VIM	FCFS	VIM	FCFS	VIM
0.25	0.184853	0.155391	0.180014	0.155803	0.180899	0.179254
0.50	0.290771	0.211629	0.280927	0.301919	0.283769	0.347363
0.75	0.382190	0.300891	0.369521	0.429263	0.374601	0.493874
1.00	0.521740	0.371445	0.506676	0.529918	0.512320	0.609679

applying the Fourier projection on the basis function $e^{i\beta_m t}$, we solved for a_{1m}, a_{2m}, and a_{3m}:

$$\begin{aligned}
a_{3m} &= \frac{1}{x_0}\left(F_2\left\langle \left.\frac{\partial^2 u}{\partial x^2}\right|_0^{x_0}\right\rangle_m - \sum_{l_0=1}^{3} d_{3l_0} x_0 F_2\left\langle \frac{t^{l_0}}{l_0!}\right\rangle_m\right),\\
a_{2m} &= \frac{1}{x_0}\\
&\quad\times\left(F_2\left\langle \left.\frac{\partial u}{\partial x}\right|_0^{x_0}\right\rangle_m - \sum_{l=2}^{3}\sum_{l_0=1}^{3} d_{ll_0}\frac{x_0^{l-1}}{(l-1)!}F_2\left\langle \frac{t^{l_0}}{l_0!}\right\rangle_m\right.\\
&\qquad\left. - a_{3m}\frac{x_0^2}{2!}\right),\\
a_{1m} &= \frac{1}{x_0}\\
&\quad\times\left(F_2\left\langle u|_0^{x_0}\right\rangle_m - \sum_{l=1}^{3}\sum_{l_0=1}^{3} d_{ll_0}\frac{x_0^{l}}{l!}F_2\left\langle \frac{t^{l_0}}{l_0!}\right\rangle_m\right.\\
&\qquad\left. - a_{2m}\frac{x_0^2}{2!} - a_{3m}\frac{x_0^3}{3!}\right).
\end{aligned} \tag{10}$$

Similar in the case with respect to t, we have

$$\begin{aligned}
b_{3n} &= \frac{1}{t_0}\left(F_1\left\langle \left.\frac{\partial^2 u}{\partial t^2}\right|_0^{t_0}\right\rangle_n - \sum_{l=1}^{3} d_{l3} F_1\left\langle \frac{x^{l}}{l!}\right\rangle_n t_0\right),\\
b_{2n} &= \frac{1}{t_0}\\
&\quad\times\left(F_1\left\langle \left.\frac{\partial u}{\partial t}\right|_0^{t_0}\right\rangle_n - \sum_{l=1}^{3}\sum_{l_0=2}^{3} d_{ll_0} F_1\left\langle \frac{x^{l}}{l!}\right\rangle_n\right.\\
&\qquad\left.\times\frac{t_0^{l_0-1}}{(l_0-1)!} - b_{3n}\frac{t_0^2}{2!}\right),\\
b_{1n} &= \frac{1}{t_0}\\
&\quad\times\left(F_1\left\langle u|_0^{t_0}\right\rangle_n - \sum_{l=1}^{3}\sum_{l_0=1}^{3} d_{ll_0} F_1\left\langle \frac{x^{l}}{l!}\right\rangle_n \frac{t_0^{l_0}}{l_0!}\right.\\
&\qquad\left. - b_{2n}\frac{t_0^2}{2!} - b_{3n}\frac{t_0^3}{3!}\right).
\end{aligned} \tag{11}$$

Table 5

t	x	$\alpha = 1$	
		VIM	CFS
0.2	0.25	0.202557	0.245396
	0.50	0.392520	0.516298
	0.75	0.558079	0.787390
	1.00	0.688938	1.058374

4. Fractional Corrected Fourier Series

In this paper, we consider the following general form of the linear time-fractional equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a_0(x)\,u(x,t) + a_1(x)\frac{\partial u}{\partial x} + a_2(x)\frac{\partial^2 u}{\partial x^2} = f(x,t) \tag{12}$$

and subject to the initial conditions

$$\begin{gathered}
u(x,0) = f(x),\\
\text{as } 0<\alpha\le 1,\quad x\in[0,x_0],\quad t\in[0,t_0],\\
u(x,0) = f(x),\quad \frac{\partial u(x,0)}{\partial t} = g(x),\\
\text{as } 1<\alpha\le 2,\quad x\in[0,x_0],\quad t\in[0,t_0].
\end{gathered} \tag{13}$$

For the case of $\alpha = 1$, the fractional equation reduces to the classical linear PDE and is similar to the case of $\alpha = 2$.

Table 6

t	x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 7.5$	$\alpha = 1$ FCFS	$\alpha = 1$ exact
1	0.25	0.077299	0.140985	0.136142	0.173193	0.169892
	0.50	0.699350	0.697982	0.597600	0.690081	0.679570
	0.75	1.350993	1.305838	1.256308	1.618244	1.529034
	1.00	2.157733	2.196163	2.187799	2.692081	2.718282

Table 7

t	x	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
		FCFS	Homotopy	FCFS	Homotopy	FCFS	Homotopy
1	0.25	0.077299	0.131454	0.140985	0.133024	0.136142	0.130504
	0.50	0.699350	0.525816	0.697982	0.532095	0.597600	0.522016
	0.75	1.350993	1.183085	1.305838	1.197213	1.256308	1.174540
	1.00	2.157733	2.103263	2.196163	2.128379	2.187799	2.088065

Definition 6. For m to be the smallest integer that exceeds α, the modified Riemann-Liouville time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\partial^{m}}{\partial t^{m}} \times \displaystyle\int_{0}^{t} (t-\tau)^{m-\alpha-1} (u(x,\tau) - u(x,0))\, d\tau, & \\ \qquad\qquad \text{for } m-1 < \alpha < m, & \\ \dfrac{\partial^{m} u(x,t)}{\partial t^{m}}, \quad \text{for } \alpha = m \in N, & \end{cases} \tag{14}$$

where $m - 1 < \alpha < m$ with $m \in N$.

To obtain the nine unknowns d_{ll_0} $(l, l_0 = 1,2,3)$ in (7), we solve the following linear equations:

$$\sum_{l=1}^{3} \sum_{l_0=1}^{3} d_{ll_0} \lambda_{l_0} \frac{x^{l-j}}{(l-j)!} t^{1+l_0-\alpha} H(l-j) H(l_0 - j_0) \Bigg|_{0}^{x_0} \Bigg|_{0}^{t_0} = \frac{\partial^{j+(j_0+\alpha)} u(x,t)}{\partial x^{j} \partial t^{j_0+\alpha}} \Bigg|_{0}^{x_0} \Bigg|_{0}^{t_0}, \tag{15}$$

where $j = 0,1,2$, $j_0 = 0,1,2$, and $\lambda_{l_0} = ((-1)^{2l_0})(\Gamma(\alpha - (l_0 + 2)) \cdot ((l_0+2) - \alpha)/\Gamma(1-\alpha)\Gamma(\alpha-1))$.

Then, we arrange the order of equations as $j = 0$, $j_0 = 0,1,2$; $j = 1$, $j_0 = 0,1,2$; and $j = 2$, $j_0 = 0,1,2$; we can have the nine unknowns $d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}, d_{31}, d_{32}$, and d_{33} in a vector form. Then, we solve this coefficient in the matrix form. By solving the matrix, we can determine the coefficients d_{ll_0} $(l = l_0 = 1,2,3)$.

Table 8

t	x	$\alpha = 1$ FCFS	$\alpha = 1$ Homotopy	$\alpha = 1$ CFS
1	0.25	0.173193	0.166667	0.170034
	0.50	0.690081	0.666667	0.612833
	0.75	1.618244	1.500000	1.592353
	1.00	2.692081	2.666667	2.682901

5. Numerical Results

Problem 7 (see [2]). We consider the linear time-fractional wave equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{2} x^{2} \frac{\partial^{2} u(x,t)}{\partial x^{2}}, \quad 1 < \alpha \leq 2, \; t > 0$$
$$u(x,0) = x, \qquad \frac{\partial u(x,0)}{\partial t} = x^{2}. \tag{16}$$

The exact solution for the case $\alpha = 2$ is given by $u(x,t) = x + x^{2} \sinh t$ (see Table 1).

For comparison between fractional corrected Fourier series (FCFS) and variational iteration method (VIM) where we take $t = 0.2$. (See Tables 2 and 3.)

To be cleared here, for $\alpha = 2$, we solve it by using original corrected Fourier series.

Problem 8 (see [2]). Consider

$$\frac{\partial^{\alpha} u}{\partial t} = \frac{\partial^{2} u}{\partial x^{2}}, \quad 0 < t < 1, \; 0 < x < 1, \; 0 < \alpha < 1, \tag{17}$$

where $u(x,0) = \sin(x)$. For $t = 0.2$, see Tables 4 and 5.

Problem 9 (see [8]). Let us first consider the following time-fractional differential equation as follows:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{2} x^{2} \frac{\partial^{2} u(x,t)}{\partial x^{2}}, \quad 0 < \alpha \leq 1,\ t > 0$$
$$u(x,0) = x^{2}, \qquad u(0,t) = 0, \tag{18}$$
$$u(1,t) = e^{t}.$$

The exact solution for the case $\alpha = 1$ is given by $u(x,t) = x^2 e^t$ (see Table 6).

For comparison between corrected Fourier series and other methods see Tables 7 and 8.

6. Conclusion

In this paper, with the presence of the modified Riemann-Liouville derivative, the corrected Fourier series has been proposed to solve the fractional partial differential problems. The solutions of problems are shown for different values in the given Tables 1–8. It is shown that there is smaller error in between CFS method and other methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors gratefully acknowledge that this research was partially supported by the University Putra Malaysia under the ERGS Grant Scheme having Project no. 5527068.

References

[1] N. Faraz, Y. Khan, H. Jafari, A. Yildirim, and M. Madani, "Fractional variational iteration method via modified Riemann-Liouville derivative," *Journal of King Saud University-Science*, vol. 23, pp. 413–417, 2011.

[2] Z. Odibat and S. Momani, "Numerical methods for nonlinear partial differential equations of fractional orders," *Applied Mathematical Modelling*, vol. 32, pp. 28–39, 2008.

[3] N. H. Zainal and A. Kilicman, "Comparison between Fourier and corrected Fourier series methods," *Malaysian Journal of Mathematical Sciences*, vol. 7, no. 2, pp. 273–282, 2013.

[4] J. Guy, "Lagrange characteristic method for solving a class of nonlinear partial differential equations of fractional order," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 873–880, 2006.

[5] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," *An International Journal Computers and Mathematics with Applications*, vol. 51, no. 9-10, pp. 1367–1376, 2006.

[6] G. Jumarie, "Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions," *Applied Mathematics Letters*, vol. 22, no. 3, pp. 378–385, 2009.

[7] G. Jumarie, "Laplace's transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1659–1664, 2009.

[8] H. Xu, S.-J. Liao, and X.-C. You, "Analysis of nonlinear fractional partial differential equations with the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1152–1156, 2009.