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Research Article

Combination-Combination Synchronization of Four Nonlinear Complex Chaotic Systems

Xiaobing Zhou,[1] Lianglin Xiong,[2] and Xiaomei Cai[3]

[1] *School of Information Science and Engineering, Yunnan University, Kunming 650091, China*
[2] *School of Mathematics and Computer Science, Yunnan University of Nationalities, Kunming 650031, China*
[3] *Bureau of Asset Management, Yunnan University, Kunming 650091, China*

Correspondence should be addressed to Xiaobing Zhou; zhouxb.cn@gmail.com

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This paper investigates the combination-combination synchronization of four nonlinear complex chaotic systems. Based on the Lyapunov stability theory, corresponding controllers to achieve combination-combination synchronization among four different nonlinear complex chaotic systems are derived. The special cases, such as combination synchronization and projective synchronization, are studied as well. Numerical simulations are given to illustrate the theoretical analysis.

1. Introduction

In 1982, Fowler et al. [1] generalized the real Lorenz model to a complex Lorenz model, which can be used to describe and simulate the physics of a detuned laser and the thermal convection of liquid flows [2, 3]. After that, many new chaotic and hyperchaotic complex systems have been reported and intensively studied, including the complex Van der Pol oscillators [4], the complex Chen and complex Lü systems [5], complex detuned laser system [6], complex hyperchaotic Lorenz system [7], complex modified hyperchaotic Lü system [8], and a novel hyperchaotic complex-variable system [9] which generates 2-, 3-, and 4-scroll attractors.

Since Pecora and Carroll [10] first proposed the drive-response concept for constructing synchronization of coupled chaotic systems, synchronization in chaotic systems has been extensively investigated due to their potential applications in the fields of secure communications; optical, chemical, physical, and biological systems; neural networks; and so forth [11–13]. When applying the complex systems in communications, the complex variables will double the number of variables and can increase the content and security of the transmitted information. Based on the Lyapunov stability theory, linear feedback controller was derived to achieve hybrid projective synchronization in a chaotic complex nonlinear system [14]. The authors [15] achieved adaptive antisynchronization of a class of chaotic complex nonlinear systems described by a united mathematical expression with fully uncertain parameters. In [16], the author investigated the modified projective phase synchronization of chaotic complex nonlinear systems. Based on the passive theory, the authors studied the projective synchronization of hyperchaotic complex nonlinear systems and its application in secure communications [17]. In [18], the authors achieved fast synchronization of a novel hyperchaotic complex system based on finite-time stability theory.

However, most of the existing synchronization schemes are based on the usual drive-response synchronization mode, which has one drive system and one response system. In [19], Luo et al. proposed the combination synchronization scheme, which has two drive systems and one response system. Zhou et al. investigated combination synchronization of three nonlinear complex hyperchaotic systems in [20]. Sun et al. [21] extended the combination synchronization scheme to the combination-combination synchronization scheme, where synchronization is achieved between two drive systems and two response systems. This synchronization scheme has advantages over the other synchronization schemes, such that it can provide greater security in secure communication.

For the nonlinear complex chaotic or hyperchaotic systems, there are no work on combination-combination synchronization for them. This paper aims to study the combination-combination synchronization of four nonlinear complex chaotic systems. The rest of this paper is organized as follows. Section 2 introduces the scheme of combination-combination synchronization. In Section 3, we investigate combination-combination synchronization of four complex nonlinear chaotic systems. Numerical simulations are conducted in Section 4. Finally, conclusions are given in Section 5.

2. The Scheme of Combination-Combination Synchronization

In the scheme of combination-combination synchronization, there are four nonlinear dynamical systems, two drive systems, and two response system.

The two drive systems are, respectively, given by

$$\dot{x}_1 = f_1(x_1), \quad (1)$$

$$\dot{x}_2 = f_2(x_2). \quad (2)$$

The two response systems are, respectively, described by

$$\dot{y}_1 = g_1(y_1) + \varphi, \quad (3)$$

$$\dot{y}_2 = g_2(y_2) + \varphi^*, \quad (4)$$

where $x_1 = (x_{11}, x_{12}, \ldots, x_{1n})^T$, $x_2 = (x_{21}, x_{22}, \ldots, x_{2n})^T$, $y_1 = (y_{11}, y_{12}, \ldots, y_{1n})^T$, and $y_2 = (y_{21}, y_{22}, \ldots, y_{2n})^T$ are the state vectors of the systems (1), (2), (3), and (4), respectively; $f_1(\cdot), f_2(\cdot), g_1(\cdot), g_2(\cdot) : R^n \to R^n$ are four continuous vector functions and $\varphi, \varphi^* : R^n \times R^n \times R^n \times R^n \to R^n$ are two controller vectors which will be designed.

Definition 1 (see [21]). If there exist four constant matrices A, B, C, and $D \in R^n$ and $C \neq 0$ or $D \neq 0$ such that

$$\lim_{t \to +\infty} \left\| Ax_1 + Bx_2 - Cy_1 - Dy_2 \right\| = 0, \quad (5)$$

the drive systems (1) and (2) are realized combination-combination synchronization with the response systems (3) and (4), where $\|\cdot\|$ represents the matrix norm.

Remark 2. The combination-combination synchronization can be reduced to combination synchronization, projective synchronization, and even control problem, if we choose specific values of A, B, C, and D.

3. Combination-Combination Synchronization of Four Nonlinear Complex Chaotic Systems

In this section, we investigate the combination-combination synchronization of four nonlinear complex chaotic systems.

The first drive system [22] is given by

$$\begin{aligned} \dot{x}_{11} &= \alpha_1 (x_{12} - x_{11}) + x_{12}x_{13}, \\ \dot{x}_{12} &= \gamma_1 x_{11} - x_{12} - x_{11}x_{13}, \\ \dot{x}_{13} &= -\beta_1 x_{13} + \frac{1}{2} (\overline{x}_{11} x_{12} + x_{11}\overline{x}_{12}), \end{aligned} \quad (6)$$

and the second drive system [23] is described as follows:

$$\begin{aligned} \dot{x}_{21} &= a_1 x_{21} + b_1 x_{22} x_{23}, \\ \dot{x}_{22} &= a_2 x_{22} + b_2 x_{21} x_{23}, \\ \dot{x}_{23} &= a_3 x_{23} + \frac{b_3}{2} (\overline{x}_{21} x_{22} + x_{21}\overline{x}_{22}). \end{aligned} \quad (7)$$

The first response system [6] takes the following form:

$$\begin{aligned} \dot{y}_{11} &= \sigma_3 y_{12} - \sigma_3 (1 - i\delta_3) y_{11} + \varphi_1 + i\varphi_2, \\ \dot{y}_{12} &= (\alpha_3 - y_{13}) y_{11} - (1 + i\delta_3) y_{12} + \varphi_3 + i\varphi_4, \\ \dot{y}_{13} &= -\beta_3 y_{13} + \frac{1}{2} (\overline{y}_{11} y_{12} + y_{11}\overline{y}_{12}) + \varphi_5, \end{aligned} \quad (8)$$

and the second response [9] is given by

$$\begin{aligned} \dot{y}_{21} &= y_{22} - \alpha_4 y_{21} + \beta_4 y_{22} y_{23} + \varphi_1^* + i\varphi_2^*, \\ \dot{y}_{22} &= \gamma_4 y_{22} - y_{21} y_{23} + y_{23} + \varphi_3^* + i\varphi_4^*, \\ \dot{y}_{23} &= \frac{\delta_4}{2} (\overline{y}_{21} y_{22} + y_{21}\overline{y}_{22}) - \sigma_4 y_{23} + \varphi_5^*, \end{aligned} \quad (9)$$

where $\alpha_1, \beta_1, \gamma_1, a_1, a_2, a_3, b_1, b_2, b_3, \alpha_3, \beta_3, \delta_3, \sigma_3, \alpha_4, \beta_4, \gamma_4, \sigma_4$, and δ_4 are system parameters; $x_{11} = u_1 + iu_2$, $x_{12} = u_3 + iu_4$, $x_{21} = v_1 + iv_2$, $x_{22} = v_3 + iv_4$, $y_{11} = w_1 + iw_2$, $y_{12} = w_3 + iw_4$, $y_{21} = \mu_1 + i\mu_2$, and $y_{22} = \mu_3 + i\mu_4$ are complex variables; $i = \sqrt{-1}$; and u_i, v_i, w_i, μ_i $(i = 1, 2, 3, 4)$, $x_{13} = u_5$, $x_{23} = v_5$, $y_{13} = w_5$, and $y_{23} = \mu_5$ are real variables. The overbar represents complex conjugate function. φ_i and φ_i^* $(i = 1, 2, 3, 4, 5)$ are real controllers to be determined. Their chaotic attractors are illustrated in Figures 1, 2, 3, and 4, respectively.

For the convenience of our discussions, we assume $A = \mathrm{diag}(k_1, k_2, k_3)$, $B = \mathrm{diag}(l_1, l_2, l_3)$, $C = \mathrm{diag}(m_1, m_2, m_3)$, and $D = \mathrm{diag}(n_1, n_2, n_3)$ in our synchronization scheme.

We define error states between the drive systems (6) and (7) and the response systems (8) and (9) as

$$\begin{aligned} e_1 + ie_2 &= k_1 x_{11} + l_1 x_{21} - m_1 y_{11} - n_1 y_{21}, \\ e_3 + ie_4 &= k_2 x_{12} + l_2 x_{22} - m_2 y_{12} - n_2 y_{22}, \\ e_5 &= k_3 x_{13} + l_3 x_{23} - m_3 y_{13} - n_3 y_{23}, \end{aligned} \quad (10)$$

such that

$$\begin{aligned} &\lim_{t \to \infty} \left\| k_1 x_{11} + l_1 x_{21} - m_1 y_{11} - n_1 y_{21} \right\| = 0, \\ &\lim_{t \to \infty} \left\| k_2 x_{12} + l_2 x_{22} - m_2 y_{12} - n_2 y_{22} \right\| = 0, \\ &\lim_{t \to \infty} \left\| k_3 x_{13} + l_3 x_{23} - m_3 y_{13} - n_3 y_{23} \right\| = 0. \end{aligned} \quad (11)$$

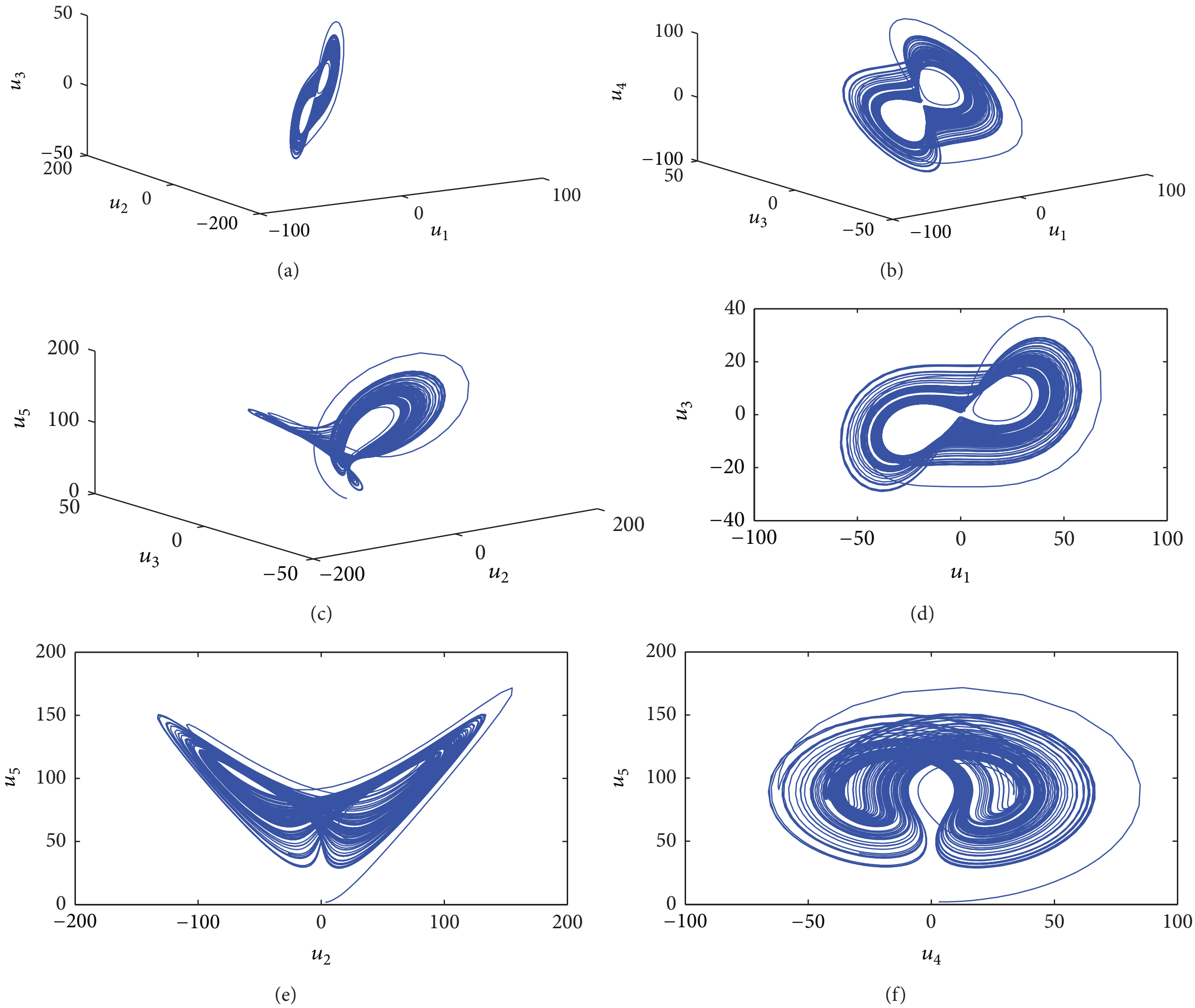

Figure 1: Chaotic attractor for system (6). (a)–(c) Projections in 3D space; (d)–(f) projections in 2D plane.

Thus, we have the following error dynamical system:

$$\begin{aligned} \dot{e}_1 + i\dot{e}_2 &= k_1\dot{x}_{11} + l_1\dot{x}_{21} - m_1\dot{y}_{11} - n_1\dot{y}_{21}, \\ \dot{e}_3 + i\dot{e}_4 &= k_2\dot{x}_{12} + l_2\dot{x}_{22} - m_2\dot{y}_{12} - n_2\dot{y}_{22}, \\ \dot{e}_5 &= k_3\dot{x}_{13} + l_3\dot{x}_{23} - m_3\dot{y}_{13} - n_3\dot{y}_{23}. \end{aligned} \quad (12)$$

Substituting (6)–(9) into (12) and separating the real and imaginary parts yields

$$\begin{aligned} \dot{e}_1 &= k_1\left[\alpha_1\left(u_3 - u_1\right) + u_3u_5\right] + l_1\left(a_1v_1 + b_1v_3v_5\right) \\ &\quad - m_1\left[\sigma_3\left(w_3 - w_1 - \delta_3w_2\right) + \varphi_1\right] \\ &\quad - n_1\left(\mu_3 - \alpha_4\mu_1 + \beta_4\mu_3\mu_5 + \varphi_1^*\right), \\ \dot{e}_2 &= k_1\left[\alpha_1\left(u_4 - u_2\right) + u_4u_5\right] + l_1\left(a_1v_2 + b_1v_4v_5\right) \\ &\quad - m_1\left[\sigma_3\left(w_4 - w_2 + \delta_3w_1\right) + \varphi_2\right] \\ &\quad - n_1\left(\mu_4 - \alpha_4\mu_2 + \beta_4\mu_4\mu_5 + \varphi_2^*\right), \\ \dot{\mathbf{e}}_3 &= \mathbf{k}_2\left(\boldsymbol{\gamma}_1\mathbf{u}_1 - \mathbf{u}_5\mathbf{u}_1 - \mathbf{u}_3\right) + \mathbf{l}_2\left(\mathbf{a}_2\mathbf{v}_3 + \mathbf{b}_2\mathbf{v}_1\mathbf{v}_5\right) \\ &\quad - \mathbf{m}_2\left[\left(\boldsymbol{\alpha}_3 - \mathbf{w}_5\right)\mathbf{w}_1 - \mathbf{w}_3 - \boldsymbol{\delta}_3\mathbf{w}_4 + \boldsymbol{\varphi}_3\right] \\ &\quad - \mathbf{n}_2\left(\boldsymbol{\gamma}_4\boldsymbol{\mu}_3 - \boldsymbol{\mu}_1\boldsymbol{\mu}_5 + \boldsymbol{\mu}_5 + \boldsymbol{\varphi}_3^*\right), \\ \dot{e}_4 &= k_2\left(\gamma_1u_2 - u_5u_2 - u_4\right) + l_2\left(a_2v_4 + b_2v_2v_5\right) \\ &\quad - m_2\left[\left(\alpha_3 - w_5\right)w_2 - w_4 - \delta_3w_3 + \varphi_4\right] \\ &\quad - n_2\left(\gamma_4\mu_4 - \mu_2\mu_5 + \varphi_4^*\right), \\ \dot{e}_5 &= k_3\left(-\beta_1u_5 + u_1u_3 + u_2u_4\right) \\ &\quad + l_3\left[a_3v_5 + b_3\left(v_1v_3 + v_2v_4\right)\right] \\ &\quad - m_3\left(-\beta_3w_5 + w_1w_3 + w_2w_4 + \varphi_5\right) \\ &\quad - n_3\left[\delta_4\left(\mu_1\mu_3 + \mu_2\mu_4\right) - \sigma_4\mu_5 + \varphi_5^*\right]. \end{aligned} \quad (13)$$

Denote $U_i = m_i\varphi_i + n_i\varphi_i^*$ $(i = 1, 2, 3, 4, 5)$; then we obtain the following results.

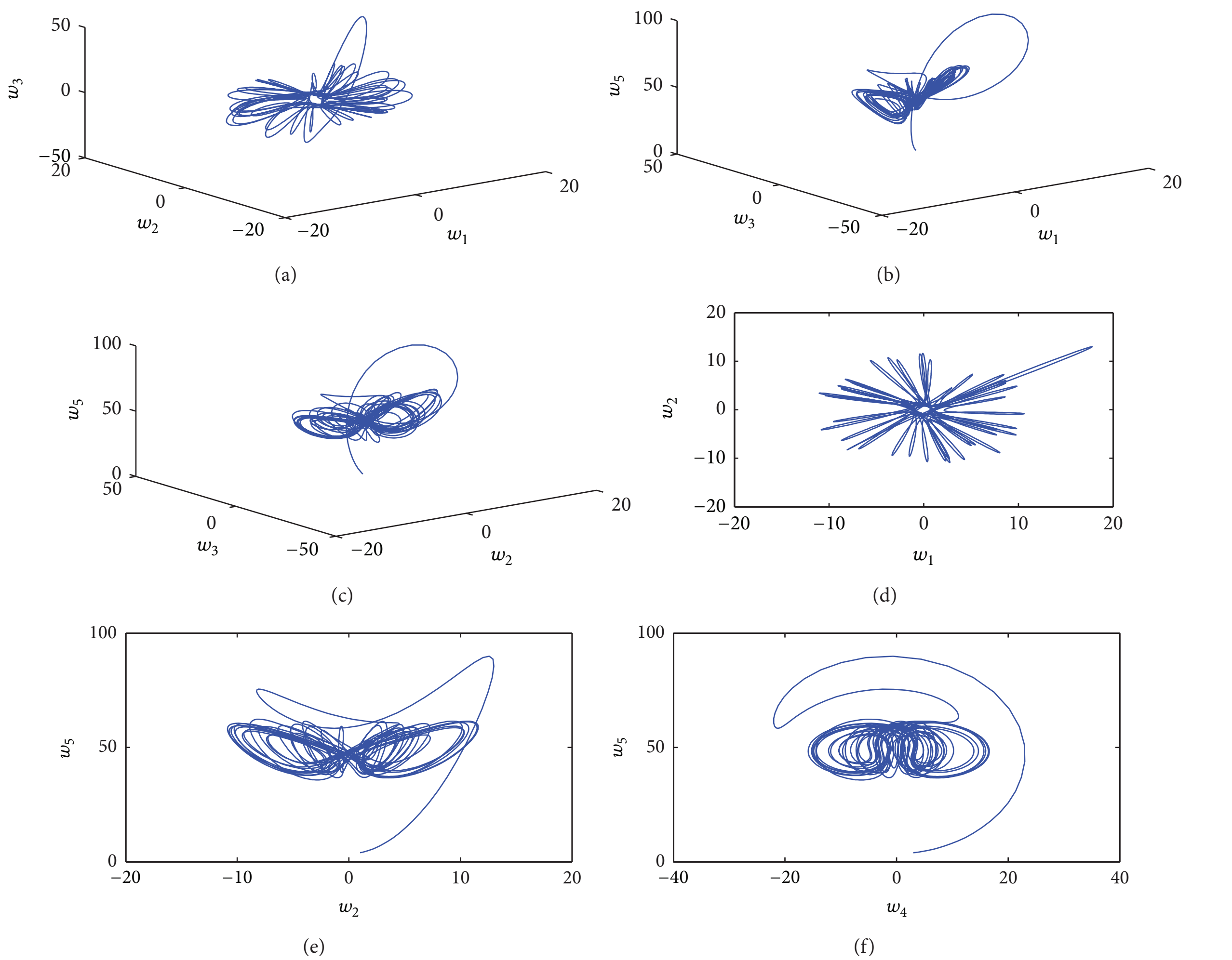

FIGURE 2: Chaotic attractor for system (7). (a)–(c) Projections in 3D space; (d)–(f) projections in 2D plane.

Theorem 3. *If the controllers are chosen as follows:*

$$\begin{aligned}
U_1 &= k_1 u_1 + l_1 v_1 - m_1 w_1 - n_1 \mu_1 \\
&\quad + a_1 \left(k_1 u_2 + l_1 v_2 - m_1 w_2 - n_1 \mu_2\right) \\
&\quad + k_1 \left[\alpha_1 \left(u_3 - u_1\right) + u_3 u_5\right] + l_1 \left(a_1 v_1 + b_1 v_3 v_5\right) \\
&\quad - m_1 \sigma_3 \left(w_3 - w_1 - \delta_3 w_2\right) - n_1 \left(\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5\right), \\
\mathbf{U_2} &= \mathbf{k_1 u_2 + l_1 v_2 - m_1 w_2 - n_1 \mu_2} \\
&\quad \mathbf{- a_1 \left(k_1 u_1 + l_1 v_1 - m_1 w_1 - n_1 \mu_1\right)} \\
&\quad \mathbf{- a_2 \left(k_2 u_3 + l_2 v_3 - m_2 w_3 - n_2 \mu_3\right)} \\
&\quad \mathbf{+ k_1 \left[\alpha_1 \left(u_4 - u_2\right) + u_4 u_5\right] + l_1 \left(a_1 v_2 + b_1 v_4 v_5\right)} \\
&\quad \mathbf{- m_1 \sigma_3 \left(w_4 - w_2 + \delta_3 w_1\right) - n_1 \left(\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5\right)}, \\
U_3 &= k_2 u_3 + l_2 v_3 - m_2 w_3 - n_2 \mu_3 \\
&\quad + a_2 \left(k_1 u_2 + l_1 v_2 - m_1 w_2 - n_1 \mu_2\right) \\
&\quad + a_3 \left(k_2 u_4 + l_2 v_4 - m_2 w_4 - n_2 \mu_4\right) \\
&\quad + k_2 \left(\gamma_1 u_1 - u_5 u_1 - u_3\right) + l_2 \left(a_2 v_3 + b_2 v_1 v_5\right) \\
&\quad - m_2 \left[\left(\alpha_3 - w_5\right) w_1 - w_3 - \delta_3 w_4\right] \\
&\quad - n_2 \left(\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5\right), \\
\mathbf{U_4} &= \mathbf{k_2 u_4 + l_2 v_4 - m_2 w_4 - n_2 \mu_4} \\
&\quad \mathbf{- a_3 \left(k_2 u_3 + l_2 v_3 - m_2 w_3 - n_2 \mu_3\right)} \\
&\quad \mathbf{- b_1 \left(k_3 u_5 + l_3 v_5 - m_3 w_5 - n_3 \mu_5\right)} \\
&\quad \mathbf{+ k_2 \left(\gamma_1 u_2 - u_5 u_2 - u_4\right) + l_2 \left(a_2 v_4 + b_2 v_2 v_5\right)} \\
&\quad \mathbf{- m_2 \left[\left(\alpha_3 - w_5\right) w_2 - w_4 - \delta_3 w_3\right]} \\
&\quad \mathbf{- n_2 \left(\gamma_4 \mu_4 - \mu_2 \mu_5\right)}, \\
U_5 &= k_3 u_5 + l_3 v_5 - m_3 w_5 - n_3 \mu_5 \\
&\quad + b_1 \left(k_2 u_4 + l_2 v_4 - m_2 w_4 - n_2 \mu_4\right) \\
&\quad + k_3 \left(-\beta_1 u_5 + u_1 u_3 + u_2 u_4\right) \\
&\quad + l_3 \left[a_3 v_5 + b_3 \left(v_1 v_3 + v_2 v_4\right)\right]
\end{aligned}$$

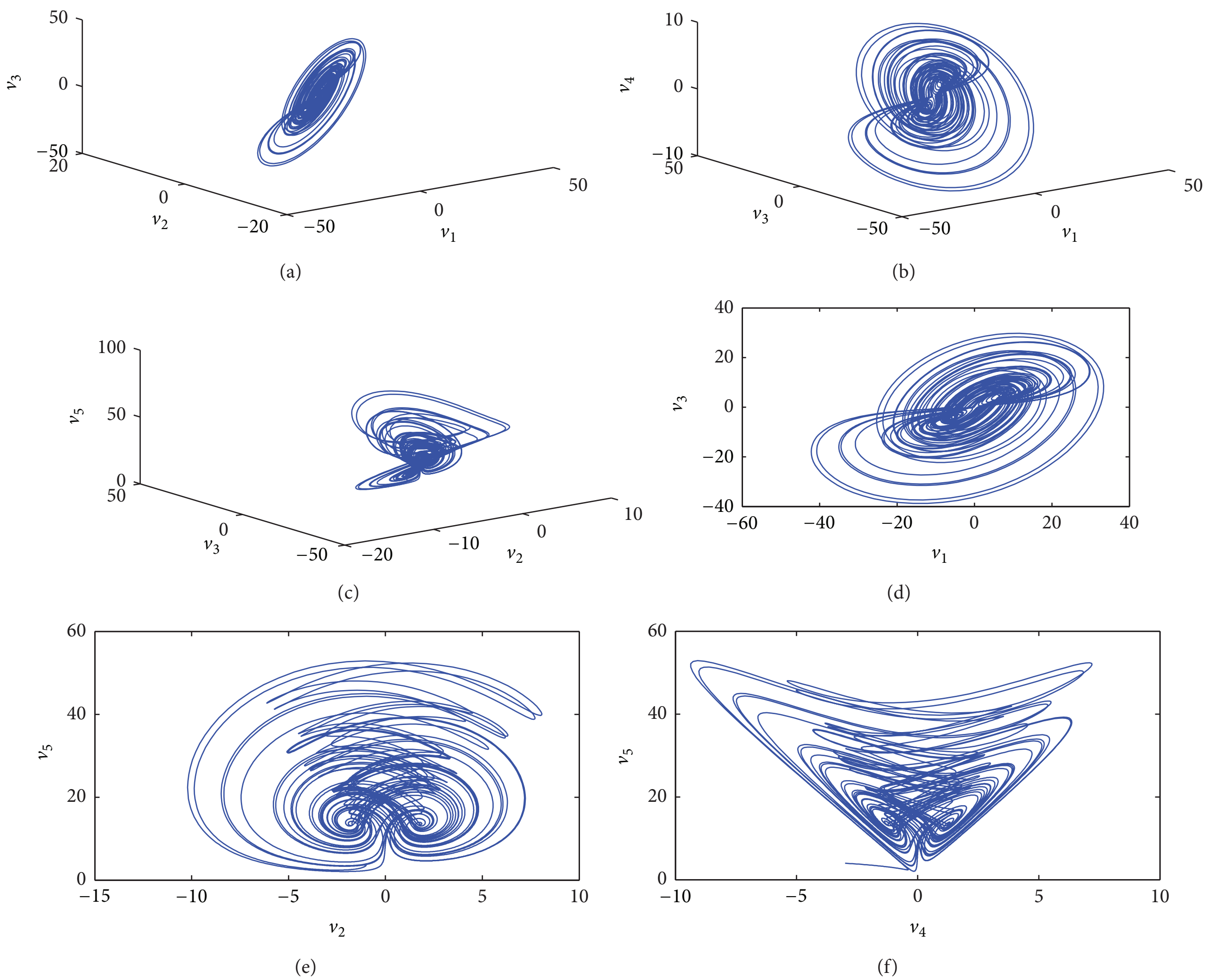

Figure 3: Chaotic attractor for system (8). (a)–(c) Projections in 3D space; (d)–(f) projections in 2D plane.

$$
\begin{aligned}
&- m_3 \left(-\beta_3 w_5 + w_1 w_3 + w_2 w_4\right) \\
&- n_3 \left[\delta_4 \left(\mu_1 \mu_3 + \mu_2 \mu_4\right) - \sigma_4 \mu_5\right],
\end{aligned} \tag{14}
$$

then the drive systems (6) *and* (7) *will achieve combination-combination synchronization with the response systems* (8) *and* (9).

Proof. Construct the following Lyapunov function:

$$
V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2\right). \tag{15}
$$

Taking the time derivative of V along the trajectory of the error dynamical system (13) yields

$$
\begin{aligned}
\dot{V} &= e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 + e_5 \dot{e}_5 \\
&= e_1 \{k_1 \left[\alpha_1 \left(u_3 - u_1\right) + u_3 u_5\right] + l_1 \left(a_1 v_1 + b_1 v_3 v_5\right) \\
&\quad - m_1 \left[\sigma_3 \left(w_3 - w_1 - \delta_3 w_2\right) + \varphi_1\right] \\
&\quad - n_1 \left(\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5 + \varphi_1^*\right)\} \\
&\quad + e_2 \{k_1 \left[\alpha_1 \left(u_4 - u_2\right) + u_4 u_5\right] \\
&\quad + l_1 \left(a_1 v_2 + b_1 v_4 v_5\right) \\
&\quad - m_1 \left[\sigma_3 \left(w_4 - w_2 + \delta_3 w_1\right) + \varphi_2\right] \\
&\quad - n_1 \left(\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5 + \varphi_2^*\right)\} \\
&\quad + e_3 \{k_2 \left(\gamma_1 u_1 - u_5 u_1 - u_3\right) + l_2 \left(a_2 v_3 + b_2 v_1 v_5\right) \\
&\quad - m_2 \left[\left(\alpha_3 - w_5\right) w_1 - w_3 - \delta_3 w_4 + \varphi_3\right] \\
&\quad - n_2 \left(\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5 + \varphi_3^*\right)\} \\
&\quad + e_4 \{k_2 \left(\gamma_1 u_2 - u_5 u_2 - u_4\right) + l_2 \left(a_2 v_4 + b_2 v_2 v_5\right) \\
&\quad - m_2 \left[\left(\alpha_3 - w_5\right) w_2 - w_4 - \delta_3 w_3 + \varphi_4\right] \\
&\quad - n_2 \left(\gamma_4 \mu_4 - \mu_2 \mu_5 + \varphi_4^*\right)\} \\
&\quad + e_5 \{k_3 \left(-\beta_1 u_5 + u_1 u_3 + u_2 u_4\right) \\
&\quad + l_3 \left[a_3 v_5 + b_3 \left(v_1 v_3 + v_2 v_4\right)\right]
\end{aligned}
$$

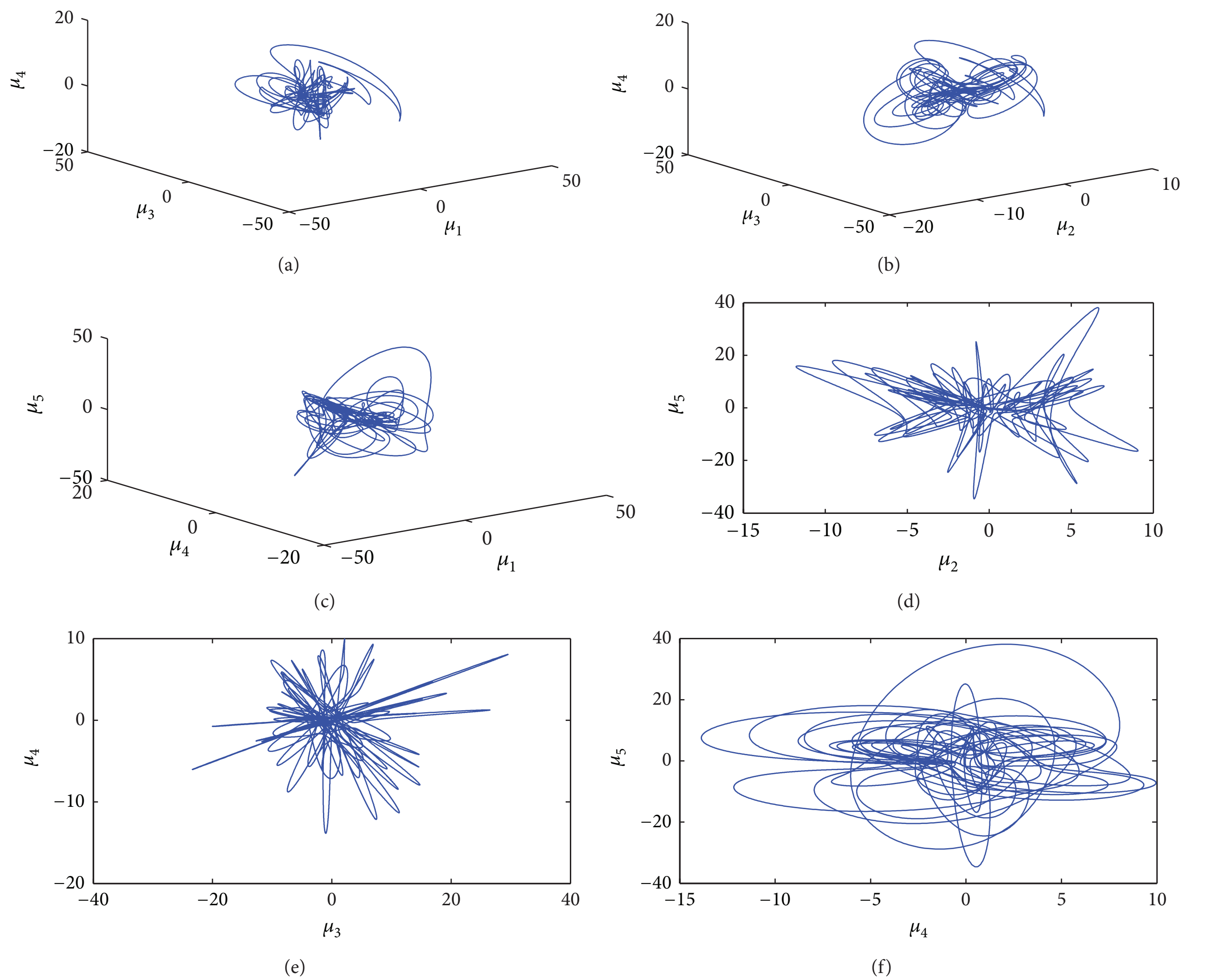

Figure 4: Chaotic attractor for system (9). (a)–(c) Projections in 3D space; (d)–(f) projections in 2D plane.

$$
\begin{aligned}
&\quad - m_3 \left(-\beta_3 w_5 + w_1 w_3 + w_2 w_4 + \varphi_5\right) \\
&\quad - n_3 \left[\delta_4 \left(\mu_1 \mu_3 + \mu_2 \mu_4\right) - \sigma_4 \mu_5 + \varphi_5^*\right]\} \\
&= e_1 \{k_1 \left[\alpha_1 \left(u_3 - u_1\right) + u_3 u_5\right] + l_1 \left(a_1 v_1 + b_1 v_3 v_5\right) \\
&\quad - m_1 \left[\sigma_3 \left(w_3 - w_1 - \delta_3 w_2\right)\right] \\
&\quad - n_1 \left(\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5\right) \\
&\quad - \left(m_1 \varphi_1 + n_1 \varphi_1^*\right)\} \\
&\quad + e_2 \{k_1 \left[\alpha_1 \left(u_4 - u_2\right) + u_4 u_5\right] + l_1 \left(a_1 v_2 + b_1 v_4 v_5\right) \\
&\quad - m_1 \left[\sigma_3 \left(w_4 - w_2 + \delta_3 w_1\right)\right] \\
&\quad - n_1 \left(\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5\right) \\
&\quad - \left(m_1 \varphi_2 + n_1 \varphi_2^*\right)\} \\
&\quad + e_3 \{k_2 \left(\gamma_1 u_1 - u_5 u_1 - u_3\right) + l_2 \left(a_2 v_3 + b_2 v_1 v_5\right) \\
&\quad - m_2 \left[\left(\alpha_3 - w_5\right) w_1 - w_3 - \delta_3 w_4\right] \\
&\quad - n_2 \left(\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5\right) \\
&\quad - \left(m_2 \varphi_3 + n_2 \varphi_3^*\right)\} \\
&\quad + e_4 \{k_2 \left(\gamma_1 u_2 - u_5 u_2 - u_4\right) + l_2 \left(a_2 v_4 + b_2 v_2 v_5\right) \\
&\quad - m_2 \left[\left(\alpha_3 - w_5\right) w_2 - w_4 - \delta_3 w_3\right] \\
&\quad - n_2 \left(\gamma_4 \mu_4 - \mu_2 \mu_5\right) - \left(m_2 \varphi_4 + n_2 \varphi_4^*\right)\} \\
&\quad + e_5 \{k_3 \left(-\beta_1 u_5 + u_1 u_3 + u_2 u_4\right) \\
&\quad + l_3 \left[a_3 v_5 + b_3 \left(v_1 v_3 + v_2 v_4\right)\right] \\
&\quad - m_3 \left(-\beta_3 w_5 + w_1 w_3 + w_2 w_4\right) \\
&\quad - n_3 \left[\delta_4 \left(\mu_1 \mu_3 + \mu_2 \mu_4\right) - \sigma_4 \mu_5\right] \\
&\quad - \left(m_3 \varphi_5 + n_3 \varphi_5^*\right)\}.
\end{aligned}
\tag{16}
$$

Substituting (14) into (16) leads to

$$\begin{aligned}
\dot{V} = e_1 \{ & k_1 [\alpha_1 (u_3 - u_1) + u_3 u_5] \\
& + l_1 (a_1 v_1 + b_1 v_3 v_5) \\
& - m_1 [\sigma_3 (w_3 - w_1 - \delta_3 w_2)] \\
& - n_1 (\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5) \\
& - [k_1 u_1 + l_1 v_1 - m_1 w_1 - n_1 \mu_1 \\
& \quad + a_1 (k_1 u_2 + l_1 v_2 - m_1 w_2 - n_1 \mu_2) \\
& \quad + k_1 [\alpha_1 (u_3 - u_1) + u_3 u_5] \\
& \quad + l_1 (a_1 v_1 + b_1 v_3 v_5) \\
& \quad - m_1 [\sigma_3 (w_3 - w_1 - \delta_3 w_2)] \\
& \quad - n_1 (\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5)]\} \\
+ e_2 \{ & k_1 [\alpha_1 (u_4 - u_2) + u_4 u_5] \\
& + l_1 (a_1 v_2 + b_1 v_4 v_5) \\
& - m_1 [\sigma_3 (w_4 - w_2 + \delta_3 w_1)] \\
& - n_1 (\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5) \\
& - [k_1 u_2 + l_1 v_2 - m_1 w_2 - n_1 \mu_2 \\
& \quad - a_1 (k_1 u_1 + l_1 v_1 - m_1 w_1 - n_1 \mu_1) \\
& \quad - a_2 (k_2 u_3 + l_2 v_3 - m_2 w_3 - n_2 \mu_3) \\
& \quad + k_1 [\alpha_1 (u_4 - u_2) + u_4 u_5] \\
& \quad + l_1 (a_1 v_2 + b_1 v_4 v_5) \\
& \quad - m_1 [\sigma_3 (w_4 - w_2 + \delta_3 w_1)] \\
& \quad - n_1 (\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5)]\} \\
+ e_3 \{ & k_2 (\gamma_1 u_1 - u_5 u_1 - u_3) + l_2 (a_2 v_3 + b_2 v_1 v_5) \\
& - m_2 [(\alpha_3 - w_5) w_1 - w_3 - \delta_3 w_4] \\
& - n_2 (\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5) \\
& - [k_2 u_3 + l_2 v_3 - m_2 w_3 - n_2 \mu_3 \\
& \quad + a_2 (k_1 u_2 + l_1 v_2 - m_1 w_2 - n_1 \mu_2) \\
& \quad + a_3 (k_2 u_4 + l_2 v_4 - m_2 w_4 - n_2 \mu_4) \\
& \quad + k_2 (\gamma_1 u_1 - u_5 u_1 - u_3) \\
& \quad + l_2 (a_2 v_3 + b_2 v_1 v_5) \\
& \quad - m_2 [(\alpha_3 - w_5) w_1 - w_3 - \delta_3 w_4] \\
& \quad - n_2 (\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5)]\} \\
+ e_4 \{ & k_2 (\gamma_1 u_2 - u_5 u_2 - u_4) + l_2 (a_2 v_4 + b_2 v_2 v_5) \\
& - m_2 [(\alpha_3 - w_5) w_2 - w_4 - \delta_3 w_3] \\
& - n_2 (\gamma_4 \mu_4 - \mu_2 \mu_5) \\
& - [k_2 u_4 + l_2 v_4 - m_2 w_4 - n_2 \mu_4 \\
& \quad - a_3 (k_2 u_3 + l_2 v_3 - m_2 w_3 - n_2 \mu_3) \\
& \quad - b_1 (k_3 u_5 + l_3 v_5 - m_3 w_5 - n_3 \mu_5) \\
& \quad + k_2 (\gamma_1 u_2 - u_5 u_2 - u_4) \\
& \quad + l_2 (a_2 v_4 + b_2 v_2 v_5) \\
& \quad - m_2 [(\alpha_3 - w_5) w_2 - w_4 - \delta_3 w_3] \\
& \quad - n_2 (\gamma_4 \mu_4 - \mu_2 \mu_5)]\} \\
+ e_5 \{ & k_3 (-\beta_1 u_5 + u_1 u_3 + u_2 u_4) \\
& + l_3 [a_3 v_5 + b_3 (v_1 v_3 + v_2 v_4)] \\
& - m_3 (-\beta_3 w_5 + w_1 w_3 + w_2 w_4) \\
& - n_3 [\delta_4 (\mu_1 \mu_3 + \mu_2 \mu_4) - \sigma_4 \mu_5] \\
& - [k_3 u_5 + l_3 v_5 - m_3 w_5 - n_3 \mu_5 \\
& \quad + b_1 (k_2 u_4 + l_2 v_4 - m_2 w_4 - n_2 \mu_4) \\
& \quad + k_3 (-\beta_1 u_5 + u_1 u_3 + u_2 u_4) \\
& \quad + l_3 [a_3 v_5 + b_3 (v_1 v_3 + v_2 v_4)] \\
& \quad - m_3 (-\beta_3 w_5 + w_1 w_3 + w_2 w_4) \\
& \quad - n_3 (\delta_4 (\mu_1 \mu_3 + \mu_2 \mu_4) - \sigma_4 \mu_5)]\},
\end{aligned} \tag{17}$$

and we get

$$\begin{aligned}
\dot{V} &= e_1 (-e_1 - a_1 e_2) + e_2 (-e_2 + a_1 e_1 + a_2 e_3) \\
&\quad + e_3 (-e_3 - a_2 e_2 - a_3 e_4) \\
&\quad + e_4 (-e_4 + a_3 e_3 + b_1 e_5) + e_5 (-e_5 - b_1 e_4) \\
&= -e_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2.
\end{aligned} \tag{18}$$

Since $\dot{V} \le 0$ as $t \to \infty$, according to the Lyapunov stability theory, we know $e_i \to 0$ ($i = 1, 2, 3, 4, 5$); that is, $\lim_{t \to \infty} \|e\| = 0$. Therefore, the drive systems (6) and (7) will achieve combination-combination synchronization with the response systems (8) and (9).

This completes the proof. □

If we choose specific values of k_i, l_i, m_i, and n_i ($i = 1, 2, 3$), then we can have the following corollaries.

Corollary 4. *(i) Suppose that $n_1 = n_2 = n_3 = 0$, and if the controllers are chosen as follows:*

$$\begin{aligned}
\varphi_1 = \frac{1}{m_1} \{ & k_1 u_1 + l_1 v_1 - m_1 w_1 \\
& + a_1 (k_1 u_2 + l_1 v_2 - m_1 w_2) \\
& + k_1 [\alpha_1 (u_3 - u_1) + u_3 u_5] \\
& + l_1 (a_1 v_1 + b_1 v_3 v_5) \\
& - m_1 \sigma_3 (w_3 - w_1 - \delta_3 w_2) \}, \\
\varphi_2 = \frac{1}{m_1} \{ & k_1 u_2 + l_1 v_2 - m_1 w_2 \\
& - a_1 (k_1 u_1 + l_1 v_1 - m_1 w_1) \\
& - a_2 (k_2 u_3 + l_2 v_3 - m_2 w_3) \\
& + k_1 [\alpha_1 (u_4 - u_2) + u_4 u_5] \\
& + l_1 (a_1 v_2 + b_1 v_4 v_5) \\
& - m_1 \sigma_3 (w_4 - w_2 + \delta_3 w_1) \}, \\
\varphi_3 = \frac{1}{m_2} \{ & k_2 u_3 + l_2 v_3 - m_2 w_3 \\
& + a_2 (k_1 u_2 + l_1 v_2 - m_1 w_2) \\
& + a_3 (k_2 u_4 + l_2 v_4 - m_2 w_4) \\
& + k_2 (\gamma_1 u_1 - u_5 u_1 - u_3) \\
& + l_2 (a_2 v_3 + b_2 v_1 v_5) \\
& - m_2 [(\alpha_3 - w_5) w_1 - w_3 - \delta_3 w_4] \}, \\
\varphi_4 = \frac{1}{m_2} \{ & k_2 u_4 + l_2 v_4 - m_2 w_4 \\
& - a_3 (k_2 u_3 + l_2 v_3 - m_2 w_3) \\
& - b_1 (k_3 u_5 + l_3 v_5 - m_3 w_5) \\
& + k_2 (\gamma_1 u_2 - u_5 u_2 - u_4) \\
& + l_2 (a_2 v_4 + b_2 v_2 v_5) \\
& - m_2 [(\alpha_3 - w_5) w_2 - w_4 - \delta_3 w_3] \}, \\
\varphi_5 = \frac{1}{m_3} \{ & k_3 u_5 + l_3 v_5 - m_3 w_5 \\
& + b_1 (k_2 u_4 + l_2 v_4 - m_2 w_4) \\
& + k_3 (-\beta_1 u_5 + u_1 u_3 + u_2 u_4) \\
& + l_3 [a_3 v_5 + b_3 (v_1 v_3 + v_2 v_4)] \\
& - m_3 (-\beta_3 w_5 + w_1 w_3 + w_2 w_4) \},
\end{aligned} \tag{19}$$

then the drive systems (6) *and* (7) *will achieve combination synchronization with the response system* (8).

(ii) Suppose that $m_1 = m_2 = m_3 = 0$, and if the controllers are chosen as follows:

$$\begin{aligned}
\boldsymbol{\varphi}_1^* = \frac{\mathbf{1}}{\mathbf{n}_1} \{ & \mathbf{k}_1 \mathbf{u}_1 + \mathbf{l}_1 \mathbf{v}_1 - \mathbf{n}_1 \boldsymbol{\mu}_1 \\
& + \mathbf{a}_1 (\mathbf{k}_1 \mathbf{u}_2 + \mathbf{l}_1 \mathbf{v}_2 - \mathbf{n}_1 \boldsymbol{\mu}_2) \\
& + \mathbf{k}_1 [\boldsymbol{\alpha}_1 (\mathbf{u}_3 - \mathbf{u}_1) + \mathbf{u}_3 \mathbf{u}_5] \\
& + \mathbf{l}_1 (\mathbf{a}_1 \mathbf{v}_1 + \mathbf{b}_1 \mathbf{v}_3 \mathbf{v}_5) \\
& - \mathbf{n}_1 (\boldsymbol{\mu}_3 - \boldsymbol{\alpha}_4 \boldsymbol{\mu}_1 + \boldsymbol{\beta}_4 \boldsymbol{\mu}_3 \boldsymbol{\mu}_5) \}, \\
\boldsymbol{\varphi}_2^* = \frac{\mathbf{1}}{\mathbf{n}_1} \{ & \mathbf{k}_1 \mathbf{u}_2 + \mathbf{l}_1 \mathbf{v}_2 - \mathbf{n}_1 \boldsymbol{\mu}_2 \\
& - \mathbf{a}_1 (\mathbf{k}_1 \mathbf{u}_1 + \mathbf{l}_1 \mathbf{v}_1 - \mathbf{n}_1 \boldsymbol{\mu}_1) \\
& - \mathbf{a}_2 (\mathbf{k}_2 \mathbf{u}_3 + \mathbf{l}_2 \mathbf{v}_3 - \mathbf{n}_2 \boldsymbol{\mu}_3) \\
& + \mathbf{k}_1 [\boldsymbol{\alpha}_1 (\mathbf{u}_4 - \mathbf{u}_2) + \mathbf{u}_4 \mathbf{u}_5] \\
& + \mathbf{l}_1 (\mathbf{a}_1 \mathbf{v}_2 + \mathbf{b}_1 \mathbf{v}_4 \mathbf{v}_5) \\
& - \mathbf{n}_1 (\boldsymbol{\mu}_4 - \boldsymbol{\alpha}_4 \boldsymbol{\mu}_2 + \boldsymbol{\beta}_4 \boldsymbol{\mu}_4 \boldsymbol{\mu}_5) \}, \\
\boldsymbol{\varphi}_3^* = \frac{\mathbf{1}}{\mathbf{n}_2} \{ & \mathbf{k}_2 \mathbf{u}_3 + \mathbf{l}_2 \mathbf{v}_3 - \mathbf{n}_2 \boldsymbol{\mu}_3 \\
& + \mathbf{a}_2 (\mathbf{k}_1 \mathbf{u}_2 + \mathbf{l}_1 \mathbf{v}_2 - \mathbf{n}_1 \boldsymbol{\mu}_2) \\
& + \mathbf{a}_3 (\mathbf{k}_2 \mathbf{u}_4 + \mathbf{l}_2 \mathbf{v}_4 - \mathbf{n}_2 \boldsymbol{\mu}_4) \\
& + \mathbf{k}_2 (\boldsymbol{\gamma}_1 \mathbf{u}_1 - \mathbf{u}_5 \mathbf{u}_1 - \mathbf{u}_3) \\
& + \mathbf{l}_2 (\mathbf{a}_2 \mathbf{v}_3 + \mathbf{b}_2 \mathbf{v}_1 \mathbf{v}_5) \\
& - \mathbf{n}_2 (\boldsymbol{\gamma}_4 \boldsymbol{\mu}_3 - \boldsymbol{\mu}_1 \boldsymbol{\mu}_5 + \boldsymbol{\mu}_5) \}, \\
\boldsymbol{\varphi}_4^* = \frac{\mathbf{1}}{\mathbf{n}_2} \{ & \mathbf{k}_2 \mathbf{u}_4 + \mathbf{l}_2 \mathbf{v}_4 - \mathbf{n}_2 \boldsymbol{\mu}_4 \\
& - \mathbf{a}_3 (\mathbf{k}_2 \mathbf{u}_3 + \mathbf{l}_2 \mathbf{v}_3 - \mathbf{n}_2 \boldsymbol{\mu}_3) \\
& - \mathbf{b}_1 (\mathbf{k}_3 \mathbf{u}_5 + \mathbf{l}_3 \mathbf{v}_5 - \mathbf{n}_3 \boldsymbol{\mu}_5) \\
& + \mathbf{k}_2 (\boldsymbol{\gamma}_1 \mathbf{u}_2 - \mathbf{u}_5 \mathbf{u}_2 - \mathbf{u}_4) \\
& + \mathbf{l}_2 (\mathbf{a}_2 \mathbf{v}_4 + \mathbf{b}_2 \mathbf{v}_2 \mathbf{v}_5) \\
& - \mathbf{n}_2 (\boldsymbol{\gamma}_4 \boldsymbol{\mu}_4 - \boldsymbol{\mu}_2 \boldsymbol{\mu}_5) \}, \\
\boldsymbol{\varphi}_5^* = \frac{\mathbf{1}}{\mathbf{n}_3} \{ & \mathbf{k}_3 \mathbf{u}_5 + \mathbf{l}_3 \mathbf{v}_5 - \mathbf{n}_3 \boldsymbol{\mu}_5 \\
& + \mathbf{b}_1 (\mathbf{k}_2 \mathbf{u}_4 + \mathbf{l}_2 \mathbf{v}_4 - \mathbf{n}_2 \boldsymbol{\mu}_4) \\
& + \mathbf{k}_3 (-\boldsymbol{\beta}_1 \mathbf{u}_5 + \mathbf{u}_1 \mathbf{u}_3 + \mathbf{u}_2 \mathbf{u}_4) \\
& + \mathbf{l}_3 [\mathbf{a}_3 \mathbf{v}_5 + \mathbf{b}_3 (\mathbf{v}_1 \mathbf{v}_3 + \mathbf{v}_2 \mathbf{v}_4)] \\
& - \mathbf{n}_3 [\boldsymbol{\delta}_4 (\boldsymbol{\mu}_1 \boldsymbol{\mu}_3 + \boldsymbol{\mu}_2 \boldsymbol{\mu}_4) - \boldsymbol{\sigma}_4 \boldsymbol{\mu}_5] \},
\end{aligned} \tag{20}$$

then the drive systems (6) *and* (7) *will achieve combination synchronization with the response system* (9).

Corollary 5. *(i) Suppose that* $k_1 = k_2 = k_3 = 0$, $n_1 = n_2 = n_3 = 0$, *and* $m_1 = m_2 = m_3 = 1$, *and if the controllers are chosen as follows:*

$$\begin{aligned}
\varphi_1 &= l_1 v_1 - w_1 + a_1 (l_1 v_2 - w_2) \\
&\quad + l_1 (a_1 v_1 + b_1 v_3 v_5) \\
&\quad - \sigma_3 (w_3 - w_1 - \delta_3 w_2), \\
\varphi_2 &= l_1 v_2 - w_2 - a_1 (l_1 v_1 - w_1) \\
&\quad - a_2 (l_2 v_3 - w_3) + l_1 (a_1 v_2 + b_1 v_4 v_5) \\
&\quad - \sigma_3 (w_4 - w_2 + \delta_3 w_1), \\
\varphi_3 &= l_2 v_3 - w_3 + a_2 (l_1 v_2 - w_2) \\
&\quad + a_3 (l_2 v_4 - w_4) + l_2 (a_2 v_3 + b_2 v_1 v_5) \\
&\quad - [(\alpha_3 - w_5) w_1 - w_3 - \delta_3 w_4], \\
\varphi_4 &= l_2 v_4 - w_4 - a_3 (l_2 v_3 - w_3) \\
&\quad - b_1 (l_3 v_5 - w_5) + l_2 (a_2 v_4 + b_2 v_2 v_5) \\
&\quad - [(\alpha_3 - w_5) w_2 - w_4 - \delta_3 w_3], \\
\varphi_5 &= l_3 v_5 - w_5 + b_1 (l_2 v_4 - w_4) \\
&\quad + l_3 [a_3 v_5 + b_3 (v_1 v_3 + v_2 v_4)] \\
&\quad - (-\beta_3 w_5 + w_1 w_3 + w_2 w_4),
\end{aligned} \tag{21}$$

then the drive system (7) *will achieve projective synchronization with the response system* (8).

(ii) Suppose that $l_1 = l_2 = l_3 = 0$, $n_1 = n_2 = n_3 = 0$, *and* $m_1 = m_2 = m_3 = 1$, *and if the controllers are chosen as follows:*

$$\begin{aligned}
\varphi_1 &= k_1 u_1 - w_1 + \alpha_3 (k_1 u_2 - w_2) \\
&\quad + k_1 [\alpha_1 (u_3 - u_1) + u_3 u_5] \\
&\quad - \sigma_3 (w_3 - w_1 - \delta_3 w_2), \\
\varphi_2 &= k_1 u_2 - w_2 - \alpha_3 (k_1 u_1 - w_1) \\
&\quad - \beta_3 (k_2 u_3 - w_3) + k_1 [\alpha_1 (u_4 - u_2) + u_4 u_5] \\
&\quad - \sigma_3 (w_4 - w_2 + \delta_3 w_1), \\
\varphi_3 &= k_2 u_3 - w_3 + \beta_3 (k_1 u_2 - w_2) \\
&\quad + \delta_3 (k_2 u_4 - w_4) + k_2 (\gamma_1 u_1 - u_5 u_1 - u_3) \\
&\quad - [(\alpha_3 - w_5) w_1 - w_3 - \delta_3 w_4], \\
\varphi_4 &= k_2 u_4 - w_4 - \delta_3 (k_2 u_3 - w_3) \\
&\quad - \sigma_3 (k_3 u_5 - w_5) + k_2 (\gamma_1 u_2 - u_5 u_2 - u_4) \\
&\quad - [(\alpha_3 - w_5) w_2 - w_4 - \delta_3 w_3], \\
\varphi_5 &= k_3 u_5 - w_5 + \sigma_3 (k_2 u_4 - w_4) \\
&\quad + k_3 (-\beta_1 u_5 + u_1 u_3 + u_2 u_4) \\
&\quad - (-\beta_3 w_5 + w_1 w_3 + w_2 w_4),
\end{aligned} \tag{22}$$

then the drive system (6) *will achieve projective synchronization with the response system* (8).

(iii) Suppose that $k_1 = k_2 = k_3 = 0$, $m_1 = m_2 = m_3 = 0$, *and* $n_1 = n_2 = n_3 = 1$, *and if the controllers are chosen as follows:*

$$\begin{aligned}
\varphi_1^* &= l_1 v_1 - \mu_1 + a_1 (l_1 v_2 - \mu_2) \\
&\quad + l_1 (a_1 v_1 + b_1 v_3 v_5) \\
&\quad - (\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5), \\
\varphi_2^* &= l_1 v_2 - \mu_2 - a_1 (l_1 v_1 - \mu_1) \\
&\quad - a_2 (l_2 v_3 - \mu_3) + l_1 (a_1 v_2 + b_1 v_4 v_5) \\
&\quad - (\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5), \\
\varphi_3^* &= l_2 v_3 - \mu_3 + a_2 (l_1 v_2 - \mu_2) \\
&\quad + a_3 (l_2 v_4 - \mu_4) + l_2 (a_2 v_3 + b_2 v_1 v_5) \\
&\quad - (\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5), \\
\varphi_4^* &= l_2 v_4 - \mu_4 - a_3 (l_2 v_3 - \mu_3) \\
&\quad - b_1 (l_3 v_5 - \mu_5) + l_2 (a_2 v_4 + b_2 v_2 v_5) \\
&\quad - (\gamma_4 \mu_4 - \mu_2 \mu_5), \\
\varphi_5^* &= l_3 v_5 - \mu_5 + b_1 (l_2 v_4 - \mu_4) \\
&\quad + l_3 [a_3 v_5 + b_3 (v_1 v_3 + v_2 v_4)] \\
&\quad - (\delta_4 (\mu_1 \mu_3 + \mu_2 \mu_4) - \sigma_4 \mu_5),
\end{aligned} \tag{23}$$

then the drive system (7) *will achieve projective synchronization with the response system* (9).

(iv) Suppose that $l_1 = l_2 = l_3 = 0$, $m_1 = m_2 = m_3 = 0$, *and* $n_1 = n_2 = n_3 = 1$, *and if the controllers are chosen as follows:*

$$\begin{aligned}
\varphi_1^* &= k_1 u_1 - \mu_1 + \alpha_3 (k_1 u_2 - \mu_2) \\
&\quad + k_1 [\alpha_1 (u_3 - u_1) + u_3 u_5] \\
&\quad - (\mu_3 - \alpha_4 \mu_1 + \beta_4 \mu_3 \mu_5), \\
\varphi_2^* &= k_1 u_2 - \mu_2 - \alpha_3 (k_1 u_1 - \mu_1) \\
&\quad - \beta_3 (k_2 u_3 - \mu_3) \\
&\quad + k_1 [\alpha_1 (u_4 - u_2) + u_4 u_5] \\
&\quad - (\mu_4 - \alpha_4 \mu_2 + \beta_4 \mu_4 \mu_5), \\
\varphi_3^* &= k_2 u_3 - \mu_3 + \beta_3 (k_1 u_2 - \mu_2) \\
&\quad + \delta_3 (k_2 u_4 - \mu_4) \\
&\quad + k_2 (\gamma_1 u_1 - u_5 u_1 - u_3) \\
&\quad - (\gamma_4 \mu_3 - \mu_1 \mu_5 + \mu_5),
\end{aligned}$$

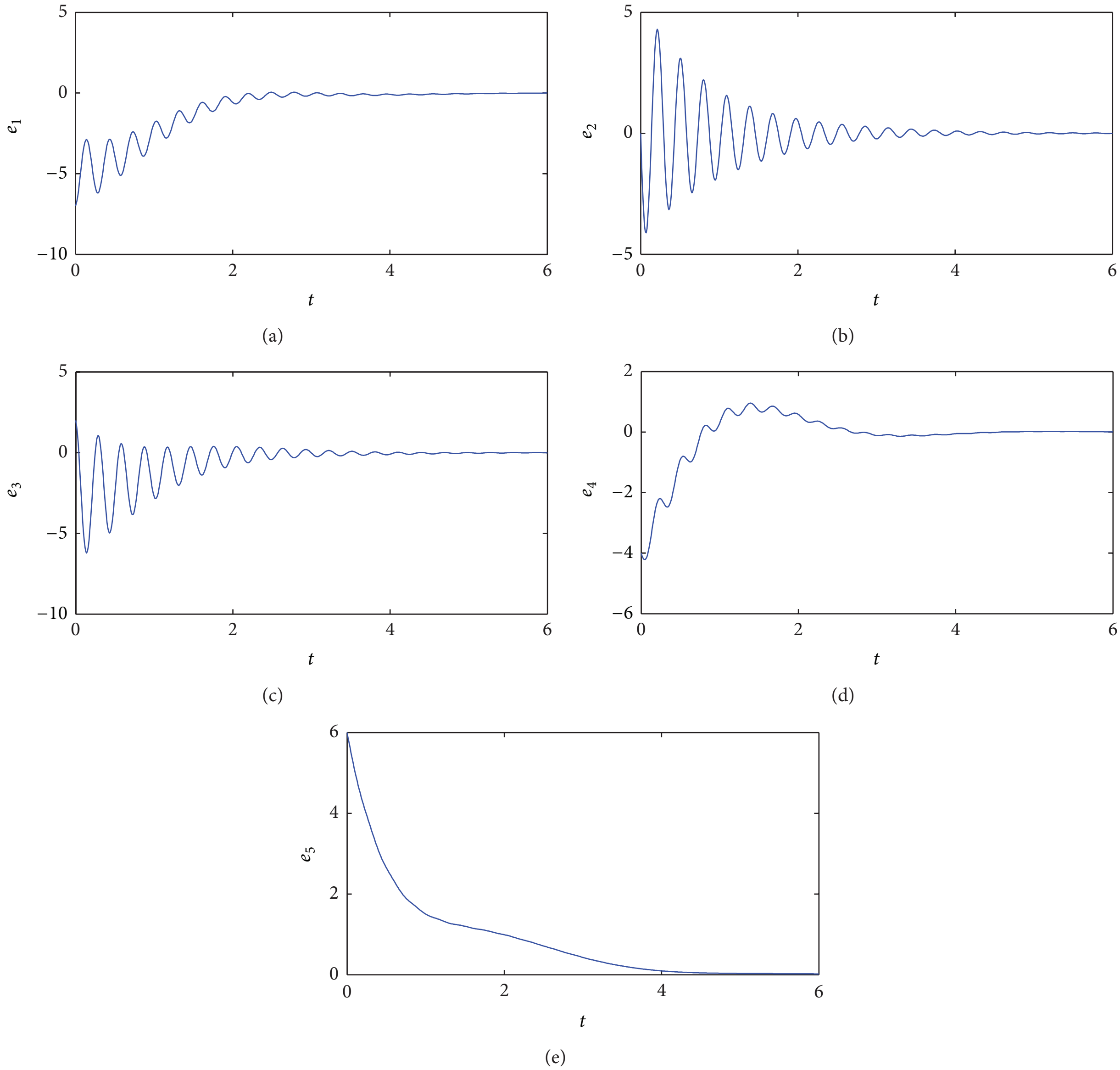

FIGURE 5: Combination-combination synchronization errors e_1, e_2, e_3, e_4, and e_5 between the drive systems (6) and (7) and the response systems (8) and (9), where $e_i = u_i + v_i - w_i - \mu_i$ $(i = 1, 2, 3, 4, 5)$.

$$\begin{aligned}\varphi_4^* &= k_2u_4 - \mu_4 - \delta_3(k_2u_3 - \mu_3)\\&\quad - \sigma_3(k_3u_5 - \mu_5) + k_2(\gamma_1u_2 - u_5u_2 - u_4)\\&\quad - (\gamma_4\mu_4 - \mu_2\mu_5),\\\varphi_5^* &= k_3u_5 - \mu_5 + \sigma_3(k_2u_4 - \mu_4)\\&\quad + k_3(-\beta_1u_5 + u_1u_3 + u_2u_4)\\&\quad - (\delta_4(\mu_1\mu_3 + \mu_2\mu_4) - \sigma_4\mu_5),\end{aligned} \tag{24}$$

then the drive system (6) *will achieve projective synchronization with the response system* (9).

Corollary 6. *(i) Suppose that $k_1 = k_2 = k_3 = 0$, $l_1 = l_2 = l_3 = 0$, $n_1 = n_2 = n_3 = 0$, and $m_1 = m_2 = m_3 = 1$, and if the controllers are chosen as follows:*

$$\begin{aligned}\varphi_1 &= -w_1 - \alpha_3w_2 - \sigma_3(w_3 - w_1 - \delta_3w_2),\\\varphi_2 &= -w_2 + \alpha_3w_1 + \beta_3w_3 - \sigma_3(w_4 - w_2 + \delta_3w_1),\\\varphi_3 &= -w_3 - \beta_3w_2 - \delta_3w_4\\&\quad - [(\alpha_3 - w_5)w_1 - w_3 - \delta_3w_4],\\\varphi_4 &= -w_4 + \delta_3w_3 + \sigma_3w_5\\&\quad - [(\alpha_3 - w_5)w_2 - w_4 - \delta_3w_3],\\\varphi_5 &= -w_5 - \sigma_3w_4 - (-\beta_3w_5 + w_1w_3 + w_2w_4),\end{aligned} \tag{25}$$

then system (8) *is stabilized to the equilibrium* $O(0,0,0,0,0)$.

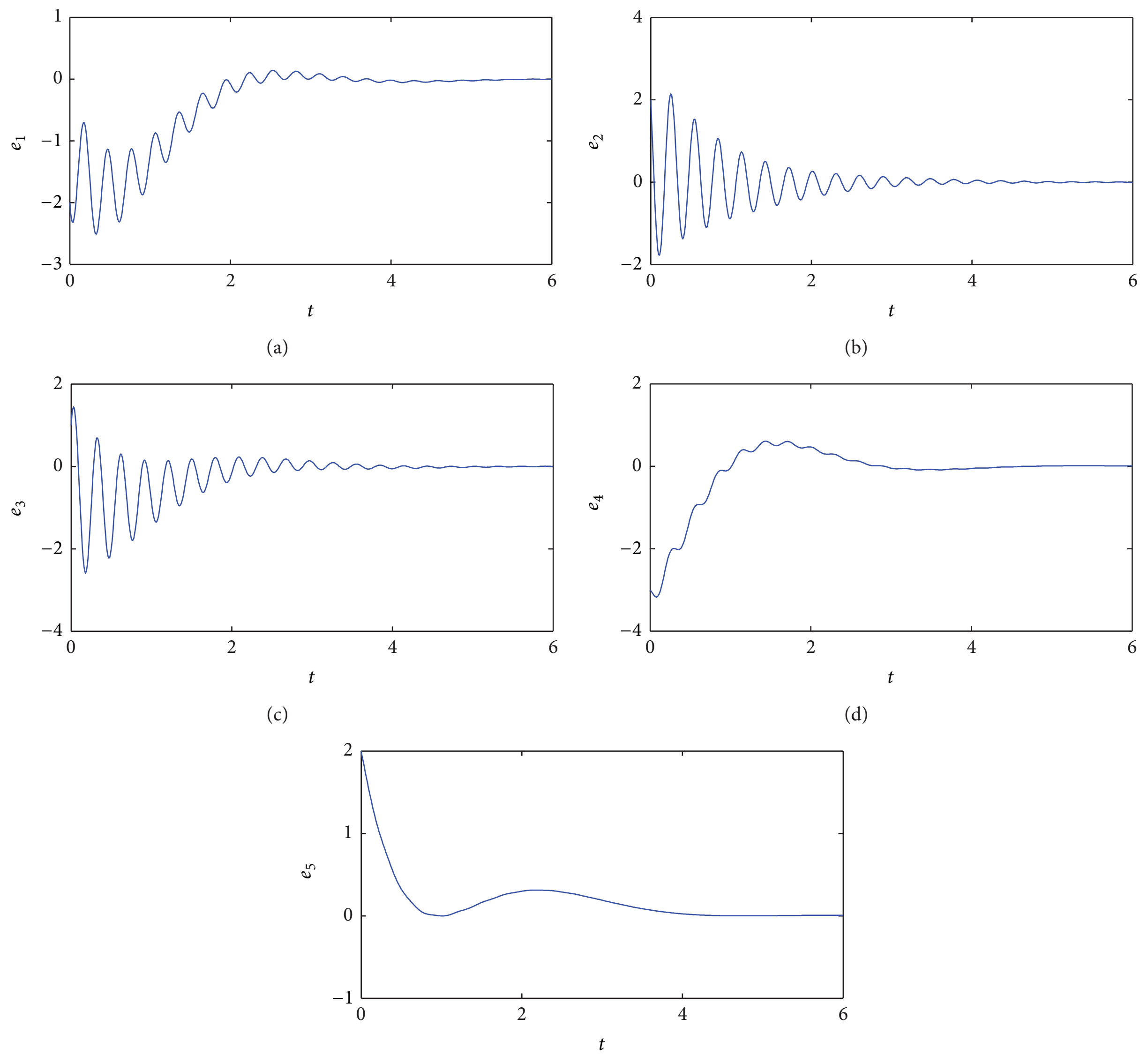

FIGURE 6: Combination synchronization errors e_1, e_2, e_3, e_4, and e_5 between the drive systems (6) and (7) and the response system (8), where $e_i = u_i + v_i - w_i$ $(i = 1, 2, 3, 4, 5)$.

(ii) Suppose that $k_1 = k_2 = k_3 = 0$, $l_1 = l_2 = l_3 = 0$, $m_1 = m_2 = m_3 = 0$, *and* $n_1 = n_2 = n_3 = 1$, *and if the controllers are chosen as follows:*

$$\begin{aligned}
\varphi_1^* &= -\mu_1 - \alpha_4\mu_2 - (\mu_3 - \alpha_4\mu_1 + \beta_4\mu_3\mu_5),\\
\varphi_2^* &= -\mu_2 + \alpha_4\mu_1 + \beta_4\mu_3 - (\mu_4 - \alpha_4\mu_2 + \beta_4\mu_4\mu_5),\\
\varphi_3^* &= -\mu_3 - \beta_4\mu_2 - \gamma_4\mu_4 - (\gamma_4\mu_3 - \mu_1\mu_5 + \mu_5),\\
\varphi_4^* &= -\mu_4 + \gamma_4\mu_3 + \sigma_4\mu_5 - (\gamma_4\mu_4 - \mu_2\mu_5),\\
\varphi_5^* &= -\mu_5 - \sigma_4\mu_4 - [\delta_4(\mu_1\mu_3 + \mu_2\mu_4) - \sigma_4\mu_5],
\end{aligned} \tag{26}$$

then system (9) *is stabilized to the equilibrium* $O(0, 0, 0, 0, 0)$.

Remark 7. The above corollaries can be easily obtained from Theorem 3, and their proofs are similar to that of Theorem 3, so we omit the proofs here.

4. Numerical Simulations

In this section, three numerical examples are presented to illustrate the theoretical analysis. In the following numerical simulations, the fourth-order Runge-kutta method is employed with time step size 0.001. The system parameters are selected as $\alpha_1 = 30$, $\gamma_1 = 90$, $\beta_1 = 11$, $a_1 = 9.5$, $a_2 = -19$, $a_3 = -3$, $b_1 = -1$, $b_2 = 1$, $b_3 = 1$, $\alpha_3 = 50.0625$, $\beta_3 = 0.75$, $\sigma_3 = 5$, $\delta_3 = 0.25$, $\alpha_4 = 3.5$, $\beta_4 = 0.599$, $\gamma_4 = 3$, $\delta_4 = 2$, and $\sigma_4 = 9$, so that the four nonlinear complex chaotic systems exhibit chaotic behaviors, respectively.

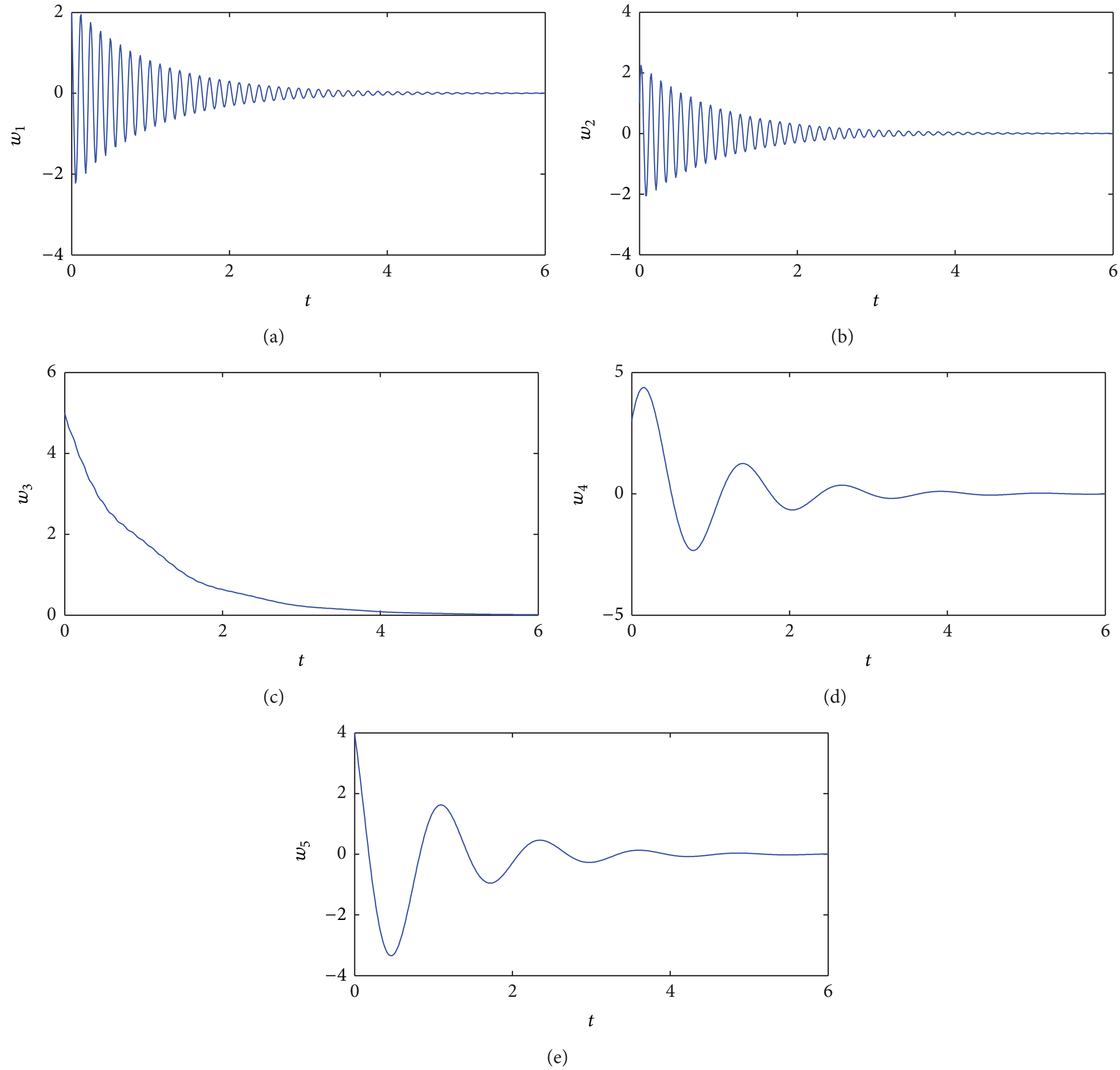

Figure 7: Time evolution of the states of system (8).

Firstly, consider the combination-combination synchronization of the two drive systems (6) and (7) and the response systems (8) and (9) with the controllers (14). We assume $k_1 = k_2 = k_3 = 1$, $l_1 = l_2 = l_3 = 1$, $m_1 = m_2 = m_3 = 1$, and $n_1 = n_2 = n_3 = 1$, and the initial states for the drive systems and response systems are arbitrarily given by $(x_{11}(0), x_{12}(0), x_{13}(0)) = (2 + 4i, 1 + 3i, 2)$, $(x_{21}(0), x_{22}(0), x_{23}(0)) = (-2 - i, 5-3i, 4)$, $(y_{11}(0), y_{12}(0), y_{13}(0)) = (2+i, 5+3i, 4)$, and $(y_{21}(0), y_{22}(0), y_{23}(0)) = (5+2i, -1+i, -4)$; that is, $(u_1(0), u_2(0), u_3(0), u_4(0), u_5(0)) = (2, 4, 1, 3, 2)$, $(v_1(0), v_2(0), v_3(0), v_4(0), v_5(0)) = (-2, -1, 5, -3, 4)$, $(w_1(0), w_2(0), w_3(0), w_4(0), w_5(0)) = (2, 1, 5, 3, 4)$, and $(\mu_1(0), \mu_2(0), \mu_3(0), \mu_4(0), \mu_5(0)) = (5, 2, -1, 1, -4)$, respectively. The corresponding numerical results are shown in Figure 5. Figure 5 displays time response of the combination-combination synchronization errors e_1, e_2, e_3, e_4, and e_5, where $e_i = u_i + v_i - w_i - \mu_i$ $(i = 1, 2, 3, 4, 5)$. The errors converge to zero which implies that the drive systems (6) and (7) and the response systems (8) and (9) have achieved combination-combination synchronization.

Secondly, consider the combination synchronization of the two drive systems (6) and (7) and the response system (8) with the controllers (19). We assume $k_1 = k_2 = k_3 = 1$, $l_1 = l_2 = l_3 = 1$, $m_1 = m_2 = m_3 = 1$, and $n_1 = n_2 = n_3 = 0$. The corresponding numerical results are shown in Figure 6. Figure 6 displays time response of the combination synchronization errors e_1, e_2, e_3, e_4, and e_5, where $e_i = u_i + v_i - w_i$ $(i = 1, 2, 3, 4, 5)$. The errors converge to zero which implies that the drive systems (6) and (7) and the response system (8) have achieved combination synchronization.

Finally, consider another special case, that is, when $k_1 = k_2 = k_3 = 0$, $l_1 = l_2 = l_3 = 0$, $m_1 = m_2 = m_3 = 1$, and $n_1 = n_2 = n_3 = 0$, system (8) will be stabilized to its equilibrium

$O(0,0,0,0,0)$. Figure 7 shows the time evolution of the states w_1, w_2, w_3, w_4, and w_5 of system (8) with controller (25), which illustrates that system (8) is stabilized to the equilibrium $O(0,0,0,0,0)$.

5. Conclusions

In this paper, we investigate the combination-combination synchronization of four nonlinear complex chaotic systems. Based on the Lyapunov stability theory, corresponding controllers to achieve combination-combination synchronization among four different nonlinear complex chaotic systems are derived. The special cases, such as combination synchronization and projective synchronization, are studied as well. This synchronization scheme has advantages over the usual drive-response synchronization, such as being able to provide greater security in secure communication. In [24], the authors applied combination synchronization in secure communication; the signal was divided into two parts, and each part was transmitted by a different chaotic system (the drive system), which implied that the signal transmitted by this model may have stronger antiattack ability and antitranslated capability than that transmitted by the usual transmission model. When applying the nonlinear complex systems in communications, the complex variables will double the number of variables and can increase the content and security of the transmitted information. Thus combination-combination synchronization of complex nonlinear systems can find better applications in security communication, such as wireless communication [25].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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