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Research Article

On Inequality Applicable to Partial Dynamic Equations

Deepak B. Pachpatte

Department of Mathematics, Dr. B.A.M. University, Aurangabad, Maharashtra 431004, India

Correspondence should be addressed to Deepak B. Pachpatte; pachpatte@gmail.com

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The main objective of the paper is to study new integral inequality on time scales which is used for the study of some partial dynamic equations. Some applications of our results are also given.

1. Introduction

During past few decades many authors have established various dynamic inequalities useful in the development of differential and integral equations. Mathematical inequalities on time scales play an important role in the theory of dynamic equations. The study of time scale was initiated by Hilger [1] in 1990 in his Ph.D. thesis which unifies continuous and discrete calculus. Since then, many authors have studied various properties of dynamic equations on time scales [2–9].

In what follows, let $\mathbb{R}$ denotes the set of real numbers and let $\mathbb{T}$ denote the arbitrary time scales. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{T}_1 = [0, a]$, and $\mathbb{T}_2 = [0, b]$ be subsets of $\mathbb{R}$ and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. Let C_{rd} denote the set of rd-continuous function. The partial delta derivative of $v(x, y)$ for $(x, y) \in \Omega$ with respect to x, y, and xy is denoted by $v^{\Delta_1}(x, y)$, $v^{\Delta_2}(x, y)$, and $v^{\Delta_1\Delta_2}(x, y) = v^{\Delta_2\Delta_1}(x, y)$. We assume here understanding of time scales calculus and notations. Further information about time scales calculus can be found in [1, 5, 10].

We require the following lemmas given in [5, 6].

Lemma 1 (see [5], Theorem 2.6). *Let $u \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}_+)$, $a \in \mathbb{R}_+$, and*

$$u^{\Delta}(t) \le a(t)\, u(t), \tag{1}$$

for all $t \in \mathbb{T}^k$; then

$$u(t) \le u(t_0)\, e_a(t, t_0), \tag{2}$$

for all $t \in \mathbb{T}^k$.

Lemma 2 (see [6], Lemma 2.1). *Let $u, a, b \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ and $a(x, y)$ is nondecreasing in $(x, y) \in \Omega$ and*

$$u(x, y) \le a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} b(s, t)\, u(s, t)\, \Delta t\, \Delta s, \tag{3}$$

for $(x, y) \in \Omega$; then

$$u(x, y) \le a(x, y)\, e_{Q(x,y)}(x, x_0), \tag{4}$$

where

$$Q(x, y) = \int_{y_0}^{y} b(x, t)\, \Delta t, \tag{5}$$

for $(x, y) \in \Omega$.

2. Main Results

Now in this section we give our main results.

Theorem 3. *Let $u(x, y)$, $w(x, y)$, $p(x, y)$, $q(x, y)$, $r(x, y) \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ and suppose that*

$$\begin{aligned} u(x, y) \le c &+ \int_{x_0}^{x} w(s, y)\, u(s, y)\, \Delta s + \int_{s_0}^{s} \int_{y_0}^{y} p(s, t) \\ &\times \Big[u(s, t) + \int_{s_0}^{s} \int_{t_0}^{t} q(\xi, \tau)\, u(\xi, \tau)\, \Delta\tau\, \Delta\xi \\ &+ \int_{a_0}^{a} \int_{b_0}^{b} r(\xi, \tau)\, u(\xi, \tau)\, \Delta\tau\, \Delta\xi \Big] \Delta t\, \Delta s, \end{aligned} \tag{6}$$

for $(x, y) \in \Omega$, *where* $c \geq 0$ *is a constant. If*

$$g = \int_{a_0}^{a} \int_{b_0}^{b} r(\xi, \tau) A(\xi, \tau) e_{H(x,y)}(\xi, \tau_0) \Delta\tau \Delta\xi, \tag{7}$$

where

$$H(x, y) = \int_{\tau_0}^{\tau} A(x, t) [p(x, t) + q(x, t)] \Delta t, \tag{8}$$

$$A(x, y) = e_{w(x,y)}(x, x_0), \tag{9}$$

for $(x, y) \in \Omega$, *then*

$$u(x, y) \leq \frac{c}{1 - g} A(x, y) e_{H(x,y)}(x_0, x), \tag{10}$$

for $(x, y) \in \Omega$.

Proof. Define a function $z(x, y)$ by

$$\begin{aligned} z(x, y) = c + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) & \\ \times \Bigg[u(s, t) \times \int_{s_0}^{s} \int_{t_0}^{t} & q(\xi, \tau) u(\xi, \tau) \Delta\tau \Delta\xi \\ + \int_{s_0}^{s} \int_{t_0}^{t} & r(\xi, \tau) u(\xi, \tau) \Delta\tau \Delta\xi \Bigg] \Delta t \Delta s. \end{aligned} \tag{11}$$

Then (6) is

$$u(x, y) \leq z(x, y) + \int_{x_0}^{x} w(s, y) u(s, y) \Delta s. \tag{12}$$

It is easy to see that $z(x, y)$ is nonnegative, rd-continuous, and nondecreasing function for $(x, y) \in \Omega$. Treating y fixed and using Lemma 1 we get

$$u(x, y) \leq A(x, y) z(x, y), \tag{13}$$

for $(x, y) \in \Omega$, where $A(x, y)$ is defined by (9). From (11), (12), and the fact that $A(x, y) \geq 1$, we have

$$\begin{aligned} z(x, y) &\leq c + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) \\ &\quad \times \Bigg[A(s, t) z(s, t) \\ &\quad + \int_{s_0}^{s} \int_{t_0}^{t} q(\xi, \tau) A(\xi, \tau) \\ &\quad \times z(\xi, \tau) \Delta\tau \Delta\xi \\ &\quad + \int_{a_0}^{a} \int_{b_0}^{b} r(\xi, \tau) A(\xi, \tau) \\ &\quad \times z(\xi, \tau) \Delta\tau \Delta\xi \Bigg] \Delta t \Delta s \\ &\leq c + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) A(s, t) \\ &\quad \times \Bigg[z(s, t) + \int_{s_0}^{s} \int_{t_0}^{t} q(\xi, \tau) A(\xi, \tau) \\ &\quad \times z(\xi, \tau) \Delta\tau \Delta\xi \\ &\quad + \int_{a_0}^{a} \int_{b_0}^{b} h(\xi, \tau) A(\xi, \tau) \\ &\quad \times z(\xi, \tau) \Delta\tau \Delta\xi \Bigg] \Delta t \Delta s. \end{aligned} \tag{14}$$

Define a function $v(x, y)$ by right hand side of (14). Then $v(0, y) = v(x, 0) = c$, $z(x, y) \leq v(x, y)$. One has

$$\begin{aligned} v^{\Delta_2\Delta_1} &= p(x, y) A(x, y) \\ &\quad \times \Bigg[z(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} q(\xi, \tau) A(\xi, \tau) z(\xi, \tau) \Delta\tau \Delta\xi \\ &\quad + \int_{a_0}^{a} \int_{b_0}^{b} r(\xi, \tau) A(\xi, \tau) z(\xi, \tau) \Delta\tau \Delta\xi \Bigg] \\ &\leq p(x, y) A(x, y) \\ &\quad \times \Bigg[v(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} q(\xi, \tau) A(\xi, \tau) v(\xi, \tau) \Delta\tau \Delta\xi \\ &\quad + \int_{a_0}^{a} \int_{b_0}^{b} r(\xi, \tau) A(\xi, \tau) v(\xi, \tau) \Delta\tau \Delta\xi \Bigg]. \end{aligned} \tag{15}$$

Define a function $f(x, y)$ by

$$\begin{aligned} f(x, y) = v(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} & q(\xi, \tau) A(\xi, \tau) v(\xi, \tau) \Delta\tau \Delta\xi \\ + \int_{a_0}^{a} \int_{b_0}^{b} & r(\xi, \tau) A(\xi, \tau) v(\xi, \tau) \Delta\tau \Delta\xi; \end{aligned} \tag{16}$$

then $v(x, y) \leq f(x, y)$, $v^{\Delta_2\Delta_1}(x, y) \leq p(x, y) A(x, y) f(x, y)$,

$$\begin{aligned} f(x_0, y) &= f(x, y_0) \\ &= c + \int_{a_0}^{a} \int_{b_0}^{b} r(\xi, \tau) A(\xi, \tau) v(\xi, \tau) \Delta\tau \Delta\xi \\ &= M \text{ (say)}, \end{aligned} \tag{17}$$

$$\begin{aligned} f^{\Delta_2\Delta_1}(x, y) &= v^{\Delta_2\Delta_1}(x, y) \\ &\quad + q(x, y) A(x, y) v(x, y) \\ &\leq p(x, y) A(x, y) f(x, y) \\ &\quad + q(x, y) A(x, y) f(x, y) \\ &= A(x, y) [p(x, y) + q(x, y)] f(x, y). \end{aligned} \tag{18}$$

By keeping **x** fixed in (18), taking $y = t$ and delta integrating with second variable from y_0 to y. Using the fact that $f^{\Delta_1}(x, y_0) = 0$ and $f(x, y)$ is nondecreasing in $(x, y) \in \Omega$, we have

$$\begin{aligned} f^{\Delta_1}(x, y) &\leq \int_{y_0}^{y} A(x, t) [p(x, t) + q(x, t)] f(x, t) \Delta t \\ &\leq f(x, y) \int_{y_0}^{y} A(x, t) [p(x, t) + q(x, t)] \Delta t. \end{aligned} \tag{19}$$

Let

$$\overline{Q}(x, y) = \int_{y_0}^{y} A(x, t) [p(x, t) + q(x, t)] \Delta t; \tag{20}$$

then (20) gives

$$f^{\Delta_1}(x,y) \le f(x,y)\overline{Q}(x,y). \quad (21)$$

Now treating y fixed in (21) and applying Lemma 1, we have

$$f(x,y) \le M e_{\overline{Q}(x,y)}(x,x_0). \quad (22)$$

From (18), (22), and (7), it is easy to see that

$$M \le \frac{c}{1-g}. \quad (23)$$

Using (23) in (22) and the fact that $z(x,y) \le v(x,y)$ and $z(x,y) \le A(x,y)v(x,y)$ we get the inequality in (10).

This completes the proof. □

3. Applications

Now we give some application of theorem to study properties of solutions of initial value problem:

$$\begin{aligned} u^{\Delta_2\Delta_1}(x,y) &= (w(x,y)u(x,y))^{\Delta_2} \\ &\quad + G\left(x,y,u(x,y),\right. \\ &\qquad \left.\int_{a_0}^{a}\int_{b_0}^{b} h(x,y,\xi,\tau,u(\xi,\tau))\,\Delta\tau\,\Delta\xi\right), \\ u(x,y_0) &= \alpha(x), \\ u(x_0,y) &= \beta(y), \\ \alpha(x_0) &= \beta(y_0) = 0, \end{aligned} \quad (24)$$

where $\alpha \in C_{rd}(\mathbb{T}_1,\mathbb{R})$, $\beta \in C_{rd}(\mathbb{T}_2,\mathbb{R})$ for $0 \le \xi \le x$, $0 \le \tau \le y$, $h \in C_{rd}(\Omega^2 \times \mathbb{R},\mathbb{R})$, $G \in C_{rd}(\Omega \times \mathbb{R}^2,\mathbb{R})$, $p \in C_{rd}(\Omega,\mathbb{R})$ is delta differentiable with respect to y.

We observe that (24) is equivalent to

$$\begin{aligned} u(x,y) &= F(x,y) + \int_{x_0}^{x} w(s,y)u(s,y)\,\Delta s \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} G\left(s,t,u(s,t),\right. \\ &\qquad \left.\int_{a_0}^{a}\int_{b_0}^{b} h(s,t,\xi,\tau,u(\xi,\tau))\,\Delta\tau\,\Delta\xi\right)\Delta t\,\Delta s, \end{aligned} \quad (25)$$

where

$$F(x,y) = \alpha(x) + \beta(y) - \int_{x_0}^{x} p(s,y_0)\alpha(s)\,\Delta s. \quad (26)$$

The following theorem deals with estimate on solution (24).

Theorem 4. *Suppose*

$$\begin{aligned} |F(x,y)| &\le c, \\ |h(x,y,s,t,u)| &\le k(x,y)r(s,t)|u|, \\ |G(x,y,u,\overline{u})| &\le p(x,y)(|u| + |\overline{u}|), \end{aligned} \quad (27)$$

where p, r, c which are as in Theorem 3 and $k(x,y)$ is rd-continuous function defined on Ω such that $k(x,y) \ge 1$. Let

$$g_0 = \int_{a_0}^{a}\int_{b_0}^{b} r(\xi,\tau)A(\xi,\tau)e_{\overline{H}(x,y)}(\xi,\tau_0)\,\Delta\tau\,\Delta\xi, \quad (28)$$

where

$$\overline{H}(x,y) = \int_{t_0}^{\tau} A(x,t)p(x,t)k(x,t)\,\Delta t, \quad (29)$$

$$\overline{A}(x,y) = e_{w(r,y)}(x,x_0), \quad (30)$$

for $(x,y) \in \Omega$. If $u(x,y)$ is any solution of (24), then

$$u(x,y) \le \frac{c}{1-g_0}\overline{A}(x,y)e_{\overline{H}(x,y)}(x,x_0), \quad (31)$$

where $(x,y) \in \Omega$.

Proof. The solution $u(x,y)$ of (24) satisfies (25). Using (27) in (25) we have

$$\begin{aligned} |u(x,y)| &\le c + \int_{x_0}^{x} |w(s,y)||u(s,y)|\,\Delta s \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t) \\ &\qquad \times \left[|u(s,t)| + \int_{a_0}^{a}\int_{b_0}^{b} k(s,t)r(\xi,\tau) \times |u(\xi,\tau)|\,\Delta\tau\,\Delta\xi\right]\Delta t\,\Delta s \\ &\le c + \int_{x_0}^{x} |w(s,y)||u(s,y)|\,\Delta s \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)k(s,t) \\ &\qquad \times \left[|u(s,t)| + \int_{a_0}^{a}\int_{b_0}^{b} r(\xi,\tau) \times |u(\xi,\tau)|\,\Delta\tau\,\Delta\xi\right]\Delta t\,\Delta s. \end{aligned} \quad (32)$$

Now an application of Theorem 3 (with $g = 0$) to (32) yields (30).

This completes the proof. □

Now we establish the uniqueness of solutions of (24).

Theorem 5. *Suppose that*

$$\begin{aligned} |h(x,y,s,t,u) - h(x,y,s,t,\overline{u})| &\le k(x,y)r(s,t)|u - \overline{u}|, \\ |G(x,y,u,\overline{u}) - G(x,y,v,\overline{v})| &\le p(x,y)(|u-v| + |\overline{u} - \overline{v}|), \end{aligned} \quad (33)$$

where k, p, and r are as in Theorem 4. Let g_0 and $\overline{A}(x,y)$ be as in (28) and (30). Then (24) has at most one solution on G.

Proof. Let $u(x, y)$ and $v(x, y)$ be two solutions of (24) on Ω; then we have

$$\begin{aligned} u(x,y) - v(x,y) &= \int_{x_0}^{x} w(x,y)\{u(s,y) - v(s,y)\}\Delta s \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y}\Big\{G\Big(s,t,u(s,t), \int_{a_0}^{a}\int_{b_0}^{b} h(s,t,\xi,\tau,u(\xi,\tau))\Delta\tau\,\Delta\xi\Big) \\ &\quad - G\Big(s,t,v(s,t), \int_{a_0}^{a}\int_{b_0}^{b} h(s,t,\xi,\tau,v(\xi,\tau))\Delta\tau\,\Delta\xi\Big)\Big\}\Delta t\,\Delta s. \end{aligned} \tag{34}$$

From (34) and (33) we obtain

$$\begin{aligned} |u(x,y) - v(x,y)| &\le \int_{x_0}^{x} |w(s,y)|\,|u(s,y) - v(s,y)|\,\Delta s \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t) \times \Big(|u(s,t) - v(s,t)| + k(s,t) \times \int_{a_0}^{a}\int_{b_0}^{b} r(\xi,\tau) \\ &\qquad \times |u(\xi,\tau) - v(\xi,\tau)|\,\Delta\tau\,\Delta\xi\Big)\Delta t\,\Delta s. \end{aligned} \tag{35}$$

Applying Theorem 3 (with $c = 0$, $g = 0$) yields

$$|u(x,y) - v(x,y)| \le 0. \tag{36}$$

Therefore $u(x, y) \le v(x, y)$; there is at most one solution of (24) in Ω. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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