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*Research Article*

# On the Strong Convergence and Complete Convergence for Pairwise NQD Random Variables

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Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$ and let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise negatively quadrant dependent random variables. The complete convergence for pairwise negatively quadrant dependent random variables is studied under mild condition. In addition, the strong laws of large numbers for identically distributed pairwise negatively quadrant dependent random variables are established, which are equivalent to the mild condition $\sum_{n=1}^{\infty} P\left(|X| > a_n\right) < \infty$. Our results obtained in the paper generalize the corresponding ones for pairwise independent and identically distributed random variables.

## 1. Introduction

Throughout the paper, let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$, and let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise i.i.d. random variables. Denote $S_n = \sum_{i=1}^{n} X_i$ for each $n \geq 1$. Now, we consider the following assumptions:

(i) $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$;

(ii) $S_n/a_n \to 0$ a.s.;

(iii) $\sum_{i=1}^{n} |X_i|/a_n \to 0$ a.s.

Recently, Sung [1] proved that the three assumptions above are equivalent for pairwise i.i.d. random variables. In addition, he presented some results on complete convergence for pairwise i.i.d. random variables. For more details about the strong law of large numbers and complete convergence for independent random variables or dependent random variables, one can refer to Etemadi [2], Wang et al. [3], Chen et al. [4], Tang [5], and so forth.

We point out that the keys to the proofs of the main results of Sung [1] are the Khintchine-Kolmogorov-type convergence theorem and the second Borel-Cantelli lemma for pairwise independent events (e.g., see Theorem 4.2.5 in [6] or Theorem 2.18.5 in [7]), while these are not proved for pairwise negatively quadrant dependent random variables (pairwise NQD, in short; see Definition 1). If we want to generalize the main results of Sung [1] to the case of pairwise NQD random variables, we should propose new methods or prove the Khintchine-Kolmogorov-type convergence theorem and the second Borel-Cantelli lemma for pairwise NQD random variables. The answer is positive.

Firstly, let us recall the concept of pairwise negatively quadrant dependent random variables as follows.

*Definition 1.* The pair $(X, Y)$ of random variables $X$ and $Y$ is said to be negatively quadrant dependent (NQD, in short), if, for all $x, y \in \mathbf{R}$,

$$P\left(X \leq x, Y \leq y\right) \leq P\left(X \leq x\right) P\left(Y \leq y\right). \quad (1)$$

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise NQD, if $(X_i, X_j)$ is NQD for every $i \neq j$, $i, j = 1, 2, \ldots$.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is called rowwise pairwise NQD random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of pairwise NQD random variables.

The concept of pairwise NQD random variables was introduced by Lehmann [8], which includes pairwise independent random sequence and some negatively dependent sequences, such as negatively associated sequences (see [9–13]), negatively orthant dependent sequences (see [9, 14–18]), and linearly negative quadrant dependent sequences (see [19–21]). Hence, studying the probability limiting behavior of pairwise NQD random variables and its applications in probability theory and mathematical statistics are of great interest. Many authors have dedicated themselves to the study of it. Matula [10] gained the Kolmogorov-type strong law of large numbers for the identically distributed pairwise NQD sequences; Wu [22] gave the generalized three-series theorem for pairwise NQD sequences and proved the Marcinkiewicz strong law of large numbers; Chen [23] discussed Kolmogorov-Chung strong law of large numbers for the nonidentically distributed pairwise NQD sequences under very mild conditions; Wan [24] and Huang et al. [25] obtained the complete convergence for pairwise NQD random sequences; Wang et al. [26], Li and Yang [27], Gan and Chen [28], Shi [29], Xu and Tang [30], and Tang [31] studied the strong convergence properties for pairwise NQD random variables; Sung [21] established the $L_r$ convergence for weighted sums of arrays of rowwise pairwise NQD random variables under weaker uniformly integrable conditions; and so on. The main purpose of the paper is to establish the second Borel-Cantelli lemma for pairwise NQD random variables and generalize the main results of Sung [1] to the case of pairwise NQD random variables without adding any extra conditions.

Our main results are as follows. The first two results are the complete convergence for pairwise NQD random variables.

**Theorem 2.** *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with identical distribution. If $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n} \left[X_i - EX_i I\left(|X_i| \leq a_n\right)\right]\right| > a_n \varepsilon\right) < \infty \quad \forall \varepsilon > 0. \tag{2}$$

**Theorem 3.** *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow \infty$. Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with identical distribution. If $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n} |S_k| > a_n \varepsilon\right) < \infty \quad \forall \varepsilon > 0. \tag{3}$$

The following two theorems are the results on strong convergence for pairwise NQD random variables.

**Theorem 4.** *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with identical distribution. Then, the following statements are equivalent:*

(i) $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$,

(ii) $(1/a_n) \sum_{i=1}^{n} [X_i - EX_i I(|X_i| \leq a_n)] \to 0$ *a.s.*

**Theorem 5.** *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow \infty$. Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with identical distribution. Then, the following statements are equivalent:*

(i) $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$,

(ii) $S_n/a_n \to 0$ *a.s.*,

(iii) $\sum_{i=1}^{n} |X_i|/a_n \to 0$ *a.s.*

With Theorem 5 and the second Borel-Cantelli lemma for pairwise NQD random variables (see Corollary 16) in hand, we can get the following result for pairwise NQD random variables.

**Corollary 6.** *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with identical distribution and $E|X| = \infty$. Then,*

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} |X_i| = 0 \quad \text{a.s. iff} \quad \sum_{n=1}^{\infty} P\left(|X| > a_n\right) < \infty;$$
$$\limsup_{n\to\infty} \frac{|S_n|}{a_n} = \infty \quad \text{a.s. iff} \quad \sum_{n=1}^{\infty} P\left(|X| > a_n\right) = \infty. \tag{4}$$

*Remark 7.* Theorems 2 and 3 deal with the complete convergence for pairwise NQD random variables. Theorems 4 and 5 deal with the strong laws of large numbers for pairwise NQD random variables, which are equivalent to the mild condition $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$. Pairwise NQD is a very wide dependence structure, which includes independent sequence as a special case. Hence, Theorems 2–5 generalize the corresponding ones for pairwise i.i.d. random variables to the case of pairwise NQD random variables.

*Remark 8.* Under the conditions of Theorem 3 and $a_{2n} \leq Ca_n$, we can get the Marcinkiewicz-Zygmund-type strong law of large numbers for pairwise NQD random variables as follows:

$$\frac{1}{a_n} \sum_{i=1}^{n} X_i \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5}$$

*Remark 9.* For a sequence $\{X, X_n, n \geq 1\}$ of pairwise i.i.d. random variables with $E|X| < \infty$, Etemadi [2] proved that $\sum_{i=1}^{n} (X_i - EX_i)/n \to 0$ a.s. Note that $E|X| < \infty$ is equivalent to $\sum_{n=1}^{\infty} P(|X| > n) < \infty$ and $E|X| < \infty$ implies $\sum_{i=1}^{n} EX_i I(|X_i| > i)/n \to 0$. Hence, Etemadi's strong law of large numbers follows from Theorem 4 with $a_n = n$.

*Remark 10.* Note that $\limsup_{n\to\infty}(|S_n|/a_n) = \infty$ a.s. is equivalent to $P(|S_n| > \alpha a_n, \text{i.o.}) = 1$ for any $\alpha > 0$. Hence, Corollary 6 improves the corresponding result of Kruglov [32].

Throughout the paper, let $I(A)$ be the indicator function of the set $A$. $C$ denotes a positive constant not depending on $n$, which may be different in various places. Denote $a_0 = 0$, $x^+ = xI(x \geq 0)$, and $x^- = -xI(x < 0)$.

## 2. Preliminaries

In this section, we will present some important lemmas which will be used to prove the main results of the paper.

The first three lemmas come from Sung [1].

**Lemma 11** (cf.[1]). *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Then the following properties hold.*

(i) $\{a_n, n \geq 1\}$ *is a strictly increasing sequence with an* $a_n \uparrow \infty$.

(ii) $\sum_{n=1}^{\infty} P(X > a_n) < \infty$ *if and only if* $\sum_{n=1}^{\infty} P(X > 2a_n) < \infty$.

(iii) $\sum_{n=1}^{\infty} P(X > a_n) < \infty$ *if and only if* $\sum_{n=1}^{\infty} P(X > \alpha a_n) < \infty$ *for any* $\alpha > 0$.

**Lemma 12** (cf. [1]). *If $\{a_n, n \geq 1\}$ is a sequence of positive constants with $a_n/n \uparrow$ and $X$ is a random variable, then*

$$\frac{n}{a_n} E|X| I(|X| \leq a_n) \leq \sum_{n=0}^{\infty} P(|X| > a_n). \quad (6)$$

**Lemma 13** (cf. [1]). *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow \infty$ and $X$ is a random variable. If $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then $(n/a_n)E|X|I(|X| \leq a_n) \to 0$.*

The next one is the basic property for pairwise NQD random variables, which was given by Lehmann [8] as follows.

**Lemma 14** (cf. [8]). *Let $X$ and $Y$ be NQD; then*

(i) $EXY \leq EXEY$;

(ii) $P(X > x, Y > y) \leq P(X > x)P(Y > y)$, *for any* $x, y \in R$;

(iii) *if $f$ and $g$ are both nondecreasing (or nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.*

The following one is the generalized Borel-Cantelli lemma, which was obtained by Matula [10].

**Lemma 15** (cf. [10]). *Let $\{A_n, n \geq 1\}$ be a sequence of events.*

(i) *If* $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then* $P(A_n, i.o.) = 0$.

(ii) *If* $P(A_k A_m) \leq P(A_k)P(A_m)$ *for* $k \neq m$ *and* $\sum_{n=1}^{\infty} P(A_n) = \infty$, *then* $P(A_n, i.o.) = 1$.

With the generalized Borel-Cantelli lemma accounted for, we can establish the second Borel-Cantelli lemma for pairwise NQD random variables as follows.

**Corollary 16** (second Borel-Cantelli lemma for pairwise NQD random variables). *Let $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n/n \uparrow$. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. Then*

$$\frac{X_n}{a_n} \longrightarrow 0 \quad a.s. \Longleftrightarrow \sum_{n=1}^{\infty} P(|X_n| > a_n) < \infty. \quad (7)$$

*Proof.* "$\Leftarrow$". By Lemma 11, $\sum_{n=1}^{\infty} P(|X_n| > a_n) < \infty$ is equivalent to $\sum_{n=1}^{\infty} P(|X_n| > a_n\varepsilon) < \infty$ for all $\varepsilon > 0$, which yields that $X_n/a_n \to 0$ a.s. by Borel-Cantelli lemma.

$\Rightarrow$. Let $X_n/a_n \to 0$ a.s., which implies that $X_n^+/a_n \to 0$ a.s. and $X_n^-/a_n \to 0$ a.s.

For any $\varepsilon > 0$, denote

$$A_n(1) = \left\{ \frac{X_n^+}{a_n} > \frac{\varepsilon}{2} \right\}, \qquad A_n(2) = \left\{ \frac{X_n^-}{a_n} > \frac{\varepsilon}{2} \right\}. \quad (8)$$

Hence,

$$P\{A_n(j), \text{i.o.}\} = 0, \quad j = 1, 2. \quad (9)$$

By Lemma 14(iii), we can see that $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ are both sequences of pairwise NQD random variables. It follows by Lemma 14(ii) that, for any $k \neq m$,

$$P(A_k(j) A_m(j)) \leq P(A_k(j)) P(A_m(j)), \quad j = 1, 2. \quad (10)$$

By Lemma 15(ii) and (9)-(10), we can see that $\sum_{n=1}^{\infty} P(A_n(j)) < \infty$ for $j = 1, 2$. Hence,

$$\sum_{n=1}^{\infty} P(|X_n| > a_n\varepsilon) \leq \sum_{n=1}^{\infty} P(A_n(1)) + \sum_{n=1}^{\infty} P(A_n(2)) < \infty \quad \text{for any } \varepsilon > 0, \quad (11)$$

which is equivalent to $\sum_{n=1}^{\infty} P(|X_n| > a_n) < \infty$ by Lemma 11. This completes the proof of the corollary. □

The last one is the Kolmogorov-type strong law of large numbers for pairwise NQD random variables obtained by Chen [23], which plays an important role in proving the main results of the paper.

**Lemma 17** (cf. [23]). *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with* $\text{Var}(X_n) < \infty$ *for each $n \geq 1$. Let $\{a_n, n \geq 1\}$ be a sequence of real numbers satisfying $0 < a_n \uparrow \infty$. Suppose that*

(i) $\sup_{n \geq 1} a_n^{-1} \sum_{i=1}^{n} E|X_i - EX_i| < \infty$;

(ii) $\sum_{n=1}^{\infty} \text{Var}(X_n)/a_n^2 < \infty$.

*Then* $a_n^{-1} \sum_{i=1}^{n} (X_i - EX_i) \to 0$ *a.s.*

## 3. Proofs of Theorems 2–5

*Proof of Theorem 2.* Note that the condition $a_n/n \uparrow$ implies that

$$\sum_{n=i}^{\infty} \frac{1}{a_n^2} \leq \sum_{n=i}^{\infty} \frac{i^2}{a_i^2 n^2} \leq \frac{i^2}{a_i^2} \sum_{n=i}^{\infty} \frac{1}{n^2} \leq \frac{i^2}{a_i^2} \cdot \frac{2}{i} = \frac{2i}{a_i^2}. \quad (12)$$

For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$Y_i = -a_n I(X_i < -a_n) + X_i I(|X_i| \leq a_n) + a_n I(X_i > a_n). \quad (13)$$

It is easily checked that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n}\left[X_i - EX_i I\left(\left|X_i\right| \le a_n\right)\right]\right| > a_n \varepsilon\right) \\
&\quad \le \sum_{n=1}^{\infty} n^{-1} P\left(\bigcup_{i=1}^{n}\left(\left|X_i\right| > a_n\right)\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n}\left(Y_i - EX_i I\left(\left|X_i\right| \le a_n\right)\right)\right| > a_n \epsilon\right) \\
&\quad \le \sum_{n=1}^{\infty} P\left(|X| > a_n\right) + \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n}\left(Y_i - EY_i\right)\right| > \frac{a_n \epsilon}{2}\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n}\left(EY_i - EX_i I\left(\left|X_i\right| \le a_n\right)\right)\right| > \frac{a_n \epsilon}{2}\right) \\
&\quad \doteq I_1 + I_2 + I_3.
\end{aligned} \tag{14}$$

To prove the desired result (2), it suffices to show $I_j < \infty$ for $j = 1, 2, 3$. Note that $I_1 < \infty$; we only need to prove $I_2 < \infty$ and $I_3 < \infty$.

Note that $\{Y_i - EY_i, 1 \le i \le n\}$ are pairwise NQD random variables by Lemma 14(iii); we have by Markov's inequality, Lemma 14(i), and the assumption $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$ that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n}\left(Y_i - EY_i\right)\right| > \frac{a_n \epsilon}{2}\right) \\
&\le C \sum_{n=1}^{\infty} n^{-1} a_n^{-2} \sum_{i=1}^{n} E\left(Y_i - EY_i\right)^2 \le C \sum_{n=1}^{\infty} a_n^{-2} EY_1^2 \\
&\le C \sum_{n=1}^{\infty} a_n^{-2} EX^2 I\left(|X| \le a_n\right) + C \sum_{n=1}^{\infty} P\left(|X| > a_n\right) \\
&\le C \sum_{n=1}^{\infty} a_n^{-2} EX^2 I\left(|X| \le a_n\right) + C.
\end{aligned} \tag{15}$$

Combining with (12) and (15), we have

$$\begin{aligned}
I_2 &\le C \sum_{n=1}^{\infty} a_n^{-2} \sum_{i=1}^{n} EX^2 I\left(a_{i-1} < |X| \le a_i\right) + C \\
&= C \sum_{i=1}^{\infty} EX^2 I\left(a_{i-1} < |X| \le a_i\right) \sum_{n=i}^{\infty} a_n^{-2} + C \\
&\le C \sum_{i=1}^{\infty} EX^2 I\left(a_{i-1} < |X| \le a_i\right) i a_i^{-2} + C \\
&\le C \sum_{i=1}^{\infty} i P\left(a_{i-1} < |X| \le a_i\right) + C \\
&\le C \sum_{i=0}^{\infty} P\left(|X| > a_i\right) + C < \infty.
\end{aligned} \tag{16}$$

Finally, we will prove $I_3 < \infty$. It is easily seen that

$$\begin{aligned}
I_3 &= \sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n}\left(EY_i - EX_i I\left(\left|X_i\right| \le a_n\right)\right)\right| > \frac{a_n \epsilon}{2}\right) \\
&\le \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^{n} P\left(\left|X_i\right| > a_n\right) > \frac{\epsilon}{2}\right) \\
&= \sum_{n=1}^{\infty} n^{-1} P\left(nP\left(|X| > a_n\right) > \frac{\epsilon}{2}\right).
\end{aligned} \tag{17}$$

In the following we prove $nP(|X| > a_n) \to 0$. Note that $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$ and $0 \le P(|X| > a_n) \downarrow$ as $n \uparrow$; we have $P(|X| > a_n) = o(1/n)$, which implies that $nP(|X| > a_n) \to 0$. Hence, $I_3 < \infty$. This completes the proof of the theorem. □

*Proof of Theorem 3.* We use the same notations as those in Theorem 2. It is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le k \le n}\left|S_k\right| > a_n \varepsilon\right) \\
&\le \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(\left|X_i\right| > a_n\right) \\
&\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le k \le n}\left|\sum_{i=1}^{k} X_i I\left(\left|X_i\right| \le a_n\right)\right| > a_n \epsilon\right) \\
&\le \sum_{n=1}^{\infty} P\left(|X| > a_n\right) \\
&\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^{n}\left|X_i I\left(\left|X_i\right| \le a_n\right)\right| > a_n \epsilon\right) \\
&= \sum_{n=1}^{\infty} P\left(|X| > a_n\right) \\
&\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^{n}\left|X_i I\left(\left|X_i\right| \le a_n\right) - Y_i + Y_i - EX_i I\right.\right. \\
&\qquad\qquad \left.\left. \times \left(\left|X_i\right| \le a_n\right) + EX_i I\left(\left|X_i\right| \le a_n\right)\right| > a_n \epsilon\right) \\
&\le C + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^{n}\left|a_n I\left(X_i < -a_n\right) - a_n I\left(X_i > a_n\right)\right|\right. \\
&\qquad\qquad \left. > \frac{a_n \epsilon}{3}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^{n}\left|Y_i - EX_i I\left(\left|X_i\right| \le a_n\right)\right| > \frac{a_n \epsilon}{3}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{-1} P\left(\sum_{i=1}^{n}\left|EX_i I\left(\left|X_i\right| \le a_n\right)\right| > \frac{a_n \epsilon}{3}\right)
\end{aligned}$$

$$\begin{aligned}&\leq C+\sum_{n=1}^{\infty}n^{-1}P\left(\sum_{i=1}^{n}I\left(\left|X_i\right|>a_n\right)>\frac{\epsilon}{3}\right)\\&\quad+\sum_{n=1}^{\infty}n^{-1}P\left(\sum_{i=1}^{n}\left|Y_i-EX_iI\left(\left|X_i\right|\leq a_n\right)\right|>\frac{a_n\epsilon}{3}\right)\\&\quad+\sum_{n=1}^{\infty}n^{-1}P\left(nE\left|X\right|I\left(\left|X\right|\leq a_n\right)>\frac{a_n\epsilon}{3}\right)\\&\doteq C+J_1+J_2+J_3.\end{aligned}\tag{18}$$

To prove the desired result (3), it remains to show $J_i<\infty$ for $i=1,2,3$.

By Markov's inequality and the assumption $\sum_{n=1}^{\infty}P(|X|>a_n)<\infty$, we have

$$J_1\leq C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}P\left(\left|X_i\right|>a_n\right)=C\sum_{n=1}^{\infty}P\left(\left|X\right|>a_n\right)<\infty.\tag{19}$$

By the assumptions of Theorem 3 and Lemma 13, we have $(n/a_n)E|X|I(|X|\leq a_n)\rightarrow 0$, which implies that $J_3<\infty$.

In the following, we will prove $J_2<\infty$. It is easily checked that

$$\begin{aligned}J_2&\leq\sum_{n=1}^{\infty}n^{-1}P\left(\sum_{i=1}^{n}\left|Y_i-EY_i\right|>\frac{a_n\epsilon}{6}\right)\\&\quad+\sum_{n=1}^{\infty}n^{-1}P\left(\sum_{i=1}^{n}\left|-P\left(X_i<-a_n\right)\right.\right.\\&\qquad\left.\left.+P\left(X_i>-a_n\right)\right|>\frac{\epsilon}{6}\right)\\&\leq\sum_{n=1}^{\infty}n^{-1}P\left(\sum_{i=1}^{n}\left(Y_i-EY_i\right)^{+}>\frac{a_n\epsilon}{12}\right)\\&\quad+\sum_{n=1}^{\infty}n^{-1}P\left(\sum_{i=1}^{n}\left(Y_i-EY_i\right)^{-}>\frac{a_n\epsilon}{12}\right)\\&\quad+\sum_{n=1}^{\infty}n^{-1}P\left(nP\left(\left|X\right|>a_n\right)>\frac{\epsilon}{6}\right)\\&\doteq J_{21}+J_{22}+J_{23}.\end{aligned}\tag{20}$$

Similar to the proof of $I_3<\infty$ in Theorem 2, we can get that $J_{23}<\infty$.

Note that, for fixed $n\geq 1$, $\{(Y_i-EY_i)^{+},1\leq i\leq n\}$ and $\{(Y_i-EY_i)^{-},1\leq i\leq n\}$ are both pairwise NQD random variables. Hence, similar to the proof of $I_2<\infty$ in Theorem 2, we have

$$\begin{aligned}J_{21}&\leq C\sum_{n=1}^{\infty}n^{-1}a_n^{-2}\sum_{i=1}^{n}E\left[\left(Y_i-EY_i\right)^{+}\right]^2\\&\leq C\sum_{n=1}^{\infty}a_n^{-2}EY_1^2<\infty.\end{aligned}\tag{21}$$

Similarly, we have $J_{22}<\infty$. Therefore, $J_2<\infty$ follows by the statements above. This completes the proof of the theorem. □

*Proof of Theorem 4.* Firstly, we will prove that (i) ⇒ (ii).

For fixed $n\geq 1$, denote

$$Y_n=-a_nI\left(X_n<-a_n\right)+X_nI\left(\left|X_n\right|\leq a_n\right)+a_nI\left(X_n>a_n\right).\tag{22}$$

Similar to the proof of $I_2<\infty$ in Theorem 2, we have

$$\begin{aligned}\sum_{n=1}^{\infty}a_n^{-2}\operatorname{Var}\left(Y_n\right)&\leq\sum_{n=1}^{\infty}a_n^{-2}EY_n^2\\&\leq\sum_{n=1}^{\infty}a_n^{-2}EX^2I\left(\left|X\right|\leq a_n\right)+\sum_{n=1}^{\infty}P\left(\left|X\right|>a_n\right)\\&\leq C\sum_{n=0}^{\infty}P\left(\left|X\right|>a_n\right)+C<\infty.\end{aligned}\tag{23}$$

It follows by Lemma 12 that

$$\begin{aligned}&\sup_{n\geq 1}a_n^{-1}\sum_{i=1}^{n}E\left|Y_i-EY_i\right|\\&\quad\leq 2\sup_{n\geq 1}a_n^{-1}\sum_{i=1}^{n}E\left|Y_i\right|\\&\quad\leq 2\sum_{i=1}^{\infty}P\left(\left|X_i\right|>a_i\right)+2\sup_{n\geq 1}na_n^{-1}E\left|X\right|I\left(\left|X\right|\leq a_n\right)\\&\quad\leq C\sum_{n=0}^{\infty}P\left(\left|X\right|>a_n\right)<\infty.\end{aligned}\tag{24}$$

Since $\operatorname{Var}(Y_n)\leq a_n^2<\infty$ for each $n\geq 1$, we have by (23) and (24) and Lemma 17 that

$$\frac{1}{a_n}\sum_{i=1}^{n}\left(Y_i-EY_i\right)\longrightarrow 0\quad\text{a.s.}\tag{25}$$

Note that

$$\begin{aligned}&\frac{1}{a_n}\sum_{i=1}^{n}\left(Y_i-EY_i\right)\\&\quad=\frac{1}{a_n}\sum_{i=1}^{n}\left[X_iI\left(\left|X_i\right|\leq a_i\right)-EX_iI\left(\left|X_i\right|\leq a_i\right)\right]\\&\qquad+\frac{1}{a_n}\sum_{i=1}^{n}\left[a_iI\left(X_i>a_i\right)-a_iI\left(X_i<-a_i\right)-a_iP\left(X_i>a_i\right)\right.\\&\qquad\left.+a_iP\left(X_i<-a_i\right)\right],\end{aligned}\tag{26}$$

and the assumption $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$ implies that $\sum_{n=1}^{\infty} I(|X_n| > a_n) < \infty$ a.s.; we can get that

$$\left|\sum_{n=1}^{\infty} \frac{1}{a_n} (a_n I(X_n > a_n) - a_n I(X_n < -a_n) - a_n P(X_n > a_n) + a_n P(X_n < -a_n))\right| \leq \sum_{n=1}^{\infty} I(|X_n| > a_n) + \sum_{n=1}^{\infty} P(|X_n| > a_n) < \infty \quad \text{a.s.,} \tag{27}$$

which together with Kronecker's lemma yield that

$$\frac{1}{a_n}\sum_{i=1}^{n} [a_i I(X_i > a_i) - a_i I(X_i < -a_i) - a_i P(X_i > a_i) + a_i P(X_i < -a_i)] \longrightarrow 0 \quad \text{a.s.} \tag{28}$$

By (26) and (28), we have

$$\frac{1}{a_n}\sum_{i=1}^{n} (X_i I(|X_i| \leq a_i) - EX_i I(|X_i| \leq a_i)) \longrightarrow 0 \quad \text{a.s.} \tag{29}$$

It follows by the assumption $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$ again that

$$\frac{1}{a_n}\sum_{i=1}^{n} X_i I(|X_i| > a_i) \longrightarrow 0 \quad \text{a.s.} \tag{30}$$

Therefore, the desired result (ii) follows by (29) and (30) immediately.

Next, we will prove that (ii) ⇒ (i). Assume that

$$a_n^{-1}\sum_{i=1}^{n} [X_i - EX_i I(|X_i| \leq a_i)] \longrightarrow 0 \quad \text{a.s.} \tag{31}$$

Then, we have

$$\begin{aligned}\frac{X_n - EX_n I(|X_n| \leq a_n)}{a_n} &= \frac{1}{a_n}\sum_{i=1}^{n} [X_i - EX_i I(|X_i| \leq a_i)] - \frac{a_{n-1}}{a_n}\frac{1}{a_{n-1}}\sum_{i=1}^{n-1} [X_i - EX_i I(|X_i| \leq a_i)] \\ &\longrightarrow 0 \quad \text{a.s.}\end{aligned} \tag{32}$$

Note that

$$\begin{aligned}\frac{E|X_n| I(|X_n| \leq a_n)}{a_n} &= \frac{E|X| (I(|X| \leq a_N) + I(a_N < |X| \leq a_n))}{a_n} \\ &\leq \frac{a_N}{a_n} P(|X| \leq a_N) + P(a_N < |X| \leq a_n) \\ &\leq \frac{a_N}{a_n} + P(|X| > a_N) \xrightarrow{n\to\infty} P(|X| > a_N) \longrightarrow 0,\end{aligned} \tag{33}$$

as $N \to \infty$, which implies that

$$\frac{EX_n I(|X_n| \leq a_n)}{a_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{34}$$

It follows by (32) and (34) that $X_n/a_n \to 0$ a.s., which is equivalent to (i) by Corollary 16. The proof is completed. □

*Proof of Theorem 5.* Firstly, we will prove that (ii) ⇒ (i). It follows by (ii) that

$$\frac{X_n}{a_n} = \frac{1}{a_n}\sum_{i=1}^{n} X_i - \frac{a_{n-1}}{a_n}\frac{1}{a_{n-1}}\sum_{i=1}^{n-1} X_i \longrightarrow 0 \quad \text{a.s.,} \tag{35}$$

which together with Corollary 16 imply that (i) holds.

On the other hand, assume that $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$; it follows by Lemma 13 that

$$\frac{1}{a_n}\left|\sum_{i=1}^{n} EX_i I(|X_i| \leq a_n)\right| \leq \frac{n}{a_n} E|X| I(|X| \leq a_n) \longrightarrow 0 \quad \text{a.s.} \tag{36}$$

The desired result (ii) follows by Theorem 4 and (36) immediately.

We have proved that (i) ⇔ (ii); next we prove (i) ⇔ (iii). It follows by Lemma 11(iii) that, for any $\epsilon > 0$,

$$\begin{aligned}&\sum_{n=1}^{\infty} P(|X| > a_n) < \infty \\ &\Longleftrightarrow \sum_{n=1}^{\infty} P(|X| > a_n\epsilon) < \infty \\ &\Longleftrightarrow \sum_{n=1}^{\infty} P(X^+ > a_n\epsilon) < \infty, \sum_{n=1}^{\infty} P(X^- > a_n\epsilon) < \infty \\ &\Longleftrightarrow \sum_{n=1}^{\infty} P(X^+ > a_n) < \infty, \sum_{n=1}^{\infty} P(X^- > a_n) < \infty.\end{aligned} \tag{37}$$

On the other hand, we have proved that (i) ⇔ (ii); hence,

$$\begin{aligned}&\sum_{n=1}^{\infty} P(X^+ > a_n) < \infty, \sum_{n=1}^{\infty} P(X^- > a_n) < \infty \\ &\Longleftrightarrow \sum_{i=1}^{n}\frac{X_i^+}{a_n} \longrightarrow 0 \quad \text{a.s.,} \quad \sum_{i=1}^{n}\frac{X_i^-}{a_n} \longrightarrow 0 \quad \text{a.s.} \\ &\Longleftrightarrow \sum_{i=1}^{n}\frac{|X_i|}{a_n} \longrightarrow 0 \quad \text{a.s.}\end{aligned} \tag{38}$$

Therefore, (i) ⇔ (iii) follows by the statements above immediately. This completes the proof of the theorem. □

*Proof of Corollary 6.* The techniques used here are the second Borel-Cantelli lemma for pairwise NQD random variables (see Corollary 16) and Theorem 5. The proof is similar to that of Corollary 2.1 of Sung [1], so the details of the proof are omitted. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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