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Research Article

$\mathcal{Q}_K$ Spaces of Several Real Variables

Guanlong Bao and Hasi Wulan

Department of Mathematics, Shantou University, Shantou 515063, China

Correspondence should be addressed to Hasi Wulan; wulan@stu.edu.cn

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We introduce a new space, $\mathcal{Q}_K(\mathbb{R}^n)$ space, of several real variables with nondecreasing functions K. By giving basic properties of the weighted function K, by establishing a Stegenga-type estimate, and by introducing the K-Carleson measure on $\mathbb{R}^{n+1}_+$, we obtain various characterizations of $\mathcal{Q}_K(\mathbb{R}^n)$ space.

1. Introduction

Recall that a locally integrable function f belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_{I\subset\mathbb{R}^n}|I|^{-1}\int_I |f(x)-f_I|\,dx<\infty, \tag{1}$$

where I denotes a cube in $\mathbb{R}^n$ with edges parallel to the coordinate axes and $|I|$ denotes the Lebesgue measure of I and

$$f_I = |I|^{-1}\int_I f(x)\,dx. \tag{2}$$

Via the John-Nirenberg inequality [1], one can show an equivalent condition of $\mathrm{BMO}(\mathbb{R}^n)$ as follows:

$$\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} \approx \sup_I |I|^{-2}\iint_I |f(x)-f(y)|^2\,dx\,dy. \tag{3}$$

C. Fefferman's famous equation, $(H^1)^* = \mathrm{BMO}$, describes a deep relation between BMO and the Hardy space (cf. [2, 3]). This leads quite naturally to increased study of these functions from the point of real variable theory and complex function theory views in the recent fifty years. See [2–9] for more results about $\mathrm{BMO}(\mathbb{R}^n)$ space.

As a generalization of $\mathrm{BMO}(\mathbb{R}^n)$, the space $\mathcal{Q}_\alpha(\mathbb{R}^n)$, $\alpha\in\mathbb{R}$, introduced by Essén et al. in [10], is defined to be the class of all locally integrable functions $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|^2_{\mathcal{Q}_\alpha(\mathbb{R}^n)} = \sup_I [\ell(I)]^{2\alpha-n}\iint_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy<\infty, \tag{4}$$

where $\ell(I)=|I|^{1/n}$ denotes the edge length of the cube I.

It is easy to see that $\mathcal{Q}_\alpha(\mathbb{R}^n)$ is always a subclass of $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathcal{Q}_\alpha(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ by choosing $\alpha=-n/2$. Moreover, we know by [10] that $\mathcal{Q}_\alpha(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$ if and only if $\alpha<0$. Also, we see that $\mathcal{Q}_\alpha(\mathbb{R})$ is trivial (containing a.e. constant functions only) if and only if $\alpha>1/2$ and $\mathcal{Q}_\alpha(\mathbb{R}^n)$, $n\ge 2$, is trivial if and only if $\alpha\ge 1$.

In this paper, we introduce and develop a more general space $\mathcal{Q}_K(\mathbb{R}^n)$ of several real variables, which can be viewed as an extension and improvement of $\mathcal{Q}_\alpha(\mathbb{R}^n)$ spaces as well as $\mathrm{BMO}(\mathbb{R}^n)$. A theory of $\mathcal{Q}_K(\mathbb{D})$ spaces on unit disc $\mathbb{D}$ has been extensively studied for recent years in the context of a wide class of function spaces; see, for example, [11–15]. Motivated by the theory of analytic $\mathcal{Q}_K(\mathbb{D})$ spaces, we define the following.

Definition 1. Let $K:[0,\infty)\to[0,\infty)$ be a nondecreasing function. A function $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the space $\mathcal{Q}_K(\mathbb{R}^n)$ if

$$\|f\|^2_{\mathcal{Q}_K(\mathbb{R}^n)} = \sup_{I\subset\mathbb{R}^n}\iint_I \frac{|f(x)-f(y)|^2}{|x-y|^{2n}}K\left(\frac{|x-y|}{\ell(I)}\right)dx\,dy<\infty. \tag{5}$$

If we take $K(t) = t^{n-2\alpha}$, for $\alpha \in \mathbb{R}$, then $\mathcal{Q}_K(\mathbb{R}^n) = \mathcal{Q}_\alpha(\mathbb{R}^n)$. Modulo constants, $\mathcal{Q}_K(\mathbb{R}^n)$ is a Banach space under the norm defined in (5).

Our paper is organized as follows.

In Section 2, we investigate the relationship between $\mathcal{Q}_K(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ and give a sufficient and necessary condition for space $\mathcal{Q}_K(\mathbb{R}^n)$ which is nontrivial.

In Section 3, we give several results about the weight function K on which $\mathcal{Q}_K(\mathbb{R}^n)$ obviously depends. In the study of $\mathcal{Q}_K(\mathbb{R}^n)$, the auxiliary function φ_K defined by

$$\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty, \tag{6}$$

still works well as the analytic $\mathcal{Q}_K(\mathbb{D})$ spaces.

In Section 4, we define the K-Carleson measure on $\mathbb{R}^{n+1}_+$. By establishing a Stegenga-type estimate, we obtain a characterization of $\mathcal{Q}_K(\mathbb{R}^n)$ spaces in terms of the K-Carleson measure.

Throughout this note, $a \lesssim b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \lesssim b$ and $b \lesssim a$ hold, then one says that $a \approx b$. For the convenience of calculation, in this paper, we always assume that $K : [0,\infty) \to [0,\infty)$ is nondecreasing and $K(2t) \approx K(t)$.

2. Basic Properties of $\mathcal{Q}_K(\mathbb{R}^n)$

Our first observation is that $\mathcal{Q}_K(\mathbb{R}^n)$ is invariant under the conformal mappings and rotations; that is, for any conformal map $\phi(x) = \lambda x + x_0$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$, or any rotation $\psi(x) = xM$ for an orthogonal matrix M of order n,

$$\begin{aligned} \|f\circ\phi\|_{\mathcal{Q}_K(\mathbb{R}^n)} &= \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)}, \\ \|f\circ\psi\|_{\mathcal{Q}_K(\mathbb{R}^n)} &\approx \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)} \end{aligned} \tag{7}$$

hold for any $f \in \mathcal{Q}_K(\mathbb{R}^n)$.

We say that the space $\mathcal{Q}_K(\mathbb{R}^n)$ is trivial if $\mathcal{Q}_K(\mathbb{R}^n)$ contains only a.e. constant functions. To discuss this problem, we recall the space $\mathrm{CIS}(\mathbb{R}^n)$, the class of all functions $f \in C^1(\mathbb{R}^n)$, with

$$\|f\|^2_{\mathrm{CIS}} = \sup_I |I|^{((2-n)/n)} \int_I |\nabla f(x)|^2 dx < \infty, \tag{8}$$

where $\nabla f(x) = (\partial f(x)/\partial x_1, \dots, \partial f(x)/\partial x_n)$. By [16], we know that

$$\left(1+|x|^2\right)^{-1} \in \mathrm{CIS}\left(\mathbb{R}^2\right), \quad \ln\left(1+|x|^2\right) \in \mathrm{CIS}\left(\mathbb{R}^n\right), \quad n > 2. \tag{9}$$

Thus $\mathrm{CIS}(\mathbb{R}^n)$ is not trivial for $n \ge 2$. However, $\mathrm{CIS}(\mathbb{R})$ is trivial.

For any cube I, if $x, y \in I$, then $|x-y| \le \sqrt{n}\ell(I)$. For $t > 0$, tI means that the cube has the same center as I and the edge length $t\ell(I)$. If $x \in I$ and $|y| < \ell(I)$, then $x + y \in 3I$. By the change of variable, $f \in \mathcal{Q}_K(\mathbb{R}^n)$ if and only if

$$\sup_I \int_{|y|<\ell(I)} \int_I |f(x+y) - f(x)|^2 \times K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} dx\, dy < \infty. \tag{10}$$

Theorem 2. *The following statements are true.*

(a) $\mathcal{Q}_{1/2}(\mathbb{R}) \subseteq \mathcal{Q}_K(\mathbb{R})$. *Moreover,* $\mathcal{Q}_K(\mathbb{R})$ *is never trivial.*

(b) *For* $n \ge 2$, $\mathcal{Q}_K(\mathbb{R}^n)$ *is not trivial if and only if*

$$\int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt < +\infty. \tag{11}$$

Moreover, if (11) *holds, then*

$$CIS(\mathbb{R}^n) \subset \mathcal{Q}_K(\mathbb{R}^n). \tag{12}$$

Proof. (a) For any cube I of $\mathbb{R}$ and $x, y \in I$, we have

$$|x-y| \le \ell(I). \tag{13}$$

By assumption on K we have

$$\begin{aligned} &\iint_I \frac{|f(x)-f(y)|^2}{|x-y|^2} K\left(\frac{|x-y|}{\ell(I)}\right) dx\, dy \\ &\quad \le K(1) \iint_I \frac{|f(x)-f(y)|^2}{|x-y|^2} dx\, dy. \end{aligned} \tag{14}$$

Hence, $\mathcal{Q}_{1/2}(\mathbb{R}) \subseteq \mathcal{Q}_K(\mathbb{R})$. $\mathcal{Q}_{1/2}(\mathbb{R})$ is not trivial and so is $\mathcal{Q}_K(\mathbb{R})$.

(b) *Necessity.* It is enough to show that if

$$\int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt = +\infty, \tag{15}$$

then $\mathcal{Q}_K(\mathbb{R}^n)$ is trivial. We will prove the necessity by two steps.

Step 1. If $f \in \mathcal{Q}_K(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ and f is nonconstant, we may assume that f is real. Then there exists a point $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$ such that $\nabla f(x_0) \neq 0$. By the Householder reflector [17, p. 71], there exists an orthogonal matrix $M = (a_{ij})$, $i, j = 1, 2, \dots, n$, such that

$$\nabla f(x_0) M = (|\nabla f(x_0)|, 0, \dots, 0), \tag{16}$$

which gives

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) a_{i1} = |\nabla f(x_0)|, \quad \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) a_{ij} = 0, \quad j = 2, 3, \dots, n. \tag{17}$$

Denote by M^T the transpose of the matrix M. Set $g(x) = f(xM^T)$. Since $\det(M^T) \neq 0$, there exists a point

$y_0 = (y_1^0, y_2^0, \ldots, y_n^0)$ such that $y_0 M^T = x_0$. Write $y = xM^T$ for convenience as follows:

$$y_j = \sum_{i=1}^{n} x_i a_{ji}, \quad j = 1, 2, \ldots, n. \tag{18}$$

Consequently,

$$\frac{\partial g}{\partial x_1}(y_0) = \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(y_0 M^T) a_{j1} = |\nabla f(x_0)|. \tag{19}$$

Similarly,

$$\frac{\partial g}{\partial x_i}(y_0) = 0, \quad i = 2, 3, \ldots, n. \tag{20}$$

Thus

$$\nabla g(y_0) = (|\nabla f(x_0)|, 0, \ldots, 0). \tag{21}$$

Note that $g \in C^1(\mathbb{R}^n)$. Then there exist a positive constant δ and a small cube I centered at y_0 on which $\partial g(x)/\partial x_1 > 2\delta$ and $\partial g(x)/\partial x_j < \delta,\ j \geq 2$. Define

$$D = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_2| + \cdots + |x_n| < x_1 < \frac{\ell(I)}{4} \right\}. \tag{22}$$

If $x, y \in I$ and $x - y \in D$, using the mean value theorem, we get

$$g(x) - g(y) > \delta(x_1 - y_1). \tag{23}$$

Thus

$$\begin{aligned} &\iint_I \frac{|g(x) - g(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\, dy \\ &\geq \int_{I/2} dx \int_{I-x} \frac{|g(x+z) - g(x)|^2}{|z|^{2n}} K\left(\frac{|z|}{\ell(I)}\right) dz \\ &\geq \int_{I/2} dx \int_D \frac{\delta^2 |z_1|^2}{|z|^{2n}} K\left(\frac{|z|}{\ell(I)}\right) dz. \end{aligned} \tag{24}$$

If $z \in D$, then $|z| \approx z_1$. Hence

$$\begin{aligned} &\iint_I \frac{|g(x) - g(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\, dy \\ &\gtrsim \int_{I/2} dx \int_D \frac{|z_1|^2}{|z_1|^{2n}} K\left(\frac{|z_1|}{\ell(I)}\right) dz \\ &\approx \int_0^{\ell(I)/4} \frac{1}{z_1^{2n-2}} K\left(\frac{z_1}{\ell(I)}\right) dz_1 \\ &\quad \times \int_{\{(z_2,\ldots,z_n): |z_2| + \cdots + |z_n| < z_1\}} dz_2 \cdots dz_n \\ &\approx \int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt = +\infty. \end{aligned} \tag{25}$$

It means that g is not an element of $\mathcal{Q}_K(\mathbb{R}^n)$. On the other hand, $g \in \mathcal{Q}_K(\mathbb{R}^n)$ since $\mathcal{Q}_K(\mathbb{R}^n)$ is invariant under rotations. This is a contraction.

Step 2. Note that $\mathcal{Q}_K(\mathbb{R}^n)$ is conformal invariant. By Minkowski's inequality, if $f \in \mathcal{Q}_K(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in \mathcal{Q}_K(\mathbb{R}^n)$ and

$$\|f * g\|_{\mathcal{Q}_K(\mathbb{R}^n)} \leq \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)} \int_{\mathbb{R}^n} |g(y)|\, dy, \tag{26}$$

where

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy. \tag{27}$$

In particular, if $g \in C_0^\infty$, the class of smooth functions with compact support, then $f * g \in \mathcal{Q}_K(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. Thus $f * g$ is constant by Step 1. By [10], there exits a sequence $\{g_n\} \subset C_0^\infty$ with $g_n \geq 0$, $\int g_n = 1$, and $\operatorname{supp} g_n$ shrinking to 0 such that $f * g_n \to f$ a.e. It follows that f is constant a.e. Thus, we complete the proof of necessity.

Sufficiency. Give a cube I and suppose $f \in \mathrm{CIS}(\mathbb{R}^n)$. By

$$|f(x+y) - f(y)| \leq \int_0^1 |\nabla f(y + tx)|\, |x|\, dt, \tag{28}$$

we have

$$\begin{aligned} &\iint_I \frac{|f(x) - f(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\, dy \\ &\leq \int_I \int_{|z| < \sqrt{n}\ell(I)} \left(\int_0^1 |\nabla f(x + tz)|\, dt\right)^2 \\ &\quad \times \frac{1}{|z|^{2n-2}} K\left(\frac{|z|}{\ell(I)}\right) dz\, dx. \end{aligned} \tag{29}$$

It follows by Minkowski's inequality that

$$\begin{aligned} &\left(\iint_I \frac{|f(x) - f(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\, dy\right)^{1/2} \\ &\leq \int_0^1 \left(\int_I \int_{|z| < \sqrt{n}\ell(I)} |\nabla f(x + tz)|^2 \frac{1}{|z|^{2n-2}} \times K\left(\frac{|z|}{\ell(I)}\right) dz\, dx\right)^{1/2} dt \\ &\leq \left(\int_{|z| < \sqrt{n}\ell(I)} \frac{1}{|z|^{2n-2}} K\left(\frac{|z|}{\ell(I)}\right) dz \times \int_{3\sqrt{n}I} |\nabla f(w)|^2 dw\right)^{1/2} \\ &\leq \|f\|_{\mathrm{CIS}} \times \left(\ell(I)^{n-2} \int_0^{\sqrt{n}\ell(I)} \frac{1}{t^{n-1}} K\left(\frac{t}{\ell(I)}\right) dt\right)^{1/2} \\ &\approx \|f\|_{\mathrm{CIS}} \left(\int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt\right)^{1/2}. \end{aligned} \tag{30}$$

Thus, $\mathrm{CIS}(\mathbb{R}^n) \subseteq \mathcal{Q}_K(\mathbb{R}^n)$ and $\mathcal{Q}_K(\mathbb{R}^n)$ is not trivial. □

Theorem 3. *The space $\mathcal{Q}_K(\mathbb{R}^n)$ is a subset of $BMO(\mathbb{R}^n)$. Furthermore,*

(a) *if $\int_0^{\sqrt{n}}(K(t)/t^{n+1})dt < \infty$, then $\mathcal{Q}_K(\mathbb{R}^n) = BMO(\mathbb{R}^n)$;*

(b) *if $\mathcal{Q}_K(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, then, for all $\beta < 1$, $\int_0^{\sqrt{n}}(K(t)/t^{n+\beta})dt < \infty$.*

Proof. Let $f \in \mathcal{Q}_K(\mathbb{R}^n)$. For any cube I and $x, y \in I$, if $r > 0$ is small enough, we know that the Lebesgue measure of the set

$$\{z \in I : \min(|x-z|, |y-z|) > r\ell(I)\} \tag{31}$$

is bigger than $|I|(1-(4r^n\pi^{n/2}/n\Gamma(n/2)))$. Since K is nondecreasing,

$$\begin{aligned}
&\int_I \min\left(K\left(\frac{|x-z|}{\ell(I)}\right), K\left(\frac{|y-z|}{\ell(I)}\right)\right)dz \\
&\quad\geq \int_{\{z\in I:\min(|x-z|,|y-z|)>r\ell(I)\}} \min\left(K\left(\frac{|x-z|}{\ell(I)}\right),\right. \\
&\qquad\qquad \left.K\left(\frac{|y-z|}{\ell(I)}\right)\right)dz \\
&\quad\geq K(r)|I|\left(1-\frac{4r^n\pi^{n/2}}{n\Gamma(n/2)}\right).
\end{aligned} \tag{32}$$

Consequently,

$$\begin{aligned}
&K(r)\left(1-\frac{4r^n\pi^{n/2}}{n\Gamma(n/2)}\right)|I|^{-2}\iint_I |f(x)-f(y)|^2 dx\,dy \\
&\quad\leq |I|^{-3}\iiint_I |f(x)-f(y)|^2 \\
&\qquad\times \min\left(K\left(\frac{|x-z|}{\ell(I)}\right),\right. \\
&\qquad\qquad \left.K\left(\frac{|y-z|}{\ell(I)}\right)\right)dx\,dy\,dz \\
&\quad\leq 2|I|^{-3}\iiint_I |f(x)-f(z)|^2 K\left(\frac{|x-z|}{\ell(I)}\right)dx\,dy\,dz \\
&\qquad+ 2|I|^{-3}\iiint_I |f(y)-f(z)|^2 K\left(\frac{|y-z|}{\ell(I)}\right)dx\,dy\,dz \\
&\quad\leq 4n^n \iint_I \frac{|f(y)-f(z)|^2}{|y-z|^{2n}} K\left(\frac{|y-z|}{\ell(I)}\right)dy\,dz.
\end{aligned} \tag{33}$$

For a small enough $r > 0$, we obtain

$$\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \sup_I \iint_I \frac{|f(y)-f(z)|^2}{|y-z|^{2n}} K\left(\frac{|y-z|}{\ell(I)}\right)dy\,dz. \tag{34}$$

Thus $\mathcal{Q}_K(\mathbb{R}^n) \subseteq \mathrm{BMO}(\mathbb{R}^n)$.

(a) Note that

$$\begin{aligned}
\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} &\approx \sup_I |I|^{-2}\iint_I |f(x)-f(y)|^2 dx dy \\
&\approx \sup_I |I|^{-1}\int_I |f(x)-f_I|^2 dx.
\end{aligned} \tag{35}$$

For a cube I and for every $y \in \mathbb{R}^n$ with $|y| < \sqrt{n}l(I)$,

$$\begin{aligned}
&\int_I |f(x+y)-f(x)|^2 dx \\
&\quad\lesssim \int_I \left|f(x+y)-f_{3\sqrt{n}I}\right|^2 + \left|f(x)-f_{3\sqrt{n}I}\right|^2 dx \\
&\quad\lesssim |I|\,\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}
&\iint_I \frac{|f(x)-f(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right)dx\,dy \\
&\quad\leq \int_{|y|<\sqrt{n}\ell(I)}\int_I |f(x+y)-f(x)|^2 \\
&\qquad\qquad \times \frac{K(|y|/\ell(I))}{|y|^{2n}}dx\,dy \\
&\quad\lesssim |I|\,\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}\int_{|y|<\sqrt{n}\ell(I)} \frac{K(|y|/\ell(I))}{|y|^{2n}}dy \\
&\quad\approx \|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}\int_0^{\sqrt{n}} \frac{K(t)}{t^{n+1}}dt.
\end{aligned} \tag{37}$$

Thus $\mathrm{BMO}(\mathbb{R}^n) \subseteq \mathcal{Q}_K(\mathbb{R}^n)$, and this deduces that $\mathcal{Q}_K(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$.

(b) Consider the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consisting of all those C^∞ functions φ on $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n}\left|x^\alpha\left(D^\beta\varphi\right)(x)\right| < \infty \tag{38}$$

for all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $\beta = (\beta_1,\ldots,\beta_n)$ of nonnegative integers, where $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $D^\beta = (\partial/\partial x_1)^{\beta_1}\cdots(\partial/\partial x_n)^{\beta_n}$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed function such that the Fourier transform $\widehat{\varphi}$ of φ has support in the unit ball and $\varphi \neq 0$ on the cube $[-3\pi, 3\pi]^n$. Let $\{a_k\}$ be a sequence of real numbers and define

$$g(x) = \sum_{k=1}^{\infty} a_k \exp\left(2^k x_1 i\right), \tag{39}$$

where x_1 is the first coordinate of x. By [10], $f = \varphi g \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $\sum_{k=1}^{\infty} a_k^2 < \infty$.

Suppose that $\mathcal{Q}_K(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. Set $a_k^2 = (1/2^{k(1-\beta)})$ for $\beta < 1$. We know that $f \in \mathcal{Q}_K(\mathbb{R}^n)$. Choosing $I = [-\pi, \pi]^n$, we have

$$\int_{|y|<2\pi}\int_I |f(x+y)-f(x)|^2 K(|y|)\frac{1}{|y|^{2n}}dx\,dy < \infty. \tag{40}$$

Since $|\varphi(x+y)| \geq c > 0$ for $x \in I$ and $|y| < 2\pi$,

$$\begin{aligned} &|g(x+y) - g(x)|^2 \\ &\quad \lesssim |f(x+y) - f(x)|^2 \\ &\qquad + |g(x)|^2 |\varphi(x+y) - \varphi(x)|^2. \end{aligned} \tag{41}$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $|\varphi(x+y) - \varphi(x)| \lesssim |y|$. Hence,

$$\begin{aligned} &\int_{|y|<2\pi} \int_I |g(x)|^2 |\varphi(x+y) - \varphi(x)|^2 K(|y|) \frac{1}{|y|^{2n}} dx\, dy \\ &\quad \lesssim \int_{|y|<2\pi} K(|y|) \frac{1}{|y|^{2n-2}} dy \\ &\qquad \times \int_{[-\pi,\pi]^n} \left| \sum_{k=1}^{\infty} a_k \exp\left(2^k x_1 i\right) \right|^2 dx \\ &\quad \approx \sum_{k=1}^{\infty} a_k^2 \int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt < \infty. \end{aligned} \tag{42}$$

Writing $y = (y_1, y')$, $y' \in \mathbb{R}^{n-1}$, we obtain

$$\begin{aligned} \infty &> \int_{|y|<2\pi} \int_I |g(x+y) - g(x)|^2 K(|y|) \frac{1}{|y|^{2n}} dx\, dy \\ &\approx \int_{|y|<2\pi} \sum_{k=1}^{\infty} \left| a_k \left(e^{2^k y_1 i} - 1 \right) \right|^2 K(|y|) \frac{1}{|y|^{2n}} dy \\ &\gtrsim \sum_{k=1}^{\infty} a_k^2 \int_{|y'|<y_1<2^{-k}} \left|2^k y_1\right|^2 K(|y|) \frac{1}{|y|^{2n}} dy \\ &\approx \sum_{k=1}^{\infty} 2^{2k} a_k^2 \int_0^{2^{-k}} \frac{K(t)}{t^{n-1}} dt \\ &\gtrsim \int_0^{1/2} \frac{K(t)}{t^{n+\beta}} dt. \end{aligned} \tag{43}$$

□

For any cube I of $\mathbb{R}^n$, when $x, y \in I$, we have that $|x - y| \leq \sqrt{n}\ell(I)$. By the definition, the $\mathcal{Q}_K(\mathbb{R}^n)$ space depends on $K(t)$ when $0 \leq t \leq \sqrt{n}$. In fact, $\mathcal{Q}_K(\mathbb{R}^n)$ depends only on $K(t)$ when t is near origin, which can be found by the following theorem. Here the proof of the theorem is left to the reader.

Theorem 4. *The following statements are true.*

(a) *Suppose that $K(r) > 0$ for some $r > 0$. One defines $K_r(t) = \min(K(r), K(t))$. Then $\mathcal{Q}_{K_r}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.*

(b) *Let*

$$K_1(t) = \begin{cases} K(t), & 0 < t \leq 1, \\ K(1)\, t^{n-1}, & t \geq 1. \end{cases} \tag{44}$$

Then $\mathcal{Q}_{K_1}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.

Remark 5. If $K(0) > 0$, we have $\mathcal{Q}_K(\mathbb{R}^n) = \mathcal{Q}_{n/2}(\mathbb{R}^n)$. Since $\mathcal{Q}_{n/2}(\mathbb{R}^n)$ is trivial for $n \geq 2$, we only pay attention to the case $K(0) = 0$.

3. Weighted Functions

The characterization of $\mathcal{Q}_K(\mathbb{R}^n)$ depends on the properties of the weight function K obviously. In this section we give several results about the weight functions that are needed for the next section.

In the analytic $\mathcal{Q}_K(\mathbb{D})$ spaces, the auxiliary function

$$\varphi_K(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty, \tag{45}$$

plays a key role; see [12, 14, 15], for example.

To study $\mathcal{Q}_K(\mathbb{R}^n)$ spaces, we need some more constraints on K as follows:

$$\int_0^1 \varphi_K(s) \frac{ds}{\min(s, s^{n-1})} < \infty, \tag{46}$$

$$\int_1^{\infty} \varphi_K(s) \frac{ds}{s^{n+1}} < \infty. \tag{47}$$

Note that (46) implies the following two conditions:

$$\int_0^1 \varphi_K(s) \frac{ds}{s} < \infty, \tag{48}$$

$$\int_0^1 \varphi_K(s) \frac{ds}{s^{n-1}} < \infty, \quad n \geq 2. \tag{49}$$

In particular, if we choose $K(t) = t^{n-2\alpha}$, then condition (47) holds if and only if $\alpha > 0$; condition (48) holds if and only if $n > 2\alpha$; and condition (49) holds if and only if $\alpha < 1$.

Lemma 6. *Let K satisfy*

$$\sup_{0<s<b} \int_s^b \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty \tag{50}$$

for some $0 < b \leq \infty$. Then one can find another nonnegative weight function K^* with $K^*(0) = 0$ such that $\mathcal{Q}_{K^*}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$, and the new weight function K^* has the following properties.

(a) *K^* is nondecreasing on $[0, \infty)$.*

(b) *$K^* \approx K$ on $[0, \infty)$, and thus K^* satisfies condition (50).*

(c) *$K^*(2t) \approx K^*(t)$ on $[0, \infty)$.*

(d) *K^* is differentiable (up to any given order) on $(0, \infty)$.*

(e) *For some small enough $c > 0$, $K^*(t)/t^{n-c}$ is nonincreasing on $(0, \infty)$. Consequently, $K^*(t)/t^n$ is also nonincreasing on $(0, \infty)$.*

Proof. By Theorem 4, we may assume that $K(t) = K(1)t^{n-1}$ for $t \geq 1$. Since K satisfies condition (50), we claim that

$$\sup_{0<s<\infty} \int_s^{\infty} \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{51}$$

If $b = \infty$, the claim is true. If $0 < b < \infty$, the claim will be confirmed by showing

$$\sup_{0<s<b} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty,$$
$$\sup_{b\le s<\infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{52}$$

For the case of $0 < s < b$, by (50),

$$\begin{aligned}\int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt &= \left(\int_s^b \frac{K(t)}{t^{n+1}} dt + \int_b^\infty \frac{K(t)}{t^{n+1}} dt\right) \int_0^s \frac{t^{n-1}}{K(t)} dt \\ &\le C + \int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^b \frac{t^{n-1}}{K(t)} dt.\end{aligned} \tag{53}$$

Taking $s = (b/2)$ in (50), we have

$$\int_0^b \frac{t^{n-1}}{K(t)} dt < \infty. \tag{54}$$

Since $K(t) = K(1)t^{n-1}$, for $t \ge 1$, we have

$$\int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^b \frac{t^{n-1}}{K(t)} dt < \infty. \tag{55}$$

Thus

$$\sup_{0<s<b} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{56}$$

Now, we prove

$$\sup_{b\le s<\infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{57}$$

In fact, if $b \le s < \infty$, $b \ge 1$,

$$\begin{aligned}\int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt &\le K(1) \int_0^b \frac{t^{n-1}}{K(t)} dt + K(1) s^{-1} \int_b^s \frac{1}{K(1)} dt \\ &\le K(1) \int_0^b \frac{t^{n-1}}{K(t)} dt + 1.\end{aligned} \tag{58}$$

If $b \le s < \infty$, $0 < b < 1$ and $s < 1$,

$$\begin{aligned}\int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt &\le \int_b^\infty \frac{K(t)}{t^{n+1}} dt \left(\int_0^b \frac{t^{n-1}}{K(t)} dt + \int_b^1 \frac{t^{n-1}}{K(t)} dt\right) \\ &< \infty.\end{aligned} \tag{59}$$

If $b \le s < \infty$, $0 < b < 1$ and $s \ge 1$,

$$\begin{aligned}\int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt &\le \int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^1 \frac{t^{n-1}}{K(t)} dt + \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_1^s \frac{t^{n-1}}{K(t)} dt \\ &\le \int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^1 \frac{t^{n-1}}{K(t)} dt + 1.\end{aligned} \tag{60}$$

Hence,

$$\sup_{b\le s<\infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{61}$$

Therefore, our claim above has been confirmed. Let

$$K^*(t) = \begin{cases} t^n \int_t^\infty \frac{K(s)}{s^{n+1}} ds, & 0 < t < \infty, \\ 0, & t = 0. \end{cases} \tag{62}$$

Then $K^*(t) = K(1)t^{n-1}$ for $t \ge 1$.

(a) Fix $0 < t_1 < t_2 < \infty$ and consider the difference

$$\begin{aligned}K^*(t_2) - K^*(t_1) = {}& (t_2^n - t_1^n) \\ &\times \int_{t_2}^\infty \frac{K(s)}{s^{n+1}} ds - t_1^n \int_{t_1}^{t_2} \frac{K(s)}{s^{n+1}} ds.\end{aligned} \tag{63}$$

Since K is nondecreasing and nonnegative, we have

$$\begin{aligned}K^*(t_2) - K^*(t_1) \ge {}& (t_2^n - t_1^n) K(t_2) \\ &\times \int_{t_2}^\infty \frac{ds}{s^{n+1}} - t_1^n K(t_2) \int_{t_1}^{t_2} \frac{ds}{s^{n+1}} = 0.\end{aligned} \tag{64}$$

(b) Using the assumption that K is nondecreasing again, we obtain

$$K^*(t) \ge t^n K(t) \int_t^\infty \frac{ds}{s^{n+1}} = \frac{K(t)}{n} \tag{65}$$

for $0 < t < \infty$. On the other hand,

$$K^*(t) \le t^n \left(\int_0^t \frac{s^{n-1}}{K(s)} ds\right)^{-1} \lesssim K(t), \quad 0 < t < \infty. \tag{66}$$

Thus, $K^* \approx K$ on $(0, \infty)$ and $\mathcal{Q}_{K^*}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.

(c) For any $t > 0$, we have

$$\frac{K^*(2t)}{K^*(t)} = 2^n \frac{\int_{2t}^\infty (K(s)/s^{n+1})\, ds}{\int_t^\infty (K(s)/s^{n+1})\, ds} \le 2^n. \tag{67}$$

Since K^* is nondecreasing, $K^*(t) \le K^*(2t)$.

(d) If we repeat the construction $K \mapsto K^*$, then we can make the new weight function differentiable up to any desired order.

(e) Note that if $c > 0$ is sufficiently small, then we have

$$\left(t^{c-n}K^*(t)\right)' = t^{c-n-1}\left(cK^*(t) - K(t)\right) < 0, \quad 0 < t < \infty. \tag{68}$$

This means that $K^*(t)/t^{n-c}$ is nonincreasing. The proof is complete. □

The following result shows that there is no essential difference between (47) and (50).

Lemma 7. *The following are equivalent.*

(a) *Equation* (50) *holds for* K.

(b) *There exists a weight* K_1, *comparable with* K, *such that, for some small enough* $c > 0$, $K_1(t)/t^{n-c}$ *is nonincreasing on* $(0,\infty)$.

(c) *Equation* (47) *holds for* K.

Proof. We assume that $K(t) = K(1)t^{n-1}$ for $t \ge 1$.

$(a) \Rightarrow (b)$ is obvious by Lemma 6.

Suppose that (b) holds. We have

$$\int_s^b \frac{K_1(t)}{t^{n+1}}dt \int_0^s \frac{t^{n-1}}{K_1(t)}dt \le \int_s^\infty t^{-1-c}dt \int_0^s t^{c-1}dt = c^{-2}. \tag{69}$$

We obtain that (50) holds for K since K is comparable with K_1. It means that $(b) \Rightarrow (a)$ holds.

For $(c) \Rightarrow (b)$, assume that (47) holds for K. We claim that

$$\liminf_{t\to 0} \frac{K(t)}{t^n} > 0. \tag{70}$$

If $s > 1$, it is clear that

$$\frac{K(1)}{K(1/s)} \le \varphi_K(s) \tag{71}$$

and by (47)

$$\int_0^1 K(s)^{-1}\frac{ds}{s^{1-n}} = \int_1^\infty K\left(\frac{1}{s}\right)^{-1}\frac{ds}{s^{n+1}} < \infty. \tag{72}$$

Thus, we have

$$\frac{t^n}{K(t)} \lesssim K(t)^{-1}\int_0^t \frac{ds}{s^{1-n}} \lesssim \int_0^1 K(s)^{-1}\frac{ds}{s^{1-n}} < \infty, \tag{73}$$

which gives the claim. We define

$$K_1(t) = t^n \int_t^\infty \frac{K(s)}{s^{n+1}}ds, \quad 0 < t < \infty. \tag{74}$$

It is easy to check that $K_1(t)$ is nondecreasing. Since K is nondecreasing, it follows that $K(t) \lesssim K_1(t)$, $0 < t < \infty$. We note that, for $0 < t < 1$,

$$\begin{gathered}\int_t^1 \frac{K(s)}{s^{n+1}}ds \le K(t)\int_t^1 \frac{\varphi_K(s/t)}{s^{n+1}}ds \le \frac{K(t)}{t^n}\int_1^\infty \frac{\varphi_K(r)}{r^{n+1}}dr,\\ \int_1^\infty \frac{K(s)}{s^{n+1}}ds \approx K(1) \lesssim \frac{K(t)}{t^n}.\end{gathered} \tag{75}$$

Thus, we obtain that

$$K_1(t) \lesssim K(t)\left(\int_1^\infty \frac{\varphi_K(s)}{s^{n+1}}ds + 1\right), \quad 0 < t < 1. \tag{76}$$

For $t \in [1,\infty)$, we have

$$K_1(t) = t^n \int_t^\infty \frac{K(s)}{s^{n+1}}ds = K(1)t^{n-1} = K(t). \tag{77}$$

Therefore, we get that $K_1 \approx K$ on $(0,\infty)$. Note that if c is sufficiently small, then we have

$$\left(t^{c-n}K_1(t)\right)' = t^{c-1-n}\left(cK_1(t) - K(t)\right) < 0, \quad 0 < t < \infty. \tag{78}$$

This means that $K_1(t)/t^{n-c}$ is nonincreasing.

Suppose that $K_1(t)/t^{n-c}$ is nonincreasing on $(0,\infty)$. For $s \ge 1$,

$$\begin{aligned}\varphi_{K_1}(s) &= \sup_{0<t\le 1}\frac{(st)^{n-c}K_1(st)(st)^{c-n}}{K_1(t)}\\ &\le \sup_{0<t\le 1}\frac{(st)^{n-c}K_1(t)t^{c-n}}{K_1(t)} = s^{n-c},\end{aligned} \tag{79}$$

which gives

$$\int_1^\infty \varphi_{K_1}(s)\frac{ds}{s^{n+1}} < \infty. \tag{80}$$

Thus $(b) \Rightarrow (c)$ holds. □

Lemma 8. *Let K satisfy* (49). *Then one can find another nonnegative weight function K_1 such that $\mathcal{Q}_{K_1}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$, and the new weight function K_1 has the following properties.*

(a) K_1 *is nondecreasing on* $(0,\infty)$.

(b) $K_1 \approx K$ *on* $(0,\infty)$, *and thus* K_1 *satisfies condition* (49).

(c) *For some small enough* $c > 0$, $K_1(t)/t^{n-2+c}$ *is nondecreasing on* $(0,\infty)$. *Consequently,* $K_1(t)/t^{n-2}$ *is also nondecreasing on* $(0,\infty)$. *Conversely, if* $K_1(t)/t^{n-2+c}$ *is nondecreasing, for some* $c > 0$, *then* (49) *holds for* K_1.

(d)

$$\sup_{0<s<\infty}\int_0^s \frac{K_1(t)}{t^{n-1}}dt \int_s^\infty \frac{t^{n-3}}{K_1(t)}dt < \infty. \tag{81}$$

Proof. Assume that $K(t) = K(1)t^{n-1}$ for $t \ge 1$. Define

$$K_1(t) = \begin{cases} t^{n-2}\displaystyle\int_0^t K(s)\frac{ds}{s^{n-1}}, & 0 < t \le 1,\\ t^{n-1}\displaystyle\int_0^1 K(s)\frac{ds}{s^{n-1}}, & t \ge 1.\end{cases} \tag{82}$$

Note that (49) is a condition for $n > 1$. Then, consider the following.

(a) is obvious.

(b) For $0 < t \le 1$,

$$K_1(t) = \int_0^1 K(st)\frac{ds}{s^{n-1}} \le K(t)\int_0^1 \varphi_K(s)\frac{ds}{s^{n-1}}. \quad (83)$$

On the other hand, since we always assume that $K(2t) \approx K(t)$, we obtain that

$$K_1(t) \ge t^{n-2}K\left(\frac{t}{2}\right)\int_{t/2}^{t}\frac{ds}{s^{n-1}} \gtrsim K(t). \quad (84)$$

Thus, $K_1 \approx K$ on $(0,1)$. For $t \ge 1$, clearly, $K_1(t) \approx K(t)$. Therefore, $K_1 \approx K$ on $(0,\infty)$ and we get that $\mathcal{Q}_{K_1}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.

(c) If $0 < t \le 1$, for some small enough $c > 0$,

$$\left(\frac{K_1(t)}{t^{n-2+c}}\right)' = t^{-c-n+1}\left(K(t) - cK_1(t)\right) > 0. \quad (85)$$

If $t \ge 1$, $K_1(t)/t^{n-2+c} = K_1(1)t^{1-c}$ is nondecreasing. Thus $K_1(t)/t^{n-2+c}$ is nondecreasing on $(0,\infty)$.

Conversely, if $K_1(t)/t^{n-2+c}$ is nondecreasing, for some $c > 0$, then, for $0 < s \le 1$,

$$\begin{aligned}\varphi_{K_1}(s) &= \sup_{0<t\le1}\frac{(st)^{n+c-2}K_1(st)(st)^{2-n-c}}{K_1(t)}\\ &\le \sup_{0<t\le1}\frac{(st)^{n+c-2}K_1(t)t^{2-n-c}}{K_1(t)} = s^{n+c-2},\end{aligned} \quad (86)$$

which gives

$$\int_0^1 \varphi_{K_1}(s)\frac{ds}{s^{n-1}} < \infty. \quad (87)$$

(d) Note that $K_1(t)/t^{n-2+c}$ is nondecreasing. For $0 < s < \infty$, we have

$$\int_0^s \frac{K_1(t)}{t^{n-1}}dt\int_s^\infty \frac{t^{n-3}}{K_1(t)}dt \le \int_0^s \frac{1}{t^{1-c}}dt\int_s^\infty \frac{1}{t^{1+c}}dt = c^{-2}. \quad (88)$$

The proof is complete. □

We end this section by giving an example. Fix $0 < \beta < 1$, and set

$$K_\beta(t) = \begin{cases}\dfrac{t^{n+\beta-1}}{|\log t|}, & 0 < t \le \dfrac{1}{e},\\ e^{-\beta}t^{n-1}, & t > \dfrac{1}{e}.\end{cases} \quad (89)$$

Since

$$\int_0^{1/e} K_\beta(t)\frac{dt}{t^{n-1}} < \infty, \qquad \int_0^{1/e} K_\beta(t)\frac{dt}{t^{n+\beta}} = \infty, \quad (90)$$

we obtain that $\mathcal{Q}_{K_\beta}(\mathbb{R}^n)$ is not trivial and

$$\mathcal{Q}_{K_\beta}(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n). \quad (91)$$

Moreover, a direct calculation shows that (46) and (47) hold for K_β.

4. Carleson-Type Measures

Let I be a cube of $\mathbb{R}^n$ and let $\mathbb{R}^{n+1}_+$ denote the upper half space based on $\mathbb{R}^n$. Define the Carleson box as follows:

$$S(I) = \left\{(x,t) \in \mathbb{R}^{n+1}_+ : x \in I, 0 < t < \ell(I)\right\}. \quad (92)$$

For $p > 0$ and a positive Borel measure μ on $\mathbb{R}^{n+1}_+$, μ is said to be a p-Carleson measure if

$$\mu(S(I)) \le M\ell(I)^{pn} \quad (93)$$

for some $M < \infty$ and all cubes $I \subseteq \mathbb{R}^n$.

Denote by $\delta(x)$ the distance of the point $x \in \mathbb{R}^{n+1}_+$ to the boundary $\partial\mathbb{R}^{n+1}_+$. Also $\tilde{y}$ stands for the symmetric point of $y \in \mathbb{R}^{n+1}_+$ with respect to $\mathbb{R}^n$; that is, if $y = (y_1,\ldots,y_n,y_{n+1})$, then $\tilde{y} = (y_1,\ldots,y_n,-y_{n+1})$.

A positive Borel measure μ is said to be a K-Carleson measure on $\mathbb{R}^{n+1}_+$, as a modification of p-Carleson measure, provided

$$\sup_{I\subseteq\mathbb{R}^n}\int_{S(I)} K\left(\frac{\delta(x)}{\ell(I)}\right)(\delta(x))^{1-n}d\mu(x) < \infty. \quad (94)$$

Clearly, if $K(t) = t^{np}$, then μ is a K-Carleson measure on $\mathbb{R}^{n+1}_+$ if and only if $(\delta(x))^{np+1-n}d\mu(x)$ is a p-Carleson measure on $\mathbb{R}^{n+1}_+$. Now, we give a characterization of K-Carleson measure as follows.

Theorem 9. *Let K satisfy* (48). *Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. Then μ is a K-Carleson measure if and only if*

$$\sup_{y\in\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x) < \infty. \quad (95)$$

Proof (sufficiency). Let I be a cube and take y to be the center of the Carleson box $S(I)$. Then $\delta(y) = \ell(I)/2$. If $x \in S(I)$, then $|x-\tilde{y}| \lesssim \ell(I)$ and hence

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x)\\ &\quad\ge \int_{S(I)} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x)\\ &\quad\gtrsim \int_{S(I)} K\left(\frac{\delta(x)}{\ell(I)}\right)(\delta(x))^{1-n}d\mu(x).\end{aligned} \quad (96)$$

Thus, if (95) holds, then μ is a K-Carleson measure.

Necessity. For $y = (y', y_{n+1}) \in \mathbb{R}^{n+1}_+$, let $I \subseteq \mathbb{R}^n$ be the cube with center y' and edge length $\delta(y)$. Set E_m to be the Carleson box $S(2^mI)$ for each positive integer m. It is clear that

$$\begin{aligned}|x-\tilde{y}| &\ge \delta(y), \quad x \in E_1,\\ |x-\tilde{y}| &\approx 2^m\delta(y), \quad x \in E_{m+1}\setminus E_m.\end{aligned} \quad (97)$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x)\\
&=\int_{E_1} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x)\\
&\quad+\sum_{m=1}^{\infty}\int_{E_{m+1}\setminus E_m} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x)\\
&\lesssim\int_{E_1} K\left(\frac{\delta(x)}{\ell(I)}\right)(\delta(x))^{1-n}d\mu(x)\\
&\quad+\sum_{m=1}^{\infty}\int_{E_{m+1}} K\left(\frac{\delta(x)}{2^{m/n}2^{m+1}\ell(I)}\right)(\delta(x))^{1-n}d\mu(x).
\end{aligned}\tag{98}$$

Since μ is a K-Carleson measure,

$$\int_{E_m} K\left(\frac{\delta(x)}{2^m\ell(I)}\right)(\delta(x))^{1-n}d\mu(x)\lesssim 1.\tag{99}$$

This together with (48) yields

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} K\left(\frac{\delta(x)(\delta(y))^{1/n}}{|x-\tilde{y}|^{(n+1)/n}}\right)(\delta(x))^{1-n}d\mu(x)\\
&\lesssim\sum_{m=1}^{\infty}\varphi_K\left(\frac{1}{2^{m/n}}\right)\approx\int_0^1\varphi_K(s)\frac{ds}{s}<\infty.
\end{aligned}\tag{100}$$

□

Let f be a measurable function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n}\frac{|f(x)|}{1+|x|^{n+1}}dx<\infty.\tag{101}$$

Its Poisson integral is defined by

$$f(x,t)=\int_{\mathbb{R}^n}P_t(x-y)f(y)\,dy,\tag{102}$$

where

$$P_t(x)=\frac{c_n t}{\left(t^2+|x|^2\right)^{(n+1)/2}},\qquad c_n=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.\tag{103}$$

The gradient of $f(x,t)$ is

$$\nabla f(x,t)=\left(\frac{\partial f(x,t)}{\partial x_1},\ldots,\frac{\partial f(x,t)}{\partial x_n},\frac{\partial f(x,t)}{\partial t}\right).\tag{104}$$

It is known that (101) holds for $f\in \mathrm{BMO}(\mathbb{R}^n)$ (see [9]).

The following main theorem generalizes the result of $\mathcal{Q}_\alpha(\mathbb{R}^n)$ in [10].

Theorem 10. *Suppose that* (47) *and* (81) *hold for K. Let $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ with* (101). *Then $f\in\mathcal{Q}_K(\mathbb{R}^n)$ if and only if $|\nabla f(x,t)|^2dxdt$ is a K-Carleson measure.*

In order to prove Theorem 10, we need the following Hardy-type inequalities.

Lemma 11 (see [18]). *Let $0<b\le\infty$, $1<p<\infty$, and $p'=(p/(p-1))$. Assume that functions μ and ν are measurable and nonnegative in the interval $(0,b)$. Then*

$$\int_0^b\left(\int_0^s f(t)\,dt\right)^p\mu(s)\,ds\le C\int_0^b f^p(s)\,\nu(s)\,ds\tag{105}$$

holds for all measurable functions $f\ge 0$ if and only if

$$\begin{aligned}
&A:=\sup_{0<s<b}\left(\int_s^b\mu(t)\,dt\right)^{1/p}\left(\int_0^s(\nu(t))^{1-p'}dt\right)^{1/p'}<\infty,\\
&\int_0^b\left(\int_s^b f(t)\,dt\right)^p\mu(s)\,ds\le C\int_0^b f^p(s)\,\nu(s)\,ds
\end{aligned}\tag{106}$$

holds for all measurable functions $f\ge 0$ if and only if

$$B:=\sup_{0<s<b}\left(\int_0^s\mu(t)\,dt\right)^{1/p}\left(\int_s^b(\nu(t))^{1-p'}dt\right)^{1/p'}<\infty.\tag{107}$$

Here C depends only on p, A, or B.

The following Stegenga-type estimate will be used in the proof of Theorem 10.

Lemma 12. *Suppose that* (81) *holds for K and*

$$\sup_{0<s<1}\int_s^1\frac{K(t)}{t^{3n+1}}dt\int_0^s\frac{t^{3n-1}}{K(t)}dt<\infty.\tag{108}$$

Let I and J be cubes in $\mathbb{R}^n$ centered at x_0 with $\ell(J)=3\ell(I)$ and let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy (101). *Then, there is a constant C independent of f, I, and J, such that*

$$\begin{aligned}
&\int_{S(I)}|\nabla f(x,t)|^2K\left(\frac{t}{\ell(I)}\right)t^{1-n}dx\,dt\\
&\le C\int_{|y|\le\sqrt{n}\ell(J)}\int_J|f(x+y)-f(x)|^2\\
&\qquad\times K\left(\frac{|y|}{\ell(J)}\right)\frac{1}{|y|^{2n}}dx\,dy\\
&\quad+C\left(\int_0^{\sqrt{n}}\frac{K(s)}{s^{n-1}}ds+\int_{1/8}^{\infty}\frac{K(s)}{s^{n+1}}ds\right)\\
&\quad\times|J|^{-1}\int_J\left|f(x)-f_J\right|^2dx\\
&\quad+C\int_0^{\sqrt{n}}\frac{K(s)}{s^{n-1}}ds\left(\ell(J)\int_{\mathbb{R}^n\setminus(2/3)J}\frac{|f(x)-f_J|}{|x-x_0|^{n+1}}dx\right)^2.
\end{aligned}\tag{109}$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. Let φ be a function with $0 \le \varphi \le 1$ such that $\varphi = 1$ on $(2/3)J$, $\operatorname{supp}\varphi \subseteq (3/4)J$, and

$$|\varphi(x) - \varphi(y)| \lesssim \ell(J)^{-1}|x-y|, \quad x, y \in \mathbb{R}^n. \tag{110}$$

Following Stegenga [19], we write

$$f = f_J + (f - f_J)\varphi + (f - f_J)(1-\varphi) = f_1 + f_2 + f_3. \tag{111}$$

Then we have

$$f(x,t) = f_1(x,t) + f_2(x,t) + f_3(x,t) \tag{112}$$

for the corresponding Poisson integrals. Since f_1 is constant, it contributes nothing to the integral with the gradient square.

Note that

$$\frac{\partial P_s(y)}{\partial y_j} = -(n+1)c_n s y_j\left(s^2 + |y|^2\right)^{-(n+3)/2}, \quad j = 1,\dots,n. \tag{113}$$

We obtain

$$\int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial y_j}dy = 0, \quad j = 1,\dots,n. \tag{114}$$

Hence,

$$\begin{aligned}\frac{\partial f(x,s)}{\partial x_j} &= \int_{\mathbb{R}^n} \frac{\partial P_s(x-y)}{\partial x_j} f(y)\,dy \\ &= \int_{\mathbb{R}^n} (n+1)c_n s\left(y_j - x_j\right) \times \left(s^2 + |y-x|^2\right)^{-(n+3)/2} f(y)\,dy \\ &= -\int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial y_j} f(x+y)\,dy \\ &= \int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial y_j}\left(f(x) - f(x+y)\right)dy.\end{aligned} \tag{115}$$

A direct calculation shows that

$$\begin{aligned}\left|\frac{\partial P_s(y)}{\partial y_j}\right| &\approx s\left|y_j\right|\left(s^2+|y|^2\right)^{-(n+3)/2} \\ &\lesssim \left(s^2+|y|^2\right)^{-(n+1)/2},\end{aligned} \tag{116}$$

which gives

$$\left|\frac{\partial P_s(y)}{\partial y_j}\right| \lesssim s^{-n-1}, \qquad \left|\frac{\partial P_s(y)}{\partial y_j}\right| \lesssim |y|^{-n-1}. \tag{117}$$

Therefore,

$$\begin{aligned}\left\|\frac{\partial f(x,s)}{\partial x_j}\right\|_{L^2(\mathbb{R}^n)} &\lesssim s^{-n-1}\int_{|y|\le s}\|f(x+y) - f(x)\|_{L^2(\mathbb{R}^n)}dy \\ &\quad + \int_{|y|>s}\|f(x+y) - f(x)\|_{L^2(\mathbb{R}^n)}|y|^{-n-1}dy.\end{aligned} \tag{118}$$

We write $y = r\xi \in \mathbb{R}^n$ with $r = |y|$ and $|\xi| = 1$. Let

$$A(r) = \int_{|\xi|=1}\|f(x+r\xi) - f(x)\|_{L^2(\mathbb{R}^n)}d\xi. \tag{119}$$

Then

$$\left\|\frac{\partial f(x,s)}{\partial x_j}\right\|_{L^2(\mathbb{R}^n)} \lesssim s^{-n-1}\int_0^s A(r)r^{n-1}dr + \int_s^\infty A(r)r^{-2}dr. \tag{120}$$

Thus,

$$\begin{aligned}&\int_{S(I)}\left|\frac{\partial f(x,s)}{\partial x_j}\right|^2 K\left(\frac{s}{\ell(I)}\right)s^{1-n}dx\,ds \\ &\quad\le \int_0^{\ell(I)}\left\|\frac{\partial f(x,s)}{\partial x_j}\right\|_{L^2(\mathbb{R}^n)}^2 K\left(\frac{s}{\ell(I)}\right)s^{1-n}ds \\ &\quad\lesssim \int_0^{\ell(I)} K\left(\frac{s}{\ell(I)}\right)s^{-3n-1}\left(\int_0^s A(r)r^{n-1}dr\right)^2 ds \\ &\qquad + \int_0^{\ell(I)} K\left(\frac{s}{\ell(I)}\right)s^{1-n}\left(\int_s^\infty A(r)r^{-2}dr\right)^2 ds \\ &\quad\approx \int_0^1 K(s)\frac{1}{s^{3n+1}\ell(I)^n}\left(\int_0^s A(\ell(I)r)r^{n-1}dr\right)^2 ds \\ &\qquad + \int_0^1 K(s)\frac{1}{s^{n-1}\ell(I)^n}\left(\int_s^\infty A(\ell(I)r)r^{-2}dr\right)^2 ds.\end{aligned} \tag{121}$$

Note that

$$\sup_{0<s<1}\int_s^1 \frac{K(t)}{t^{3n+1}}dt\int_0^s \frac{t^{3n-1}}{K(t)}dt < \infty. \tag{122}$$

By Lemma 11,

$$\begin{aligned}&\int_0^1 K(s)\frac{1}{s^{3n+1}}\left(\int_0^s A(\ell(I)r)r^{n-1}dr\right)^2 ds \\ &\quad\lesssim \int_0^1 K(s)\frac{1}{s^{n+1}}A^2(\ell(I)s)\,ds.\end{aligned} \tag{123}$$

Since (81) holds for K, by Lemma 11 again,

$$\begin{aligned}&\int_0^1 K(s)\frac{1}{s^{n-1}}\left(\int_s^\infty A(\ell(I)r)r^{-2}dr\right)^2 ds \\ &\quad\lesssim \int_0^\infty K(s)\frac{1}{s^{n+1}}A^2(\ell(I)s)\,ds.\end{aligned} \tag{124}$$

Therefore,

$$\int_{S(I)} \left|\frac{\partial f(x,s)}{\partial x_j}\right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n}\,dx\,ds \lesssim \int_0^\infty K\left(\frac{s}{\ell(I)}\right) s^{-1-n} A^2(s)\,ds. \quad (125)$$

By Hölder's inequality,

$$\begin{aligned}\int_{S(I)} \left|\frac{\partial f(x,s)}{\partial x_j}\right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n}\,dx\,ds &\lesssim \int_0^\infty \int_{|\xi|=1} \|f(x+s\xi)-f(x)\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad \times K\left(\frac{s}{\ell(I)}\right) s^{-1-n}\,d\xi\,ds \\ &\approx \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(x+y)-f(x)|^2 dx\right] \\ &\quad \times K\left(\frac{|y|}{\ell(I)}\right)|y|^{-2n} dy.\end{aligned} \quad (126)$$

Note that

$$\frac{\partial P_s(y)}{\partial s} = c_n\left[s^2+|y|^2-s^2(n+1)\right]\left(s^2+|y|^2\right)^{-(n+3)/2}. \quad (127)$$

Hence,

$$\left|\frac{\partial P_s(y)}{\partial s}\right| \lesssim \left(s^2+|y|^2\right)\left(s^2+|y|^2\right)^{-(n+3)/2}. \quad (128)$$

It follows that

$$\left|\frac{\partial P_s(y)}{\partial s}\right| \lesssim s^{-n-1}, \qquad \left|\frac{\partial P_s(y)}{\partial s}\right| \lesssim |y|^{-n-1}. \quad (129)$$

Since

$$\int_{\mathbb{R}^n} P_s(y)\,dy = 1, \quad (130)$$

this deduces that

$$\int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial s} dy = 0. \quad (131)$$

We have

$$\frac{\partial f(x,s)}{\partial s} = \int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial s}\left(f(x+y)-f(x)\right) dy. \quad (132)$$

Repeating the procedure above, we also can obtain

$$\begin{aligned}&\int_{S(I)} \left|\frac{\partial f(x,s)}{\partial s}\right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n}\,dx\,ds \\ &\quad\lesssim \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(x+y)-f(x)|^2 dx\right] K\left(\frac{|y|}{\ell(I)}\right)|y|^{-2n} dy.\end{aligned} \quad (133)$$

Therefore,

$$\begin{aligned}\int_{S(I)} |\nabla f_2(x,s)|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n}\,dx\,ds &\lesssim \iint_{\mathbb{R}^n} |f_2(x+y)-f_2(x)|^2 K\left(\frac{|y|}{\ell(I)}\right)|y|^{-2n} dx\,dy \\ &\approx \iint_{\mathbb{R}^n} \frac{|f_2(x)-f_2(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\,dy \\ &\approx \int\int_{x,y\in J}\cdots + \int\int_{x\notin J, y\in 3/4J}\cdots + \int\int_{y\notin J, x\in 3/4J} \\ &\approx B_1+B_2+B_3.\end{aligned} \quad (134)$$

To estimate B_1, we note that

$$|\varphi(x)-\varphi(y)| \lesssim \ell(J)^{-1}|x-y|, \quad x,y\in\mathbb{R}^n. \quad (135)$$

Thus

$$\begin{aligned}|f_2(x)-f_2(y)| &\lesssim |f(x)-f(y)| \\ &\quad + (\ell(J))^{-1}|x-y|\,|f(y)-f_J|.\end{aligned} \quad (136)$$

We have

$$\begin{aligned}&\iint_J \frac{|f(x)-f(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\,dy \\ &\quad\lesssim \int_{|y|\le\sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right)|y|^{-2n} \\ &\qquad\times \int_J |f(x+y)-f(x)|^2 dx\,dy, \\ &(\ell(J))^{-2}\iint_J \frac{|f(y)-f_J|^2}{|x-y|^{2n-2}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\,dy \\ &\quad\lesssim (\ell(J))^{-2}\int_{|y|\le\sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right)|y|^{2-2n} \\ &\qquad\times \int_J |f(x+y)-f_J|^2 dx\,dy \\ &\quad\lesssim \int_{|y|\le\sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right)|y|^{-2n} \\ &\qquad\times \int_J |f(x+y)-f(x)|^2 dx\,dy \\ &\qquad + (\ell(J))^{-2}\int_{|y|\le\sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right)|y|^{2-2n} \\ &\qquad\times \int_J |f(x)-f_J|^2 dx\,dy\end{aligned}$$

$$\lesssim \int_{|y|\le\sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right)|y|^{-2n} \times \int_J |f(x+y)-f(x)|^2 dx\,dy + \int_0^{\sqrt{n}} \frac{K(s)}{s^{n-1}} ds |J|^{-1} \int_J |f(x)-f_J|^2 dx. \tag{137}$$

Hence,

$$B_1 \lesssim \int_{|y|\le\sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right)|y|^{-2n} \int_J |f(x+y)-f(x)|^2 dx\,dy + \int_0^{\sqrt{n}} \frac{K(s)}{s^{n-1}} ds |J|^{-1} \int_J |f(x)-f_J|^2 dx. \tag{138}$$

To handle B_2, note that

$$|x-y| > \frac{1}{8}\ell(J), \quad |f_2(x)-f_2(y)| \lesssim |f(y)-f_J|, \quad x\notin J,\ y\in 3/4J. \tag{139}$$

We obtain

$$B_2 = \int_{x\notin J}\int_{y\in 3/4J} \frac{|f_2(x)-f_2(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx\,dy \lesssim \int_{y\in 3/4J} |f(y)-f_J|^2 dy \int_{|z|>(1/8)\ell(J)} K\left(\frac{|z|}{\ell(J)}\right)|z|^{-2n} dz \lesssim \int_{1/8}^{\infty} \frac{K(s)}{s^{n+1}} ds |J|^{-1} \int_J |f(y)-f_J|^2 dy. \tag{140}$$

Similarly,

$$B_3 \lesssim \int_{1/8}^{\infty} \frac{K(s)}{s^{n+1}} ds |J|^{-1} \int_J |f(y)-f_J|^2 dy. \tag{141}$$

Moreover, if $(x,s)\in S(I)$ and $y\in\mathbb{R}^n\setminus(2/3)J$, then

$$\frac{1}{(s+|x-y|)^{n+1}} \lesssim |y|^{-n-1}, \tag{142}$$

and, by $|\nabla P_s(y)| \lesssim (s+|y|)^{-n-1}$, we have

$$\int_{S(I)} |\nabla f_3(x,s)|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx\,ds \lesssim \int_{S(I)} \left(\int_{\mathbb{R}^n} |\nabla P_s(x-y)||f_3(y)|\,dy\right)^2 \times K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx\,ds \lesssim \int_{S(I)} \left(\int_{\mathbb{R}^n\setminus(2/3)J} \frac{|f(y)-f_J|}{(s+|x-y|)^{n+1}} dy\right)^2 \times K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx\,ds \lesssim \int_0^1 \frac{K(s)}{s^{n-1}} ds \left(\ell(I)\int_{\mathbb{R}^n\setminus(2/3)J} \frac{|f(y)-f_J|}{|y|^{n+1}} dy\right)^2. \tag{143}$$

Combining the inequalities above, Lemma 12 is proved. □

Proof of Theorem 10. We assume that $K(t)=K(1)t^{n-1}$ for $t\ge 1$. By Lemma 7, K satisfies (47) if and only if

$$\sup_{0<s<1} \int_s^1 \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{144}$$

Sufficiency. Let I be a cube and $f(x,t)$ is the Poisson integral of f. Note that

$$f(x,|y|) - f(x) = \int_0^{|y|} \frac{\partial f(x,t)}{\partial t} dt. \tag{145}$$

By Minkowski's inequality, we have

$$\left(\int_I |f(x,|y|)-f(x)|^2 dx\right)^{1/2} \le \int_0^{|y|} \left(\int_I \left|\frac{\partial f(x,t)}{\partial t}\right|^2 dx\right)^{1/2} dt \le \int_0^{|y|} \left(\int_I |\nabla f(x,t)|^2 dx\right)^{1/2} dt. \tag{146}$$

Hence,

$$\int_{|y|<\ell(I)} K\left(\frac{|y|}{\ell(I)}\right)\frac{1}{|y|^{2n}} \left(\int_I |f(x,|y|)-f(x)|^2 dx\right) dy \lesssim (\ell(I))^{2-n} \int_0^1 \frac{K(r)}{r^{n+1}} \left(\int_0^r \left(\int_I |\nabla f(x,\ell(I)s)|^2 dx\right)^{1/2} ds\right)^2 dr \lesssim \int_{S(I)} |\nabla f(x,t)|^2 K\left(\frac{t}{\ell(I)}\right) t^{1-n} dx\,dt. \tag{147}$$

The last inequality above holds by Lemma 11 since K satisfies

$$\sup_{0<s<1} \int_s^1 \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{148}$$

Thus,

$$\sup_I \int_{|y|<\ell(I)} K\left(\frac{|y|}{\ell(I)}\right)\frac{1}{|y|^{2n}} \times \left(\int_I |f(x,|y|)-f(x)|^2 dx\right) dy < \infty. \tag{149}$$

For $|y| < \ell(I)$,

$$\int_I |f(x+y,|y|) - f(x+y)|^2 dx \leq \int_{3I} |f(x,|y|) - f(x)|^2 dx. \tag{150}$$

Similarly, we get

$$\sup_I \int_{|y|<\ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \times \left(\int_I |f(x+y,|y|) - f(x+y)|^2 dx\right) dy < \infty. \tag{151}$$

Note that

$$|f(x+y,|y|) - f(x,|y|)| \leq \int_0^{|y|} \left|\nabla f\left(x + te_y, |y|\right)\right| dt, \qquad e_y = \frac{y}{|y|}. \tag{152}$$

When $|y| < \ell(I)$, we employ Minkowski's inequality to get

$$\begin{aligned}\left(\int_I |f(x+y,|y|) - f(x,|y|)|^2 dx\right)^{1/2} &\leq \int_0^{|y|} \left(\int_I \left|\nabla f\left(x + te_y, |y|\right)\right|^2 dx\right)^{1/2} dt \\ &\lesssim |y| \left(\int_{3I} |\nabla f(x,|y|)|^2 dx\right)^{1/2}.\end{aligned} \tag{153}$$

Hence,

$$\begin{aligned}\sup_I \int_{|y|<\ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} &\times \left(\int_I |f(x+y,|y|) - f(x,|y|)|^2 dx\right) dy \\ &\lesssim \sup_I \int_{S(I)} |\nabla f(x,t)|^2 K\left(\frac{t}{\ell(I)}\right) t^{1-n} dx\, dt < \infty.\end{aligned} \tag{154}$$

By the triangle inequality, we get

$$\sup_I \int_{|y|<\ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \times \left(\int_I |f(x+y) - f(x)|^2 dx\right) dy < \infty. \tag{155}$$

Thus $f \in \mathcal{Q}_K(\mathbb{R}^n)$.

Necessity. Note that

$$\sup_{0<s<1} \int_s^1 \frac{K(t)}{t^{3n+1}} dt \int_0^s \frac{t^{3n-1}}{K(t)} dt \leq \sup_{0<s<1} \int_s^1 \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt. \tag{156}$$

Thus K satisfies the conditions of Lemma 12. Let I and J be cubes in $\mathbb{R}^n$ centered at x_0 with $\ell(J) = 3\ell(I)$. Since $f \in \mathcal{Q}_K(\mathbb{R}^n) \subseteq \mathrm{BMO}(\mathbb{R}^n)$, by [10, p. 590], we have

$$\ell(J) \int_{\mathbb{R}^n \setminus (2/3)J} \frac{|f(x) - f_J|}{|x - x_0|^{n+1}} dx \lesssim \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{157}$$

Note that

$$|J|^{-1} \int_J |f(x) - f_J|^2 dx \lesssim \int_{|y| \leq \sqrt{n}\ell(J)} \int_J |f(x+y) - f(x)|^2 K\left(\frac{|y|}{\ell(J)}\right) \frac{1}{|y|^{2n}} dx\, dy. \tag{158}$$

Lemma 12 gives

$$\int_{S(I)} |\nabla f(x,t)|^2 K\left(\frac{t}{\ell(I)}\right) t^{1-n} dx\, dt \lesssim \|f\|^2_{\mathcal{Q}_K(\mathbb{R}^n)} + \|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{159}$$

The proof of Theorem 10 is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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