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Research Article

A Note on the Normal Index and the *c*-Section of Maximal Subgroups of a Finite Group

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Let M be a maximal subgroup of finite group G. For each chief factor H/K of G such that $K \leq M$ and $G = MH$, we called the order of H/K the normal index of M and $(M \cap H)/K$ a section of M in G. Using the concepts of normal index and c-section, we obtain some new characterizations of p-solvable, 2-supersolvable, and p-nilpotent.

1. Introduction

In this paper, all groups considered are finite. Let $\pi(G)$ denote the set of prime divisors of $|G|$, and for $p \in \pi(G)$ let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Write $M \lessdot G$ to indicate that M is a maximal subgroup of G. For convenience, we cite the following relative definitions. For a fixed prime $p \in \pi(G)$,

(1) $\mathcal{F}_c(G) = \{M \mid M \lessdot G \text{ and } |G : M| \text{ is composite}\}$,

(2) $\mathcal{F}_p(G) = \{M \mid M \lessdot G \text{ and } |G : M|_p = 1\}$,

(3) $\mathcal{F}_{pc}(G) = \mathcal{F}_p(G) \cap \mathcal{F}_c(G)\}$,

(4) $\mathcal{F}^p(G) = \{M \mid M \lessdot G \text{ and } N_G(P) \leq M\}$, where $P \in \mathrm{Syl}_p(G)$,

(5) $\mathcal{F}^{pc}(G) = \mathcal{F}^p(G) \cap \mathcal{F}_c(G)$.

The remaining notation and terminology in this paper are standard, as in Huppert [1].

In 1959, Deskins [2] introduced the concept of normal index. For a maximal subgroup M of a group G, the order of a chief factor H/K of G, where H is minimal in the set of normal supplements of M in G, is known as the normal index of M of G, denoted by $\eta(G : M)$. If H/K is such a chief factor, then $G = MH$, $K \leq M$, and $|G : M| = |H/K : (H/K) \cap (M/K)|$, so $|H/K| = |(H \cap M)/K||G : M|$. The intersection $(M \cap H)/K$ is called a c-section of M. Li and Wang in [3] proved that every maximal subgroup M of G has a unique c-section up to isomorphism. Let $\mathrm{Sec}(M)$ denote a group which is isomorphic to a c-section of M. Then $\eta(G : M) = |\mathrm{Sec}(M)| \cdot |G : M|$. Deskins [2] showed that G is solvable if and only if $\eta(G : M) = |G : M|$ for every maximal subgroup M of G. The investigations on the normal index have been developed by many scholars; see [3–7]. But the earlier results concern the cases where p is either the largest prime dividing $|G|$ or an odd prime. In 2010, Zhang and Li analyzed the case when $p = 2$ and obtained some interesting results. In particular we note the following theorems.

Theorem 1 (see [8, Theorem 3.1]). *A group G is solvable if and only if $\eta(G : M)_2 = 1$ for every $M \in \mathcal{F}^2(G)$.*

Theorem 2 (see [8, Theorem 3.4]). *A group G is solvable if and only if $\mathrm{Sec}(M)$ is either a $2'$-group or an abelian 2-group for every $M \in \mathcal{F}^2(G)$.*

We observe that Theorems 1 and 2 still hold by replacing 2 with another prime p. For example, let $G = S_4$ and let $M \in \mathcal{F}^3(G)$. Since the order of M is 6 or 12, $\eta(G : M)_3 = 1$. So G satisfies the hypotheses of Theorem 1. But G is 3-solvable. It is natural to ask that the theorems above hold or not for any prime p. In part 3, we give positive answer and relative results.

2. Preliminary Results

Lemma 3 (see [8, Lemma 2.2]). *Let G be a group, N a normal subgroup of G, and $p \in \pi(G)$. Let M be a maximal subgroup of G and $N \leq M$.*

(1) *We have* $\eta(G/N : M/N) = \eta(G : M)$ *and* $\mathrm{Sec}\,(M/N) \cong \mathrm{Sec}\,(M)$.

(2) *If* $M/N \in \mathcal{F}_p(G/N)$, *then* $M \in \mathcal{F}_p(G)$.

(3) *If* $M/N \in \mathcal{F}^p(G/N)$, *then* $M \in \mathcal{F}^p(G)$.

(4) *If* $p = \max \pi(G)$, *then* $\mathcal{F}^p(G) = \mathcal{F}^{pc}(G)$.

Lemma 4 (see [6, Theorem 7]). *G is p-supersolvable if and only if, for each maximal subgroup M of G, $\eta(G : M)_p = |G : M|_p = 1$ or p.*

3. Main Results

Theorem 5. *G is p-solvable if and only if $\eta(G : M)_p = 1$ for every $M \in \mathcal{F}^p(G)$.*

Proof. ⇒: Suppose that G is p-solvable and let N be a minimal normal subgroup. If a maximal subgroup $M \in \mathcal{F}^p(G)$ containing N, then, by induction, it follows that $\eta(G/N : M/N)_p = \eta(G : M)_p = 1$. If $N \nsubseteq M$, then we must have $|N|_p = 1$, since N is a p'-group.

⇐: Conversely, let $\eta(G : M)_p = 1$ hold for each maximal subgroup $M \in \mathcal{F}^2(G)$. Only we need to consider that G is not simple. Otherwise, $|G|_p = \eta(G : M)_p = 1$. Certainly, G is p-solvable.

Now let N be a minimal normal subgroup of G. Observe the quotient group G/N. For every maximal subgroup $M/N \in \mathcal{F}^p(G/N)$, it is easy to see $M \in \mathcal{F}^p(G)$. By Lemma 3 and hypothesis, $\eta(G/N : M/N)_p = \eta(G : M)_p = 1$. Hence G/N is p-solvable by induction. Since the class of all p-solvable groups is a saturated formation, we may suppose that N is the unique minimal normal subgroup of G. If $|N|_p = 1$, then N is a p'-group. Moreover, G/N is p-solvable, and so is G. Now consider $|N|_p \neq 1$. Let P be a Sylow p-subgroup of G and $K = P \cap N$. Then K is a Sylow p-subgroup of N. Clearly, $N_G(P) \leq N_G(K) < G$. So $N_G(K)$ is contained in some maximal subgroup T of G. Hence $T \in \mathcal{F}^p(G)$. By Frattini argument, $G = NN_G(K) = NT$. It follows that $|N|_p = \eta(G : T)_p = 1$, a contradiction, and we are done. □

Corollary 6. *G is solvable if and only if, for every $M \in \mathcal{F}^p(G)$, $\eta(G : M) = 1$, where p is an arbitrary divisor of $|G|$.*

It was announced by Zhang and Li in [8, Theorem 5] that a group G is solvable if and only if $\mathrm{Sec}(M)$ is a $2'$-group or an abelian 2-group for $M \in \mathcal{F}^2(G)$. We extend this theorem by proving the following.

Theorem 7. *G is p-solvable if and only if, for any $M \in \mathcal{F}^p(G)$, $\mathrm{Sec}\,(M)$ is an abelian p-group or a p'-group, where p is a prime divisor of $|G|$.*

Proof. ⇒: Suppose that G is p-solvable and let N be a minimal normal subgroup. If a maximal subgroup $M \in \mathcal{F}^p(G)$ containing N, then, by induction, it follows that $\mathrm{Sec}(M/N) \cong \mathrm{Sec}(M)$ is an abelian p-group or a p'-group in view of Lemma 3. If $N \nsubseteq M$, then $G = MN$. If $|N|_p = 1$, then $\eta(G : M)_p = |N|_p = 1$, and so $\mathrm{Sec}(M)$ is a p'-group. Now consider $|N|_p = 1$. By the p-solvability of G, it implies that N is an elementary abelian p-group. It follows that $\mathrm{Sec}\,(M) \cong M \cap N$ is an abelian p-group.

⇐: Conversely, suppose $\mathrm{Sec}(M)$ is an abelian p-group or a p'-group. Let N be a minimal normal subgroup of G. By Lemma 3, G/N satisfies the hypotheses of the theorem. Then by induction, G/N is p-solvability. If $|N|_p = 1$, then G is p-solvable. Now assume that $|N|_p \neq 1$, then G is p-solvable. Let P be a Sylow p-solvable of G and $K = P \cap N$. Then, K is a Sylow p-subgroup of N. Obviously, $N_G(P) \leq N_G(K) < G$. So $N_G(K)$ is contained in some maximal subgroup T of G, and consequently, $T \in \mathcal{F}^p(G)$. By Frattini argument, $G = NN_G(K) = NT$. Then the minimal normality of N shows $\mathrm{Sec}(M) \cong M \cap N$. On the other hand, $K \leq N_N(K) \leq M \cap N$. Combining the hypothesis, $\mathrm{Sec}(M)$ is an abelian p-group, and so is $M \cap N$. It follows that $N_N(K) = C_N(K)$. By Burnside Theorem, N is p-nilpotent, which contradicts the minimal normality of N. Therefore, the conclusion holds. □

In view of Theorem 7 it is natural to ask if a group G is p-solvable when $|\mathrm{Sec}(M)|_p = p^\alpha$ or 1, for $M \in \mathcal{F}^p(G)$, where p is a prime divisor of $|G|$. The answer of the question is negative. For example, set $G = PSL(2,7)$ and $p = 3$; every maximal subgroup M satisfies that $|\mathrm{Sec}(M)|_3 = 3$, but G is not 3-solvable. For p-solvable, the condition that $\mathrm{Sec}(M)$ is an abelian p-group is crucial.

It is proved in [6, Theorem 7] that a group G is p-supersolvable if and only if, for each maximal subgroup M of G, $\eta(G : M)_p = |G : M|_p = 1$ or p. It is natural to ask if a group G is p-supersolvable when $\eta(G : M)_p = 1$ or p for any maximal subgroup M of G. The answer of the question is negative. For example, set $G = PSL(2,7)$ and $p = 3$; every maximal subgroup M satisfies that $\eta(G : M)_3 = 3$, but G is not 3-supersolvable. But assuming that $p = 2$, the result holds or not. For the question, we give the positive answer. Next, we prove the result.

Theorem 8. *G is 2-supersolvable if and only if, for any maximal subgroup M of G, $\eta(G : M)_2 = 1$ or 2.*

Proof. ⇒: Suppose that G is 2-supersolvable. Certainly, G is solvable. By Lemma 4, the necessity holds.

⇐: Conversely, assume the result is not true and let G be a counterexample of minimal order. Now, we assert G is not simple. If not, then $\eta(G : M)_2 = |G|_2 = 1$ or 2. For $|G|_2 = 1$, it is clear that G is 2-supersolvable, a contradiction. Assume that $|G|_2 = 2$. Then G is a cyclic group of order 2, and so G is 2-supersolvable, a contradiction. This contradiction shows G is not simple. Let N be the minimal normal subgroup of G. By Lemma 3, G/N satisfies the hypotheses of the theorem. The minimal choice of G implies that G/N is 2-supersolvable. If N is contained in each maximal subgroup M of G, then $N \subseteq \Phi(G)$, and consequently, $G/\Phi(G)$ is 2-supersolvable, and so is G, a contradiction. Hence there is a maximal subgroup M of G, such that $G = MN$. Suppose that $|N|_2 = 1$. It follows that G is 2-supersolvable, a contradiction. So $|N|_2 \neq 1$. By hypothesis, $\eta(G : M)_2 = |N|_2 = 2$. Moreover, N is solvable. Therefore, $|N| = 2$, and so G is 2-supersolvable, which contradicts the assumption. Now the proof of theorem is completed. □

Theorem 9. *Suppose G is a group and p is the smallest prime divisor of $|G|$. Then G is p-nilpotent if and only if the following conditions are satisfied:*

(1) *$\eta(G : M)_p = 1$ or p for every maximal subgroup M of G;*

(2) *if $\eta(G : M)_p = p$ for some maximal subgroup M, then $M \trianglelefteq G$.*

Proof. ⇒: Assume that G is p-nilpotent. Then G is p-supersolvable and (1) holds by Lemma 4. Now let M be a maximal subgroup of G with $\eta(G : M)_p = p$ and $G = PT$, where P is a Sylow p-subgroup and T is a normal Hall p'-subgroup of G. Suppose $T \nsubseteq M$ and let $1 \trianglelefteq \cdots \trianglelefteq T_2 \trianglelefteq T_1 \trianglelefteq \cdots \trianglelefteq T \trianglelefteq G$ be a chief group series, where $T_1 \nleq M$ and $T_2 \leq M$. Then $\eta(G : M)_p = |T_1/T_2|_p = 1$, a contradiction. Hence $T \subseteq M$. Since $\eta(G : M)_p = p$, $|G : M|_p = 1$ or p. If $|G : M|_p = 1$, then some Sylow p-subgroup of G, say P_1, is contained in M, and it follows that $G = P_1T \subseteq M$, a contradiction. Therefore, $|G : M|_p = p$. Since $M = T(M \cap P)$, $|P|/|M \cap P| = |G : M|_p = p$, which leads to $P \cap M \trianglelefteq P$. Hence $M \trianglelefteq PM = G$.

⇐: Now suppose (1) and (2) hold. If, for each maximal subgroup M of G, $\eta(G : M)_p = 1$, then by Theorem 8, G is p-solvable. Combining condition (2), we have that G is not simple. Let N be a minimal normal subgroup of G. By Lemma 3, G/N satisfies the hypotheses. By induction, G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we may regard N as the unique minimal normal subgroup of G and $\Phi(G) = 1$. So there exists a maximal subgroup M of G such that $G = NM$ and $\eta(G : M)_p = |N|_p = 1$ or p.

Suppose $|N|_p = 1$. Then N is a p'-group. Since G/N is p-nilpotent, G is p-nilpotent.

Assume $|N|_p = p$. Since p is the smallest prime divisor of $|G|$, N is p-nilpotent, and so $|N| = p$. It now follows that $M \cap N = 1$ and $M \cong G/N$ is p-nilpotent. Note $M = M_pM_{p'}$, where M_p is a Sylow p-subgroup and $M_{p'}$ is a normal Hall p'-subgroup of M. Then by (2), $M_{p'}$ char $M \trianglelefteq G$, and so $M_{p'} \trianglelefteq G$. Consequently, $G = MN = (M_pN)M_{p'}$ and $M_{p'}$ is a normal Hall p'-subgroup of G. □

The proof of the theorem has been done.

Obviously, in Theorem 9, removing the condition "p is the smallest prime divisor of $|G|$", and the result does not hold.

Theorem 10. *G has a p-nilpotent maximal subgroup M with prime power normal index; then G is p-solvable.*

Proof. Assume that the theorem is false and let G be a minimal counterexample. Let M be a p-solvable maximal subgroup of G with $\eta(G : M) = q^{\alpha}$, where q is a prime. Now we assert that G is not simple. Otherwise, $\eta(G : M) = |G| = q^{\alpha}$, a contradiction. Let N be a minimal normal subgroup of G. Next, we consider the following two cases.

Case 1 ($N \subseteq M$). Then by Lemma 3, $\eta(G/N : M/N) = \eta(G : M) = q^{\alpha}$. Since M is p-solvable, M/N and N are p-solvable. By the minimal choice of G, it implies that G/N is p-solvable, so is G, a contradiction.

Case 2 ($N \nsubseteq M$). Then $G = MN$ and $G/N \cong M/(M \cap N)$ is p-solvable. On the other hand, $\eta(G : M) = |N|$ is a q-group. Thus G is p-solvable, a final contradiction. This contradiction completes the proof of the theorem. □

Theorem 11. *If G has a p-supersolvable maximal subgroup M such that $\eta(G : M)$ is a prime and $M_G = 1$, then G is p-supersolvable.*

Proof. Assume the result is not true and let G be a counterexample of minimal order. By Theorem 10, G is p-solvable. Let N be an arbitrary minimal normal subgroup of G. Then N is an abelian p-group or a p'-group. Moreover, since $M_G = 1$, $G = MN$. Suppose every minimal normal subgroup N of G is a p'-group. But $G/N = MN/N \cong M/M \cap N$ is p-supersolvable; it follows that G is p-supersolvable, a contradiction. This contradiction shows that there exists some minimal normal subgroup K of G which is an abelian p-group. Then $G = MK$ and $M \cap K = 1$. From this it follows that $\eta(G : M) = |K|$ is a prime. Since $G/K \cong M$ is p-supersolvable, G is p-supersolvable, a contradiction. Hence the result holds. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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