Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 920348, 5 pages
http://dx.doi.org/10.1155/2014/920348

Research Article

On a System Modelling a Population with Two Age Groups

Hongliang Gao and Ruyun Ma

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Hongliang Gao; gaohongliang101@163.com

Received 4 December 2013; Accepted 11 June 2014; Published 23 June 2014

Academic Editor: Youyu Wang

A system of first order ordinary differential equations describing a population divided into juvenile and adult age groups is studied. The system is not cooperative but its linear part is, and this makes it possible to establish the existence and nonexistence results of positive solutions for the system in terms of the principal eigenvalue of the corresponding linearized system.

1. Introduction

In this paper, we will study the following problem:

$$\begin{aligned} u' &= a(t)v - c(t)u - eu[u+v], \\ v' &= b(t)u - d(t)v - fv[u+v], \end{aligned} \tag{1}$$

where a, b, c, and $d \in C(\mathbb{R}, (0, \infty))$ are ω-periodic functions and e and f are positive constants.

We are interested in dividing the individuals within a population into two age groups. The first group contains all newborns in addition to all young individuals who are unable to produce newborns; such group will be referred to as the juvenile group. The second group, which we will call the adult group, contains all individuals who can produce newborns in addition to old individuals who may not be able to produce newborns. The functions u and v represent, respectively, the total number of individuals who belong to the juvenile and adult groups. As adults give birth to juveniles, the function a corresponds to the birth rate of the population. Juveniles are lost both through death and through becoming adults; the function c corresponds to this overall loss. The function b gives the rate at which juveniles become adults and the function d corresponds to the death rate of adult population. The terms $-eu[u+v]$ and $-fv[u+v]$ correspond to decrease in population size due to competition for limited resources.

In natural environments the number of individuals of a population changes in time in different ways. Many observations show that the number of individuals of a population can have large oscillations in nature. In the earlier models the population is characterized by its size which is the total number of individuals within the population or the total biomass. One of such models is the Malthus model for the human population growth. P. F. Verhulst in 1838 introduced another model which is known as the realistic model; see [1, 2]. Models presenting qualitatively this type of behavior are density dependent unstructured population models; the most well-known model for interspecific competition has been proposed by Lotka and Volterra and has been studied extensively by Bonhoeffer, Borrelli, and Murray; see also [1–3]. In fact, up to the mid of the 20th century most models characterize the population by its size or total biomass. In such models the population is considered as homogeneous; that is, the models do not distinguish between the individuals within the population. Models that involve structured population are called structured population models. A structured model describes how individuals move in time among different groups and thus describes the dynamics of population groups and as a result it describes the dynamics of the whole population. For other related results, we refer the readers to [4–7] and the references therein.

Recently, a model for the growth of a population of two age groups (adult and juvenile) in which there is competition for limited resources has been considered in [8], where the authors assumed that the population is homogenous with common birth rate, common death rate, and common inhibiting constants. They established a time-invariant structure under general conditions and discussed the stability of the equilibrium points. More concretely, at time t the net rate of change in the populations of the two groups is modeled by the system

$$\begin{aligned} u' &= av - cu - eu[u+v], \\ v' &= bu - dv - fv[u+v]; \end{aligned} \tag{2}$$

the authors showed in [8] that if $cd < ab$, then (2) has a unique positive equilibrium point value in addition to the trivial equilibrium point $u = v = 0$.

Obviously, in [8], since a, b, c, and d are positive constants, the solutions of (2) can be explicitly given and some estimates can be carried out easily. However, when a, b, c, and d are positive functions, the method of [8] cannot be applied to deal with the system (1) any more. If a, b, c, and d are not constants, whether the system (1) has a positive solution or not is a natural question. Inspired by above considerations, in the present paper, we will first establish the lower and upper solutions method for more general system

$$\begin{aligned} u' + p(t)u &= f(u,v), \\ v' + q(t)v &= g(u,v), \end{aligned} \tag{3}$$

where f and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions and p, q are ω-periodic continuous functions, and then we will prove the existence of positive solutions for system (1) by applying above method.

Our main results can be stated as below.

Theorem 1. *Suppose that the functions $\underline{u}, \underline{v}, \overline{u}, \overline{v} \in X \cap C^1(\mathbb{R},\mathbb{R})$. $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are ordered coupling lower and upper solutions of systems* (3)*; the following condition is hold (H). There exists M_1, $N_1 > 0$ such that, for any $\underline{v} \le v \le \overline{v}$ and $\underline{u} \le u \le \overline{u}$,*

$$\begin{aligned} f(\overline{u},v) - f(u,v) &\ge -M_1(\overline{u}-u), \\ f(u,v) - f(\underline{u},v) &\ge -M_1(u-\underline{u}), \\ g(u,\overline{v}) - g(u,v) &\ge -N_1(\overline{v}-v), \\ g(u,v) - g(u,\underline{v}) &\ge -N_1(v-\underline{v}). \end{aligned} \tag{4}$$

Then, the problem (3) *has at least one solution (u^*, v^*) with $\underline{u} \le u^* \le \overline{u}$, $\underline{v} \le v^* \le \overline{v}$.*

Theorem 2. *There exists a positive periodic solution of systems* (1) *if and only if $\lambda_1(M) < 0$.*

Remark 3. To overcome the difficulties caused by the spatially heterogeneous, we discuss the system (3) by lower and upper solutions method established in Theorem 1 and obtain the necessary and sufficient conditions for the existence of positive periodic solutions of (1) in terms of the principal eigenvalue of the associated linear system. For other related results on the study of differential systems via lower and upper solutions method, we refer the readers here to [9–11] and the references listed therein.

The rest of the paper is organized as follows. In Section 2, we establish the lower and upper solutions methods for the system (3). In Section 3, we obtain the necessary and sufficient conditions for the existence of a positive periodic solution of (1).

2. Lower and Upper Solutions Method

In this section, we will develop lower and upper solutions method for system (3).

Let X be a Banach space defined as

$$X := \{u \in C(\mathbb{R},\mathbb{R}) \mid u(t) = u(t+\omega),\ t \in \mathbb{R}\}. \tag{5}$$

Definition 4. Assume that the functions $\underline{u}$, $\underline{v}$, $\overline{u}$, $\overline{v} \in X \cap C^1(\mathbb{R},\mathbb{R})$. Then, $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are called ordered coupling lower and upper solutions of systems (3), respectively, if $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$ satisfying

$$\begin{aligned} \underline{u}' + p(t)\underline{u} &\le f(\underline{u},v), & \underline{v} \le v \le \overline{v}, \\ \underline{v}' + q(t)\underline{v} &\le g(u,\underline{v}), & \underline{u} \le u \le \overline{u}, \\ \overline{u}' + p(t)\overline{u} &\ge f(\overline{u},v), & \underline{v} \le v \le \overline{v}, \\ \overline{v}' + q(t)\overline{v} &\ge g(u,\overline{v}), & \underline{u} \le u \le \overline{u}. \end{aligned} \tag{6}$$

Proof of Theorem 1. By the condition (H), there exist $M \ge M_1$ and $N \ge N_1$ such that $\int_0^\omega [p(t)+M]dt > 0$ and $\int_0^\omega [q(t)+N]dt > 0$.

For any $\varphi, \psi \in X$, we consider the following linear problem:

$$\begin{aligned} u' + p(t)u + Mu &= f(\varphi,\psi) + M\varphi, \\ v' + q(t)v + Nv &= g(\varphi,\psi) + N\psi. \end{aligned} \tag{7}$$

It is well known that the system (7) is equivalent to the equation

$$\begin{aligned} (u(t), v(t)) = \Bigg(&\int_t^{t+w} G_1(t,s)\left[f(\varphi(s),\psi(s)) + M\varphi(s)\right] ds, \\ &\int_t^{t+w} G_2(t,s)\left[g(\varphi(s),\psi(s)) + N\psi(s)\right] ds \Bigg), \end{aligned} \tag{8}$$

where

$$\begin{aligned} G_1(t,s) &= \frac{e^{\int_t^s [p(\theta)+M]d\theta}}{e^{\int_0^\omega [p(\theta)+M]d\theta} - 1}, \\ G_2(t,s) &= \frac{e^{\int_t^s [q(\theta)+N]d\theta}}{e^{\int_0^\omega [q(\theta)+N]d\theta} - 1}, \end{aligned} \tag{9}$$

and, consequently,

$$(u,v) = T(\varphi,\psi) \tag{10}$$

with

$$T(\varphi,\psi) = \left(\int_t^{t+w} G_1(t,s)\left[f(\varphi(s),\psi(s)) + M\varphi(s)\right]ds, \int_t^{t+w} G_2(t,s)\left[g(\varphi(s),\psi(s)) + N\psi(s)\right]ds\right). \tag{11}$$

It is easy to see that $T : X^2 \to X^2$ is completely continuous.

Let $P = \{(u,v) \in X^2 \mid \underline{u} \le u \le \overline{u}, \underline{v} \le v \le \overline{v}\}$; then P is bounded closed convex subset of X^2. So, $T : P \to X^2$ is also completely continuous. We will show that $T : P \to P$.

Let $(\varphi,\psi) \in P$, $(u,v) = T(\varphi,\psi)$, and $w = \overline{u} - u$, $z = \overline{v} - v$. Since $\underline{u} \le \varphi \le \overline{u}$ and $\underline{v} \le \psi \le \overline{v}$, we have

$$\begin{aligned} \overline{u}' + p(t)\overline{u} + M\overline{u} &\ge f(\overline{u},\psi) + M\overline{u}, \\ \overline{v}' + q(t)\overline{v} + N\overline{v} &\ge g(\varphi,\overline{v}) + N\overline{v}. \end{aligned} \tag{12}$$

Hence, w, z satisfy

$$\begin{aligned} w' + p(t)w + Mw &\ge f(\overline{u},\psi) - f(\varphi,\psi) + M(\overline{u} - \varphi) \ge 0, \\ z' + q(t)z + Nz &\ge g(\varphi,\overline{v}) - g(\varphi,\psi) + N(\overline{v} - \psi) \ge 0. \end{aligned} \tag{13}$$

T is strongly positive and implies $w \ge 0$, $z \ge 0$; that is, $u \le \overline{u}$, $v \le \overline{v}$. By a similar method, we have $u \ge \underline{u}$ and $v \ge \underline{v}$. Consequently, $T : P \to P$. By the Schauder fixed point theorem, T has a fixed point (u^*, v^*) in P. Therefore, the problem (3) has at least one solution (u^*, v^*) with $\underline{u} \le u^* \le \overline{u}$ and $\underline{v} \le v^* \le \overline{v}$. □

3. Existence and Nonexistence of Positive Periodic Solutions

We consider the system

$$\begin{aligned} u' &= A(t)u + B(t)v, \\ v' &= C(t)u + D(t)v, \end{aligned} \tag{14}$$

where A, B, C, and D are ω periodic functions, $B, C \in C(\mathbb{R}[0,\infty))$.

Lemma 5. *Suppose that there exist functions $u_0, v_0 \in X \cap C^1(\mathbb{R},\mathbb{R})$ such that $u_0 > 0$, $v_0 > 0$, and*

$$\begin{aligned} u_0' &\ge A(t)u_0 + B(t)v_0, \\ v_0' &\ge C(t)u_0 + D(t)v_0, \end{aligned} \tag{15}$$

where equality does not hold in all of the equations in (15). *Then,* (14) *satisfies the strong maximum principle; that is, if u, $v \in X \cap C^1(\mathbb{R},\mathbb{R})$ such that*

$$\begin{aligned} u' &\ge A(t)u + B(t)v, \\ v' &\ge C(t)u + D(t)v, \end{aligned} \tag{16}$$

then either (i) *$u, v \equiv 0$ or* (ii) *$u > 0$ and $v > 0$.*

Proof. Suppose that the result is false. Then, there exist u_1, v_1 not both identically zero satisfying inequalities (16) but not satisfying (ii) in the conclusion of the theorem. For $0 \le \tau \le 1$ define $u_\tau = (1-\tau)u_0 + \tau u_1$ and $v_\tau = (1-\tau)v_0 + \tau v_1$. Then, there exists τ_0, $0 < \tau_0 \le 1$, such that $u_\tau, v_\tau > 0$ for $0 \le \tau < \tau_0$ and either u_{τ_0} or v_{τ_0} has a zero. We may assume without loss of generality that there exists t_1 such that $u_{\tau_0}(t_1) = 0$. Then,

$$u_{\tau_0}' \ge A(t)u_{\tau_0} + B(t)v_{\tau_0}, \tag{17}$$

and so

$$u_{\tau_0}' - A(t)u_{\tau_0} \ge 0. \tag{18}$$

Moreover, K is chosen sufficiently large to ensure that $\int_0^\omega [K - A(t)]dt > 0$, so it follows from the strongly positive that $u_{\tau_0} \equiv 0$ or $u_{\tau_0} > 0$. Since $u_{\tau_0}(t_1) = 0$, it follows that $u_{\tau_0} \equiv 0$. Hence, by (17), it also follows that $v_{\tau_0} \equiv 0$. Since both u_1, v_1 are not identically zero and $\tau_0 < 1$, from $u_{\tau_0} = (1-\tau_0)u_0 + \tau_0 u_1 \equiv 0$ and $v_{\tau_0} = (1-\tau_0)v_0 + \tau_0 v_1 \equiv 0$, we have $u_0 = -(\tau_0/(1-\tau_0))u_1$ and $v_0 = -(\tau_0/(1-\tau_0))v_1$. This is impossible as u_0, v_0 satisfy (15) and u_1, v_1 satisfy (16) and so the proof is complete. □

Corollary 6. *Suppose $A(t) + B(t) < 0$ and $C(t) + D(t) < 0$. Then, the system* (14) *satisfies the maximum principle.*

Proof. The result follows from Lemma 5 by choosing $u_0 = v_0 = K$ where K is any positive number. □

Lemma 7. *The system* (14) *has a principal eigenvalue; that is, there exists $\Lambda \in \mathbb{R}$ and functions $u, v \in X \cap C^1(\mathbb{R},\mathbb{R})$ such that $u, v > 0$ and*

$$\begin{aligned} u' - A(t)u - B(t)v &= \Lambda u, \\ v' - C(t)u - D(t)v &= \Lambda v. \end{aligned} \tag{19}$$

Proof. Define $L : [X \cap C^1(\mathbb{R},\mathbb{R})]^2 \to X^2$ by

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix}, \tag{20}$$

and define the matrix $M(t)$ by

$$M(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}. \tag{21}$$

By essentially the same argument as in [12, Lemma 12], if $K > 0$ is sufficiently large, then $L - M + K$ is an invertible operator such that $(L - M + K)^{-1}$ is compact. If, moreover, K is chosen sufficiently large to ensure that $A(t) + B(t) - K < 0$ and $C(t) + D(t) - K < 0$, it follows from Corollary 6 that $(L - M + K)^{-1}$ is strongly positive.

Since $(L - M + K)^{-1}$ is compact and strongly positive, $(L - M + K)^{-1}$ has a positive principal eigenvalue (μ). Thus, there exists $U_0 = \binom{u_0}{v_0}$ with $u_0, v_0 > 0$ such that $(L - M + K)^{-1}U_0 = \mu U_0$. Hence, $(L - M)U_0 = (1/\mu - K)U_0$ and so $L - M$ has a principal eigenvalue. □

We will denote the principal eigenvalue of $L - M$ by $\lambda_1(M)$.

Corollary 8. *Suppose $M_1(t)$ and $M_2(t)$ are cooperative matrices (i.e., matrices with positive entries in the off-diagonal elements) such that $M_1(t) \geq M_2(t)$ (i.e., the (i,j)th element of $M_1 \geq$ the (i,j)th element of M_2 but $M_1 \not\equiv M_2$). Then, $\lambda_1(M_1) < \lambda_1(M_2)$.*

Proof. There exists $U_0 = \binom{u_0}{v_0}$ such that $u_0, v_0 > 0$ and $[L - M_1 - \lambda_1(M_1)]U_0 = 0$. Then,

$$\begin{aligned}[L - M_2 - \lambda_1(M_1)]U_0 &= [L - M_1 - \lambda_1(M_1)]U_0 + (M_1 - M_2)U_0 \\ &= (M_1 - M_2)U_0 \geq 0.\end{aligned} \tag{22}$$

But $(M_1 - M_2)U_0 \not\equiv 0$ and so by Lemma 5 the system $[L - M_2 - \lambda_1(M_1)]U = 0$ satisfies the strong maximum principle. Hence, if γ denotes the principal eigenvalue for the system $L - M_2 - \lambda_1(M_1)I$, it follows easily that $\gamma > 0$. Clearly, $L - M_2$ has principal eigenvalue $\lambda_1(M_1) + \gamma > \lambda_1(M_1)$ and so $\lambda_1(M_1) < \lambda_1(M_2)$, and the proof is complete. □

System (1) can be rewritten as

$$L\begin{pmatrix}u\\v\end{pmatrix} = M(t)\begin{pmatrix}u\\v\end{pmatrix} - N\begin{pmatrix}u\\v\end{pmatrix}, \tag{23}$$

where $M(t) = \begin{pmatrix}-c(t) & a(t)\\ b(t) & -d(t)\end{pmatrix}$ and $N : X^2 \to X^2$ such that

$$N\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}eu[u+v]\\ fv[u+v]\end{pmatrix}. \tag{24}$$

Although $M(t)$ is a cooperative matrix, system (1) is not a cooperative system. We can give necessary and sufficient conditions for the existence of a positive solution.

Proof of Theorem 2. Suppose $\lambda_1(M) < 0$. Then, there exists $\phi_1, \phi_2 > 0$ such that

$$(L - M)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \lambda_1(M)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}; \tag{25}$$

that is,

$$\begin{aligned}\phi_1' &= a(t)\phi_2 - c(t)\phi_1 + \lambda_1(M)\phi_1,\\ \phi_2' &= b(t)\phi_1 - d(t)\phi_2 + \lambda_1(M)\phi_2.\end{aligned} \tag{26}$$

Let $\binom{\underline{u}}{\underline{v}} = \varepsilon\binom{\phi_1}{\phi_2}$ and $\binom{\overline{u}}{\overline{v}} = \binom{K}{K}$. We will show that $\binom{\underline{u}}{\underline{v}}$ and $\binom{\overline{u}}{\overline{v}}$ satisfy the hypotheses of Theorem 1 provided that $\varepsilon > 0$ is chosen sufficiently small and K is chosen sufficiently large. Let $K = \max\{\max a(t)/e, \max b(t)/f\}$. Then, $a(t) - eK \leq 0$ and so

$$\begin{aligned}a(t)v - c(t)\overline{u} - e\overline{u}[\overline{u} + v] &= [a(t) - eK]v - c(t)K - eK^2 \\ &\leq 0 = \overline{u}'\end{aligned} \tag{27}$$

whenever $v \geq 0$. Similarly,

$$\overline{v}' \geq b(t)u - d(t)\overline{v} - f\overline{v}[u + \overline{v}], \tag{28}$$

whenever $u \geq 0$.

Let $\varepsilon_0 = \min\{\min a(t)/e\max\phi_1,\ \min b(t)/f\max\phi_2\}$. Then, when $\varepsilon < \varepsilon_0$, $a(t) - \varepsilon e\phi_1 \geq 0$ and $b(t) - \varepsilon f\phi_2 \geq 0$.

Hence, when $\varepsilon < \varepsilon_0$, $\underline{u} = \varepsilon\phi_1$, and $v \geq \varepsilon\phi_2$, we have

$$\begin{aligned}&\underline{u}' - a(t)v + c(t)\underline{u} + e\underline{u}[\underline{u} + v]\\ &\quad= \varepsilon\left[\phi_1' + c(t)\phi_1 - a(t)\phi_2\right] + a(t)[\varepsilon\phi_2 - v]\\ &\qquad+ e\varepsilon\phi_1[\varepsilon\phi_1 + v]\\ &\quad= \varepsilon\lambda_1(M)\phi_1 + \varepsilon a(t)\phi_2 - [a(t) - \varepsilon e\phi_1]v + \varepsilon^2 e[\phi_1]^2\\ &\quad\leq \varepsilon\lambda_1(M)\phi_1 + \varepsilon a(t)\phi_2 - [a(t) - \varepsilon e\phi_1]\varepsilon\phi_2 + \varepsilon^2 e[\phi_1]^2\\ &\quad= \varepsilon\lambda_1(M)\phi_1 + \varepsilon^2 e\phi_1[\phi_1 + \phi_2] < 0,\end{aligned} \tag{29}$$

when ε is sufficiently small.

Similarly, when $\underline{v} = \varepsilon\phi_2$, $u \geq \varepsilon\phi_1$, and ε is sufficiently small, we have

$$\underline{v}' - b(t)u + d(t)\underline{v} + f\underline{v}[u + \underline{v}] < 0. \tag{30}$$

Hence, by Theorem 1, there exists a positive solution of system (1).

Suppose now that system (1) has a solution $\binom{u_0}{v_0}$ with u_0, $v_0 > 0$. Then, $\binom{u_0}{v_0}$ is a solution of the system

$$\begin{aligned}u' + [c(t) + eq(t)]u - a(t)v &= 0,\\ v' - b(t)u + [d(t) + fq(t)]v &= 0,\end{aligned} \tag{31}$$

where $q(t) = u_0(t) + v_0(t)$. Hence, $\binom{u_0}{v_0}$ may be regarded as the principal eigenfunction corresponding to the principal eigenvalue $\lambda = 0$ of the system $(L - M_q(t))U = 0$, where

$$M_q(t) = \begin{pmatrix}-c(t) - eq(t) & a(t)\\ b(t) & -d(t) - fq(t)\end{pmatrix}, \qquad U = \begin{pmatrix}u\\v\end{pmatrix}. \tag{32}$$

Hence, $\lambda_1(M_q) = 0$. As $M_q \leq M$ but $M_q \not\equiv M$, it follows from Corollary 8 that $\lambda_1(M) < \lambda_1(M_q) = 0$ and the proof is complete. □

Remark 9. Let $M_1 = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$. Then $\lambda_1(M_1) = 0$. If $M > M_1$, then by Corollary 8 and Theorem 2 we know that system (1) has a positive solution. In fact, we can deduce from $M > M_1$ that $ab > cd$. Hence, our main results extend and complement the corresponding ones of [8] to some extent.

Example 10. Let $a(t) = 2 + \sin(\pi t/2)$; $b(t) = e^{t/2}$ and $t \in [0, 2]$ with $b(t+4) = b(t)$ and $b(t) = b(-t)$; $c(t) = (1/2)|\cos(\pi t/4)| + 1/2$; $d(t) = (1/4)t + 1/2$ and $t \in [0, 2]$ with $d(t+4) = d(t)$ and $d(-t) = d(t)$. We consider the following problem:

$$\begin{aligned}u' &= a(t)v - c(t)u - u[u+v],\\ v' &= b(t)u - d(t)v - v[u+v].\end{aligned} \tag{33}$$

It is easy to verify that $M > M_1$; then $\lambda_1(M) < 0$ by Corollary 8. Therefore, we know that system (33) has a positive solution by Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper is supported by the NSFC (nos. 11361054 and 11201378), SRFDP (no. 20126203110004) and Gansu Provincial National Science Foundation of China (no. 1208RJZA258).

References

[1] S. Bonhoeffer, *Ecology and Evolution II: Populations, Theoretical Biology*, Institute of Integrative Biology, ETH Zurich, 2005.

[2] R. L. Borrelli and C. S. Coleman, *Differential Equations: A Modeling Perspective*, John Wiley & Sons, New York, NY, USA, 1998.

[3] J. D. Murray, *Mathematical Biology*, Springer, New York, NY, USA, 2nd edition, 1993.

[4] P. H. Leslie, "In the use of matrices in certain population mathematics," *Biometrika*, vol. 33, pp. 183–212, 1945.

[5] M. B. Usher, "A matrix model for forest management," *Biometrics*, vol. 25, pp. 309–315, 1969.

[6] H. Caswell, *Matrix Population Models*, Sinauer Associates, Sunderland, UK, 2nd edition, 2001.

[7] J. M. Cushing, *An Introduction to Structured Population Dynamics*, vol. 71, The Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1998.

[8] M. Y. M. Alzoubi and A. A. Al-Basheer, "Adult-juvenile population growth model with density dependence," *International Journal of Pure and Applied Mathematics*, vol. 52, no. 5, pp. 701–709, 2009.

[9] R. Chen, R. Ma, and Z. He, "Positive periodic solutions of first-order singular systems," *Applied Mathematics and Computation*, vol. 218, no. 23, pp. 11421–11428, 2012.

[10] X. Yang, "Upper and lower solutions for periodic problems," *Applied Mathematics and Computation*, vol. 137, no. 2-3, pp. 413–422, 2003.

[11] X. Yang, "The method of lower and upper solutions for systems of boundary value problems," *Applied Mathematics and Computation*, vol. 144, no. 1, pp. 169–172, 2003.

[12] S. M. Bouguima, S. Fekih, and W. Hennaoui, "Spacial structure in a juvenile-adult model," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 3, pp. 1184–1201, 2008.