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Research Article

Homotopy Perturbation Method to Obtain Positive Solutions of Nonlinear Boundary Value Problems of Fractional Order

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We use the homotopy perturbation method for solving the fractional nonlinear two-point boundary value problem. The obtained results by the homotopy perturbation method are then compared with the Adomian decomposition method. We solve the fractional Bratu-type problem as an illustrative example.

1. Introduction

In the last three decades, extensive work has been done using the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM), which provide analytical approximation for nonlinear differential equation [1–5]. These methods have been implemented in several boundary value problems with all types of boundary conditions. In the literature, nonlinear boundary value problems (BVP) have been studied extensively [4–9]. In the present paper, we obtain positive solutions of nonlinear fractional order BVP using the HPM. We investigate the following type of fractional BVP:

$$D^{\alpha}u(x) + \mu F(x, u(x)) = 0, \quad 0 < x < 1,\ 1 < \alpha \le 2,\ \mu > 0,\ u(0) = 0,\ u(1) = c, \tag{1}$$

where D^{α} is Caputo fractional derivative, c is a constant, $\mu > 0$, and $F : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. Bai and Lü have discussed existence and multiplicity results of positive solution of (1) by means of fixed-point theorems [10]. Jafari and Daftardar-Gejji have applied the ADM to obtain positive solution of the above equation [11].

In this paper, we employ the HPM to obtain positive solutions of (1) and then we compare the obtained result with those obtained by the Adomian decomposition method.

2. Basic Definition

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_{α}, $\alpha \in R$, if there exists a real number $p(> a)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0,\infty)$. Clearly $C_{\alpha} \subset C_{\beta}$ if $\beta \le \alpha$.

Definition 2. A function $f(x)$, $x > 0$, is said to be in the space C_{α}^{m}, $m \in N \cup \{0\}$, if $f^{(m)} \in C_{\alpha}$.

Definition 3. The left sided Riemann-Liouville fractional integral of order $\mu \ge 0$ [12–14] of a function $f \in C_{\alpha}$, $\alpha \ge -1$ is defined as

$$I^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad \mu > 0,\ x > 0, \qquad I^0 f(x) = f(x). \tag{2}$$

Definition 4. Let $f \in C_{-1}^{m}$, $m \in N \cup \{0\}$. Then the (left sided) Caputo fractional derivative of f is defined as [1, 12, 14]

$$D^{\mu} f(x) = \begin{cases} I^{m-\mu} f^{(m)}(x) & m-1 < \mu \le m,\ m \in N, \\ \dfrac{d^m f(x)}{dx^m} & \mu = m. \end{cases} \tag{3}$$

Note that [1, 12–14]

$$I^{\mu}I^{\nu}f = I^{\mu+\nu}f, \quad \mu,\nu \geq 0,$$
$$f \in C_{\alpha}, \quad \alpha \geq -1,$$
$$I^{\mu}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}x^{\gamma+\mu}, \quad \mu > 0,\ \gamma > -1,\ x > 0, \tag{4}$$
$$I^{\mu}D^{\mu}f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{x^k}{k!}, \quad m-1 < \mu \leq m.$$

3. Homotopy Perturbation Method

In this section we employ the HPM to obtain positive solution of nonlinear fractional order BVP (1). We construct the following homotopy for (1):

$$\mathscr{H}(v;p) = D^{\alpha}u(x) + p\mu F(x,u(x)) = 0. \tag{5}$$

The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $D^{\alpha}u(x) = 0$ is continuously transformed to the original problem $D^{\alpha}u(x) + \mu F(x,u(x)) = 0$. If $p = 0$, then (5) becomes a linear equation

$$D^{\alpha}u(x) = 0 \tag{6}$$

and when $p = 1$, then (5) turns out to be the original equation (1).

The HPM uses the embedding parameter p as a "small parameter," and writes the solution of (5) as a power series of p; that is,

$$v = v_0 + v_1 p + v_2 p^2 + \cdots, \tag{7}$$

$$F(x,v(x)) = H_0(v_0) + H_1(v_0,v_1)p + H_2(v_0,v_1,v_2)p^2 + \cdots, \tag{8}$$

where $H_j(v_0,v_1,\ldots,v_j)$ depend upon $v_0,v_1,\ldots,v_j$. In view of (8), $H_j(v_0,v_1,\ldots,v_j)$ will be calculated like Adomian polynomials [15, 16]

$$H_j(v_0,v_1,\ldots,v_j) = \frac{1}{j!}\frac{\partial^j}{\partial p^j}F\left(x,\sum_{i=0}^{j}v_i p^i\right)\Bigg|_{p=0}. \tag{9}$$

Substituting (7) and (8) into (5) leads to

$$\mathscr{H}(v;p) = \sum_{m=0}^{\infty}D^{\alpha}v_m p^m + p\mu\sum_{m=0}^{\infty}H_m(v_0,\ldots,v_m)p^m = 0. \tag{10}$$

Now equating the terms with identical powers of p, we can obtain a series of equations of the following form:

$$\begin{aligned}
&p^0: D^{\alpha}u_0 = 0, \quad u_0(0) = 0,\ u_0'(0) = \beta,\\
&p^1: D^{\alpha}u_1 + \mu H_0(v_0) = 0, \quad u_1(0) = 0,\ u_1'(0) = 0,\\
&p^2: D^{\alpha}u_2 + \mu H_1(v_0,v_1) = 0, \quad u_2(0) = 0,\ u_2'(0) = 0,\\
&p^3: D^{\alpha}u_3 + \mu H_2(v_0,v_1,v_2) = 0, \quad u_3(0) = 0,\ u_3'(0) = 0.\\
&\ \vdots
\end{aligned} \tag{11}$$

It is obvious that the system of nonlinear equations in (11) is easy to solve and the components v_i, $i \geq 0$, of the homotopy perturbation method can be completely determined and the series solutions are thus entirely determined.

Setting $p = 1$ results in the approximate solution of (7):

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{12}$$

We can approximate the solution u by accelerating $p = 1$ and the truncated series:

$$\varphi_k = \sum_{m=0}^{k-1} v_m, \qquad \lim_{k\to\infty}\varphi_k = u(x), \tag{13}$$

where $\beta = v'(0)$ will be determined by applying suitable boundary conditions of (1).

4. Illustrative Examples

In this section, to give a clear overview of the HPM for fractional nonlinear BVP, we present the following examples. We apply the HPM and compare the results with the ADM.

Example 1. Consider the following nonlinear boundary value problem [11]:

$$D^{\alpha}u + u^2(x) - x^4 - 2 = 0, \quad 1 < \alpha \leq 2, \quad 0 \leq x \leq 1, \quad u(0) = 0, \quad u(1) = 1. \tag{14}$$

According to the modified HPM [17], we construct the following homotopy:

$$D^{\alpha}v - 2 + p\left(v^2(x) - x^4\right) = 0. \tag{15}$$

Substitution of (7) and (8) into (15) and then equating the terms with same powers of p yield the following series of equations:

$$\begin{aligned}
&p^0: D^{\alpha}v_0 - 2 = 0, \quad v_0(0) = 0,\ v_0'(0) = \beta,\\
&p^1: D^{\alpha}v_1(x) - x^4 + H_0 = 0, \quad v_1(0) = 0,\ v_1'(0) = 0,\\
&p^2: D^{\alpha}v_2(x) + H_1 = 0, \quad v_2(0) = 0,\ v_2'(0) = 0,\\
&\ \vdots\\
&p^{m+1}: D^{\alpha}v_{m+1}(x) + H_m = 0, \quad v_m(0) = 0,\ v_m'(0) = 0,
\end{aligned} \tag{16}$$

where H_m are defined as

$$H_m = \sum_{j=0}^{m} v_j v_{m-j}. \tag{17}$$

In view of (4) and (17), by applying the inverse operator I^{α} on both sides of (16) and solving corresponding integrals we get

$$v_0 = x\beta + \frac{2x^{\alpha}}{\alpha\Gamma(\alpha)},$$

$$v_1 = x\beta + \frac{2x^{\alpha}}{\alpha\Gamma(\alpha)} - \frac{2x^{\alpha+2}\beta^2}{\alpha(2+3\alpha+\alpha^2)\Gamma(\alpha)} + \frac{24x^{\alpha+4}}{\Gamma(\alpha+5)} - \frac{4x^{2\alpha+1}\beta\Gamma(2+\alpha)}{\alpha\Gamma(\alpha)\Gamma(2+2\alpha)} - \frac{8x^{3\alpha}\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+3\alpha)},$$

$$
\begin{aligned}
v_2 = {} & x\beta + \frac{2x^{\alpha}}{\alpha\Gamma(\alpha)} - \frac{2x^{\alpha+2}\beta^2}{\alpha(2+3\alpha+\alpha^2)\Gamma(\alpha)} + \frac{24x^{\alpha+4}}{\Gamma(\alpha+5)} \\
& - \frac{4x^{2\alpha+1}\beta\Gamma(2+\alpha)}{\alpha\Gamma(\alpha)\Gamma(2+2\alpha)} + \frac{8x^{3+2\alpha}\Gamma(4+\alpha)}{(\alpha+\alpha^2)\Gamma(\alpha)\Gamma(5+2\alpha)} \\
& - \frac{48x^{5+2\alpha}\beta\Gamma(6+\alpha)}{\Gamma(5+\alpha)\Gamma(6+2\alpha)} - \frac{8x^{3\alpha}\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\
& + \frac{16x^{2+3\alpha}\beta^2\Gamma(2+\alpha)}{\Gamma(\alpha)\Gamma(3+3\alpha)} + \frac{16x^{2+3\alpha}\beta^2\Gamma(2+\alpha)}{\alpha\Gamma(\alpha)\Gamma(3+3\alpha)} \\
& + \frac{8x^{2+3\alpha}\beta^2\Gamma(3+2\alpha)}{\alpha\Gamma(\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} \\
& - \frac{96x^{4+3\alpha}\Gamma(5+2\alpha)}{\alpha\Gamma(\alpha)\Gamma(5+\alpha)\Gamma(5+3\alpha)} \\
& + \frac{16x^{1+4\alpha}\beta\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(2+4\alpha)} \\
& + \frac{48x^{1+4\alpha}\alpha\beta\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(2+4\alpha)} \\
& + \frac{16x^{1+4\alpha}\beta\Gamma(2+3\alpha)}{\Gamma(\alpha)\Gamma(2+2\alpha)\Gamma(2+4\alpha)} \\
& + \frac{16x^{1+4\alpha}\beta\Gamma(2+3\alpha)}{\alpha\Gamma(\alpha)\Gamma(2+2\alpha)\Gamma(2+4\alpha)} \\
& + \frac{32x^{5\alpha}\Gamma(2\alpha)\Gamma(1+4\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)^2\Gamma(1+3\alpha)\Gamma(1+5\alpha)}.
\end{aligned}
\tag{18}
$$

Other components are determined similarly. Further we compute $u(x)$ for various values of α.

For $\alpha = 1.2$,

$$
\begin{aligned}
u(x) = l_1 \approx {} & 1.81521x^{1.2} - 0.734086x^{3.6} + 0.141671x^{5.2} \\
& + 0.316924x^{6} - 0.0458495x^{7.6} + x\beta \\
& - 0.868184x^{3.4}\beta + 0.57265x^{5.8}\beta - 0.0260918x^{7.4}\beta \\
& - 0.257842x^{3.2}\beta^2 + 0.345771x^{5.6}\beta^2 \\
& + 0.0697577x^{5.4}\beta^3.
\end{aligned}
\tag{19}
$$

For $\alpha = 1.5$,

$$
\begin{aligned}
u(x) = l_2 \approx {} & 1.50451x^{1.5} - 0.259467x^{4.5} + 0.0833665x^{5.5} \\
& + 0.0400539x^{7.5} - 0.0105982x^{8.5} + x\beta \\
& - 0.416667x^{4}\beta + 0.101256x^{7}\beta - 0.0077381x^{8}\beta \\
& - 0.171943x^{3.5}\beta^2 + 0.0866187x^{6.5}\beta^2 + 0.025x^{6}\beta^3.
\end{aligned}
\tag{20}
$$

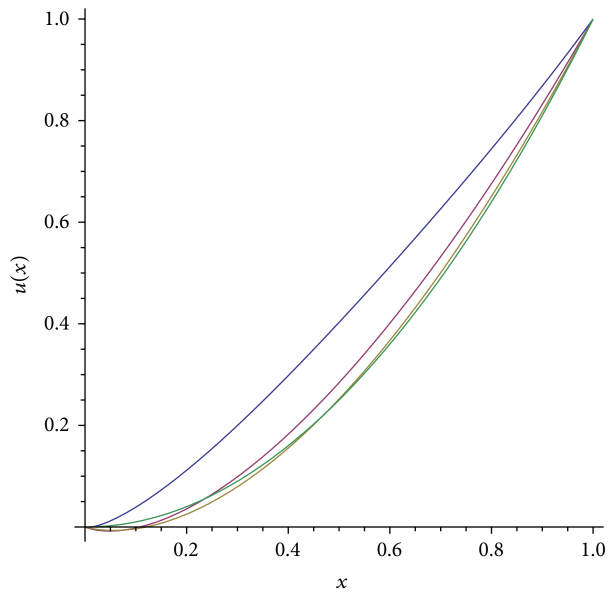


Figure 1

For $\alpha = 1.7$,

$$
\begin{aligned}
u(x) = l_3 \approx {} & 1.2947x^{1.7} - 0.119284x^{5.1} + 0.0580541x^{5.7} \\
& + 0.008744x^{8.5} - 0.0037705x^{9.1} + x\beta \\
& - 0.242159x^{4.4}\beta + 0.0285576x^{7.8}\beta \\
& - 0.00335666x^{8.4}\beta - 0.129606x^{3.7}\beta^2 \\
& + 0.0320066x^{7.1}\beta^2 + 0.0121972x^{6.4}\beta^3.
\end{aligned}
\tag{21}
$$

For $\alpha = 2$,

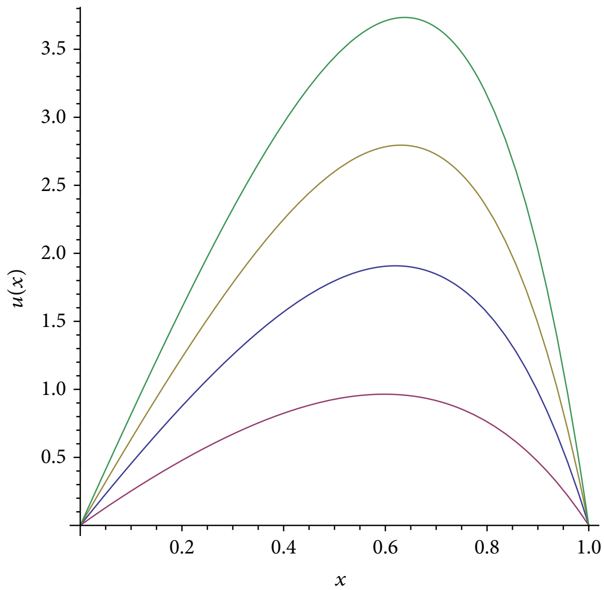
$$
\begin{aligned}
u(x) = l_4 \approx {} & x^2 + x\beta - \frac{x^5\beta}{10} + \frac{x^9\beta}{360} - \frac{x^4\beta^2}{12} \\
& + \frac{11x^8\beta^2}{1680} + \frac{x^7\beta^3}{252}.
\end{aligned}
\tag{22}
$$

To determine β, we impose the boundary condition at $x = 1$ and using $u(1) = 1$, then

$$
\begin{aligned}
\beta &= -0.757698 \quad \text{for } \alpha = 1.2, \\
\beta &= -0.493564 \quad \text{for } \alpha = 1.5, \\
\beta &= -0.29346 \quad \text{for } \alpha = 1.7, \\
\beta &= 0 \quad \text{for } \alpha = 2.
\end{aligned}
\tag{23}
$$

In Figure 1 we plot $l_1, \ldots, l_4$.

It should be remarked that the graphs drawn here using the HPM are in excellent agreement with those drawn using the ADM [11].



$u(x) = \varphi_5$

FIGURE 2

Example 2. Consider the one-dimensional fractional planar Bratu-type problem [4]

$$D^{\alpha}u(x) + \mu e^{u(x)} = 0,$$
$$0 \le x \le 1, \quad 1 < \alpha \le 2, \quad u(0) = 0, \quad u(1) = 0. \tag{24}$$

In view of (5) we construct the following homotopy:

$$\mathscr{H}(v; p) = D^{\alpha}v(x) + p\mu e^{v(x)} = 0. \tag{25}$$

Substituting (7) and (8) into (25) and proceeding as before we have

$$p^0: D^{\alpha}v_0 = 0, \quad v_0(0) = 0, \ v_0'(0) = \beta,$$
$$p^{m+1}: D^{\alpha}v_{m+1}(x) + \mu H_m = 0, \quad v_m(0) = 0, \ v_m'(0) = 0, \tag{26}$$

where the first few Adomian polynomials H_n that represent the nonlinear term $e^{v(x)}$ are defined as

$$H_0 = e^{v_0},$$
$$H_1 = e^{v_0}v_1,$$
$$H_2 = \frac{e^{v_0}v_1^2}{2} + e^{v_0}v_2. \tag{27}$$
$$\vdots$$

In Figure 2 we plot l_i, $i = 1, 2, 3, 4$ where l_i are as defined in Example 2 and $\mu = 1$.

Remark 5. The graphs drawn in Figure 2 are in excellent agreement with those drawn in [11] using the ADM.

Theorem 6. *The homotopy perturbation method for solving nonlinear fractional BVP (1) is Adomian's decomposition method with the homotopy $\mathscr{H}(v; p)$ given by*

$$\mathscr{H}(v; p) = D^{\alpha}v(x) + p\mu F(x, v(x)). \tag{28}$$

Proof. Substituting (7) and (8) into (5) and equating the terms with the identical powers of p, we have

$$\mathscr{H}(v; p) = \sum_{i=0}^{\infty} D^{\alpha}v_i p^i + p\mu \sum_{i=0}^{\infty} H_i p^i = 0, \tag{29}$$

$$p^0: D^{\alpha}v_0 = 0, \quad v_0(0) = 0, \ v_0'(0) = \beta,$$
$$p^{n+1}: D^{\alpha}v_{n+1} + \mu H_n = 0, \tag{30}$$
$$v_n(0) = 0, \quad v_n'(0) = 0, \quad n = 0, 1, 2, \ldots.$$

Applying the inverse operator I^{α} on both sides of (30), we have

$$v_0 = \beta x$$
$$v_{n+1} = \mu I^{\alpha}H_n, \quad n = 0, 1, 2, \ldots. \tag{31}$$

We know $H_n = A_n$. Substituting (31) in (7) leads us to

$$v = v_0 + v_1 p + v_2 p^2 + \cdots = \beta x + I^{\alpha}A_0 p + I^{\alpha}A_1 p^2 + \cdots. \tag{32}$$

So

$$\lim_{p\to 1} v = \beta x + I^{\alpha}A_1 + I^{\alpha}A_2 + \cdots$$
$$= \beta x + I^{\alpha}\sum_{i=0}^{\infty} A_i = \sum_{i=0}^{\infty} u_i = u. \tag{33}$$

Therefore, by letting

$$\mathscr{H}(v; p) = v - f(x) - pN(v), \tag{34}$$

we observe that the power series $v_0 + v_1 p + v_2 p^2 + \cdots$ corresponds to the solution of the equation $\mathscr{H}(v; p) = D^{\alpha}v + p\mu F(x, v(x)) = 0$ and becomes the approximate solution of (1) if $p \to 1$. This shows that the homotopy perturbation method is Adomian's decomposition method with the homotopy $\mathscr{H}(v; p)$ given by (34). The proof of Theorem 6 is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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