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Research Article

Weighted Dual Covariance Moore-Penrose Inverses with respect to an Invertible Element in C^*-Algebras

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We study several algebraic properties of dual covariance and weighted dual covariance sets in rings with involution and C^*-algebras. Moreover, we show that the weighted dual covariance set, seen as a multivalued map, has some kind of continuity. Also, we prove weighed dual covariance set invariant under the bijection multiplicative $*$-functions.

1. Introduction

Suppose $\mathfrak{R}$ is a ring with unity $1 \neq 0$. A mapping $* : x \mapsto x^*$ of $\mathfrak{R}$ into itself is called an *involution* if

$$(x^*)^* = x, \qquad (x+y)^* = x^* + y^*, \qquad (xy)^* = y^* x^*, \tag{1}$$

for all x and y in $\mathfrak{R}$. Throughout this paper $\mathfrak{R}$ will be a ring $\mathfrak{R}$ with an involution.

An element $a \in \mathfrak{R}$ is called *regular* if it has a generalized inverse in $\mathfrak{R}$.

It is well known that every regular element in a C^*-algebra has the Moore-Penrose inverse (denoted by MP-inverse from this point on). Generally MP-inverse is uniquely determined in $\mathfrak{R}$ if it exists. We will denote the MP-inverse of a by $a^\dagger$.

In the following, we will denote by $\mathfrak{R}^{-1}$ and $\mathfrak{R}^\dagger$ the set of an invertible and MP-invertible elements of $\mathfrak{R}$, respectively.

Assume that a is an element in $\mathfrak{R}^{-1}$. Its inverse a^{-1} is *covariant* with respect to $\mathfrak{R}^{-1}$; that is, for all $b \in \mathfrak{R}^{-1}$, we have

$$\left(bab^{-1}\right)^{-1} = ba^{-1}b^{-1}. \tag{2}$$

In general, the elements of $\mathfrak{R}^\dagger$ are not covariant under $\mathfrak{R}^{-1}$ (see [1]). For a given element $a \in \mathfrak{R}^\dagger$ with MP-inverse $a^\dagger$, we will denote the *covariance set* by $\mathfrak{C}(a)$ and define

$$\mathfrak{C}(a) = \left\{ b \in \mathfrak{R}^{-1} : \left(bab^{-1}\right)^\dagger = ba^\dagger b^{-1} \right\}. \tag{3}$$

For more definitions and notations we refer the interested readers to [2]. Covariance set was studied by [1, 3–5].

We define the *dual covariance set* by reversing the roles of a and b in $\mathfrak{C}(a)$ and denote it by $\mathfrak{D}(b)$. In fact,

$$\mathfrak{D}(b) = \left\{ a \in \mathfrak{R}^\dagger : \left(bab^{-1}\right)^\dagger = ba^\dagger b^{-1} \right\}. \tag{4}$$

This notion was studied by Robinson in [6] for matrices.

Note that if $a \in \mathfrak{R}^\dagger$ with MP-inverse $a^\dagger$ and $b \in \mathfrak{R}^{-1}$, then, from (3) and (4), we obtain

$$b \in \mathfrak{C}(a) \quad \text{iff } a \in \mathfrak{D}(b). \tag{5}$$

Also, it should be noted that $\mathfrak{C}(a) \subset \mathfrak{R}^{-1} \subset \mathfrak{D}(b) \subset \mathfrak{R}^\dagger$ for every $a \in \mathfrak{R}^\dagger$ and for each $b \in \mathfrak{R}^{-1}$. Moreover, this inclusion can be proper; for instance, $0 \in \mathfrak{D}(b)$ but $0 \notin \mathfrak{C}(a)$.

The aim of this paper is to investigate the properties of dual covariance and weighted dual covariance set. In Section 2, we define and characterize the weighted dual covariance in terms of commutators. Also we prove that the dual covariance sets of b^{-1} and b^* coincide. Moreover, we collect some interesting properties of $\mathfrak{D}_{e,f}(b)$ and $\mathfrak{D}(b)$ in C^*-algebras and rings with an involution. In addition, we show that the weighted dual covariance set, seen as a multivalued map, has some kind of continuity. In Section 3 we study the relations of $\mathfrak{D}_{e,f}(b)$ and multiplicative $*$-functions. Also, we prove that weighed dual covariance sets are invariant under the bijection multiplicative $*$-functions.

2. Weighted Dual Covariance Set

The weighted Moore-Penrose inverse (weighted MP-inverse from this point on) for matrices was introduced by Chipman in [7]. For some historical notes of weighted MP-inverse see Rao and Mitra [8] and references therein. In the next definition, it will be introduced in C^*-algebras (see also [9]). In what follows, we will only consider unital C^*-algebras. Indeed, $\mathfrak{A}$ and $\mathfrak{B}$ are unital C^*-algebras; the nonzero elements, $1_{\mathfrak{A}}$ and $1_{\mathfrak{B}}$, are the units of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. We will denote by $\mathfrak{A}^{-1}$ and $\mathfrak{A}^{\dagger}$ the subset of invertible elements and MP-invertible elements of $\mathfrak{A}$, respectively.

Definition 1. Let $\mathfrak{A}$ be a C^*-algebra and e, f two positive elements in $\mathfrak{A}^{-1}$. We say that an element $a \in \mathfrak{A}$ has a weighted MP-inverse with weights e, f if there exists $b \in \mathfrak{A}$ such that

$$\begin{aligned} aba = a, \qquad & bab = b, \\ e^{-1}(ab)^* e = ab, \qquad & f^{-1}(ba)^* f = ba. \end{aligned} \tag{6}$$

As already observed in [9], if weighted MP-inverse with weights e, f exists, then it is unique, and so we will denote it by $a^{\dagger}_{e,f}$. Every regular element in a C^*-algebra has a weighted MP-inverse [9, Theorem 4] and it can be written as

$$a^{\dagger}_{e,f} = f^{-1/2}\left(e^{1/2}af^{-1/2}\right)^{\dagger} e^{1/2}. \tag{7}$$

Next, we extend the definition of dual covariance set to *weighted dual covariance set*.

Definition 2. Suppose $b \in \mathfrak{A}^{-1}$ and e, f are positive elements in $\mathfrak{A}^{-1}$. We define weighted dual covariance set by

$$\begin{aligned} \mathfrak{D}_{e,f}(b) = \{ a \in \mathfrak{A}^{\dagger} : b^{-1}a^{\dagger}b \text{ is weighted MP-inverse} \\ \text{of } b^{-1}ab \text{ with weights } e, f \}. \end{aligned} \tag{8}$$

In the following theorem we characterize $\mathfrak{D}_{e,f}(b)$ in terms of commutators.

Theorem 3. *Assume $b \in \mathfrak{A}^{-1}$ and e, f are positive elements in $\mathfrak{A}^{-1}$. Then the following statements are equivalent:*

(i) $a \in \mathfrak{D}_{e,f}(b)$;

(ii) $[a^{\dagger}a, b^*eb] = 0$ *and* $[aa^{\dagger}, b^*fb] = 0$.

Proof. (i)⇒(ii): suppose $a \in \mathfrak{C}_{e,f}(b)$. Then $b^{-1}a^{\dagger}b$ is a weighted MP-inverse of $b^{-1}ab$ with weights e, f. Thus, $f^{-1}(ba^{\dagger}ab^{-1})^* f = ba^{\dagger}ab^{-1}$. Therefore, $(b^*f)^{-1}a^{\dagger}ab^*fb = ba^{\dagger}a$. This implies that $[a^{\dagger}a, b^*fb] = 0$. In a similar manner from $e^{-1}(baa^{\dagger}b^{-1})^*e = ba^{\dagger}ab^{-1}$ we conclude that $[aa^{\dagger}, b^*eb] = 0$.

(ii)⇒(i): since $a^{\dagger}$ is MP-inverse of a, it suffices to show that $e^{-1}(ba^{\dagger}ab^{-1})^*e = ba^{\dagger}ab^{-1}$ and $f^{-1}(ba^{\dagger}ab^{-1})^*f = ba^{\dagger}ab^{-1}$. By the assumptions $[a^{\dagger}a, b^*fb] = 0$. From this we obtain $f^{-1}(b^*)^{-1}a^{\dagger}ab^*fb = ba^{\dagger}a$. Hence, $f^{-1}(ba^{\dagger}ab^{-1})^*f = ba^{\dagger}ab^{-1}$. In a similar manner from $[aa^{\dagger}, b^*eb] = 0$ we get $e^{-1}(baa^{\dagger}b^{-1})^*e = baa^{\dagger}b^{-1}$. □

The following result is obtained from Theorem 3 by setting $e = f = 1$ for C^*-algebras. However, it is true for a generalized case in rings with involution. In fact, consider the following.

Proposition 4. *Assume that $b \in \mathfrak{R}^{-1}$. Then*

$$a \in \mathfrak{D}(b) \quad iff \quad \left[a^{\dagger}a, b^*b\right] = 0, \ \left[aa^{\dagger}, b^*b\right] = 0. \tag{9}$$

Proposition 5. *Assume that $b \in \mathfrak{R}^{-1}$. Then $\mathfrak{D}(b^*) = \mathfrak{D}(b^{-1})$.*

Proof. By Proposition 4,

$$a \in \mathfrak{D}(b^*) \quad \text{iff} \quad \left[a^{\dagger}a, bb^*\right] = 0, \ \left[aa^{\dagger}, bb^*\right] = 0. \tag{10}$$

This is equivalent to

$$a^{\dagger}abb^* = bb^*a^{\dagger}a, \qquad aa^{\dagger}bb^* = bb^*aa^{\dagger}. \tag{11}$$

Multiply (11) from left and right by $(bb^*)^{-1}$; we get

$$\left[a^{\dagger}a, \left(b^{-1}\right)^* b^{-1}\right] = 0, \qquad \left[aa^{\dagger}, \left(b^{-1}\right)^* b^{-1}\right] = 0. \tag{12}$$

Again Proposition 4 shows that (12) holds if and only if $a \in \mathfrak{D}(b^{-1})$. □

Proposition 6. *Assume that $b \in \mathfrak{R}^{-1}$ and b is normal. Then $\mathfrak{D}(b) = \mathfrak{D}(b^{-1}) = \mathfrak{D}(b^n)$, where n is an integer number.*

Proof. Since b is normal, the first equality is an immediate consequence of Proposition 5. For proof of the second equality, obviously $\mathfrak{D}(b^n) \subset \mathfrak{D}(b)$. For the converse suppose that $a \in \mathfrak{D}(b)$; using Proposition 4, normality of b, and induction, one can get $b^n \in \mathfrak{D}(b)$ for all integer n. Thus $a \in \mathfrak{D}(b^n)$.

The following example shows that the normality hypothesis cannot be omitted from the above proposition. □

Example 7. Set $a = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 2\sqrt{7} & -2 & \sqrt{7} \\ 0 & 1 & 2\sqrt{7} \\ \sqrt{7} & 4 & 0 \end{bmatrix}$. Then $a = a^{\dagger} \in \mathfrak{D}(b)$ since $b^*baa^{\dagger} = aa^{\dagger}bb^* = a^{\dagger}abb^* = bb^*a^{\dagger}a = \begin{bmatrix} 14 & -21 & 35 \\ 21 & 42 & -21 \\ 35 & 21 & 14 \end{bmatrix}$. But $a \notin \mathfrak{D}(b^{-1})$, because

$$\begin{aligned} (b^*)^{-1}baa^{\dagger} &= \begin{bmatrix} -\frac{6}{343} & -\frac{61}{1029} & \frac{43}{1029} \\ \frac{37}{1029} & \frac{30}{343} & -\frac{53}{1029} \\ \frac{43}{1029} & \frac{15}{343} & -\frac{2}{1029} \end{bmatrix} \\ &\neq \begin{bmatrix} \frac{2}{343} & -\frac{15}{343} & \frac{17}{343} \\ \frac{13}{1029} & \frac{74}{1029} & -\frac{61}{1029} \\ \frac{19}{1029} & \frac{29}{1029} & -\frac{10}{1029} \end{bmatrix} = aa^{\dagger}(b^*)^{-1}b. \end{aligned} \tag{13}$$

We recall that a *cone* is a set of rays; in other words $K \subset \mathfrak{A}$ is a cone if $x \in K$ implies $\lambda x \in K$ for each $\lambda \geq 0$. Also, an element $a \in \mathfrak{R}$ is called simply polar [10] if it has a commuting generalized inverse (in the sense of von Neumann); that is, there exists a generalized inverse c of a, such that $[a, c] = 0$.

In the next result we collect some noteworthy properties of weighted dual covariance set.

Proposition 8. *Assume that $b \in \mathfrak{A}^{-1}$. Then, the following statements are equivalent:*

(i) $a \in \mathfrak{D}_{e,f}(b)$;

(ii) $a^\dagger \in \mathfrak{D}_{f,e}(b)$;

(iii) $a^* \in \mathfrak{D}_{f,e}(b)$;

(iv) $aa^\dagger \in \mathfrak{D}_{e,f}(b)$ *and* $a^\dagger a \in \mathfrak{D}_{e,f}(b)$;

(v) $\lambda a \in \mathfrak{D}_{e,f}(b)$ *for any nonzero scalar* λ.
Moreover, if a is simply polar, then the following statements are equivalent to the above statements:

(vi) $aa^\dagger \in \mathfrak{D}_{e,f}(b)$;

(vii) $a^\dagger a \in \mathfrak{D}_{e,f}(b)$.

Proof. First we show that (i)⇔(ii) by Theorem 3 $a \in \mathfrak{D}_{e,f}(b)$ if and only if

$$\left[a^\dagger a, b^* eb\right] = 0, \qquad \left[aa^\dagger, b^* fb\right] = 0. \tag{14}$$

Since $(a^\dagger)^\dagger = a$, thus (14) is equivalent to

$$\left[a^\dagger \left(a^\dagger\right)^\dagger, b^* eb\right] = 0, \qquad \left[\left(a^\dagger\right)^\dagger a^\dagger, b^* fb\right] = 0. \tag{15}$$

Again, Theorem 3 shows that (15) holds if and only if $a^\dagger \in \mathfrak{D}_{f,e}(b)$.

(i)⇔(iii): in a similar manner, since $a^\dagger a = a^*(a^*)^\dagger$ and $aa^\dagger = (a^*)^\dagger a^*$ by applying Theorem 3, we get $a \in \mathfrak{D}_{e,f}(b) \Leftrightarrow a^* \in \mathfrak{D}_{f,e}(b)$.

(i)⇒(iv): $a \in \mathfrak{D}_{e,f}(b)$ if and only if (14) holds. Since $aa^\dagger = aa^\dagger aa^\dagger$ and $(aa^\dagger)^\dagger = aa^\dagger$, thus (14) is equivalent to

$$\left[aa^\dagger \left(aa^\dagger\right)^\dagger, b^* eb\right] = 0, \qquad \left[\left(aa^\dagger\right)^\dagger aa^\dagger, b^* fb\right] = 0. \tag{16}$$

This implies that $aa^\dagger \in \mathfrak{D}_{e,f}(b)$. Similarly we get

$$\left[a^\dagger a \left(a^\dagger a\right)^\dagger, b^* eb\right] = 0, \qquad \left[\left(a^\dagger a\right)^\dagger a^\dagger a, b^* fb\right] = 0. \tag{17}$$

Thus, $a^\dagger a \in \mathfrak{D}_{e,f}(b)$.

For the proof of (iv)⇒(i), it is easy to verify that (iv) is satisfied if and only if (16) and (17) hold. These together imply (14); that is, (i) holds.

(i)⇔(v): since $\lambda \neq 0$, $(\lambda a)^\dagger = (1/\lambda) a^\dagger = (a\lambda)^\dagger$. Now applying Theorem 3 we obtain the result.

(vi)⇔(vii): since a is simply polar, thus $aa^\dagger = a^\dagger a$ and so (vi) and (vii) are equivalent.

(iv)⇔(vi) is obvious. □

Corollary 9. *If $b \in \mathfrak{A}^{-1}$, then $\mathfrak{D}_{e,f}(b)$ is a cone.*

It is well known that every normal element is simply polar. Hence, consider the following.

Corollary 10. *If a is normal, then*

$$a \in \mathfrak{D}_{e,f}(b) \Longleftrightarrow aa^\dagger \in \mathfrak{D}_{e,f}(b) \Longleftrightarrow a^\dagger a \in \mathfrak{D}_{e,f}(b). \tag{18}$$

Corollary 11. *Assume that $b \in \mathfrak{R}^{-1}$. Then for each $\lambda \neq 0$,*

$$\begin{aligned} a \in \mathfrak{D}(b) &\Longleftrightarrow a^\dagger \in \mathfrak{D}(b) \Longleftrightarrow a^* \in \mathfrak{D}(b) \\ &\Longleftrightarrow \lambda a \in \mathfrak{D}(b) \Longleftrightarrow aa^\dagger \in \mathfrak{D}(b), \\ & a^\dagger a \in \mathfrak{D}(b). \end{aligned} \tag{19}$$

Proposition 12. *Assume that $b \in \mathfrak{A}^{-1}$ and $\lambda \neq 0$ is any scalar. Then $\mathfrak{D}_{e,f}(b) = \mathfrak{D}_{e,f}(\lambda b)$.*

Proof. By Theorem 3, $a \in \mathfrak{D}_{e,f}(b)$ if and only if (14) is satisfied which is equivalent to

$$\left[a^\dagger a, (\lambda b)^* e (\lambda b)\right] = 0, \qquad \left[aa^\dagger, (\lambda b)^* f (\lambda b)\right] = 0. \tag{20}$$

This holds if and only if $a \in \mathfrak{D}_{e,f}(\lambda b)$. □

Corollary 13. *If $b \in \mathfrak{R}^{-1}$ and λ is a nonzero scalar, then $\mathfrak{D}(b) = \mathfrak{D}(\lambda b)$.*

Proposition 14. *Let $a, c \in \mathfrak{A}^\dagger$ with MP-inverses $a^\dagger$ and $c^\dagger$, respectively. Assume that $c^\dagger a = 0 = a^\dagger c$ and $ac^\dagger = 0 = ca^\dagger$. If $a, c \in \mathfrak{D}_{e,f}(b)$, then $a + c \in \mathfrak{D}_{e,f}(b)$.*

Proof. It is easy to verify that $a^\dagger + b^\dagger$ is the MP-inverse of $a+b$. Then, $(a+b)^\dagger = a^\dagger + b^\dagger$. Since $a, c \in \mathfrak{D}_{e,f}(b)$,

$$\begin{aligned} \left[a^\dagger a, b^* eb\right] &= 0, \qquad \left[aa^\dagger, b^* fb\right] = 0, \\ \left[c^\dagger c, b^* eb\right] &= 0, \qquad \left[cc^\dagger, b^* fb\right] = 0. \end{aligned} \tag{21}$$

By using the linearity of commutator and the assumptions, from (21), we conclude that

$$\begin{aligned} \left[(a+c)\left(a^\dagger + c^\dagger\right), b^* eb\right] &= 0, \\ \left[\left(a^\dagger + c^\dagger\right)(a+c), b^* fb\right] &= 0. \end{aligned} \tag{22}$$

Now, Theorem 3 implies that $a + c \in \mathfrak{D}_{e,f}(b)$. □

Corollary 15. *Let a, $c \in \mathfrak{A}^\dagger$ with MP-inverses $a^\dagger$ and $c^\dagger$, respectively. Assume that $c^\dagger a = 0 = a^\dagger c$. If a, $c \in \mathfrak{D}_{e,f}(b)$ are self-adjoint, then $a + c \in \mathfrak{D}_{e,f}(b)$.*

Corollary 16. *Let $a, c \in \mathfrak{R}^\dagger$ with MP-inverses $a^\dagger$ and $c^\dagger$, respectively. Assume that $c^\dagger a = 0 = a^\dagger c$. If $a, c \in \mathfrak{D}(b)$ and $[a, c^\dagger] = [a^\dagger, c] = 0$, then $a + c \in \mathfrak{D}(b)$.*

It should be noted that $\mathfrak{D}_{e,f}(b)$ is not closed under addition even $e = f = 1$. For example, set $b = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $a = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, and $c = \left[\begin{smallmatrix} -1 & -1 \\ 0 & -2 \end{smallmatrix}\right]$; then b, a, and c are invertible. Thus $a, c \in \mathfrak{D}(b)$. But $(a + c)(a + c)^\dagger bb^* = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $bb^*(a + c)(a + c)^\dagger = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. This means that $a + c \notin \mathfrak{D}(b)$.

The next theorem shows that the weighted dual covariance set, seen as a multivalued map, has some kind of continuity.

Theorem 17. *Assume that $\{b_n\}$ is a sequence in $\mathfrak{A}^{-1}$ and $b_n \to b \in \mathfrak{A}^{-1}$. Let $a_n \in \mathfrak{D}_{e,f}(b_n)$ be such that $a_n \to a \in \mathfrak{A}^\dagger$. If $\sup_n \|a_n^\dagger\| < \infty$, then $a \in \mathfrak{D}_{e,f}(b)$.*

Proof. By assumption, $\sup_n \|a_n^\dagger\| < \infty$. Therefore [11, Theorem 1.6] implies that $a_n^\dagger \to a^\dagger$, $a_n^\dagger a_n \to a^\dagger a$, and $a_n a_n^\dagger \to aa^\dagger$. Since $a_n \in \mathfrak{D}_{e,f}(b_n)$,

$$\left[a_n^\dagger a_n, b_n^* e b_n\right] = 0, \qquad \left[a_n a_n^\dagger, b_n^* f b_n\right] = 0. \tag{23}$$

Letting $n \to \infty$ in (23), we obtain

$$\left[a^\dagger a, b^* e b\right] = 0, \qquad \left[a a^\dagger, b^* f b\right] = 0. \tag{24}$$

Now, Theorem 3 implies that $a \in \mathfrak{D}_{e,f}(b)$. □

3. Weighted Dual Covariance and $*$-Functions

Let us start with a definition.

Definition 18. Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras. A multiplicative $*$-function $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a map satisfying

(i) $\varphi(ab) = \varphi(a)\varphi(b)$ for all a and b in $\mathfrak{A}$;

(ii) $\varphi(a^*) = (\varphi(a))^*$ for all a in $\mathfrak{A}$;

(iii) $\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

Note that every C^*-algebras homomorphism is a linear multiplicative $*$-function.

It is easy to verify that if a is a positive element, then $\varphi(a)$ is positive. In addition, if $a \in \mathfrak{A}^\dagger$ is positive with MP-inverse $a^\dagger$, then $\varphi(a^\dagger)$ is positive because

$$\begin{aligned} \varphi\left(a^\dagger\right) &= \varphi\left(a^\dagger a a^\dagger\right) = \varphi\left(\left(a^{1/2} a^\dagger\right)^* a^{1/2} a^\dagger\right) \\ &= \varphi\left(\left(a^{1/2} a^\dagger\right)\right)^* \varphi\left(a^{1/2} a^\dagger\right). \end{aligned} \tag{25}$$

Moreover, formula (7) shows that if a is weighted MP-invertible with the weights e and f, then $a_{e,f}^\dagger$ is positive and so $\varphi(a_{e,f}^\dagger)$ is positive.

Proposition 19. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras and suppose that $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a multiplicative $*$-function.*

(i) *If a is a regular element in $\mathfrak{A}$, then $\varphi(a)$ is regular in $\mathfrak{B}$.*

(ii) *If a is weighted MP-invertible in $\mathfrak{A}$ with weights e, f, then $\varphi(a)$ is weighted MP-invertible in $\mathfrak{B}$ with weights $\varphi(e)$, $\varphi(f)$. Moreover*

$$\varphi\left(a_{e,f}^\dagger\right) = (\varphi(a))_{\varphi(e)\varphi(f)}^\dagger. \tag{26}$$

Proof. The first assertion is a consequence of the second one. So we only prove (ii). Let $a_{e,f}^\dagger$ be the weighted MP-inverse of a in $\mathfrak{A}$ with weights e, f. Then by definition,

$$\begin{aligned} a &= a a_{e,f}^\dagger a, & a_{e,f}^\dagger a a_{e,f}^\dagger &= a_{e,f}^\dagger, \\ \left(a a_{e,f}^\dagger\right)^* &= e^{-1} a a_{e,f}^\dagger e, & \left(a_{e,f}^\dagger a\right)^* &= f^{-1} a_{e,f}^\dagger a f. \end{aligned} \tag{27}$$

Since φ is a multiplicative $*$-function, so (27) implies that

$$\begin{aligned} &\varphi(a) = \varphi(a)\varphi\left(a_{e,f}^\dagger\right)\varphi(a), \\ &\varphi\left(a_{e,f}^\dagger\right)\varphi(a)\varphi\left(a_{e,f}^\dagger\right) = \varphi\left(a_{e,f}^\dagger\right), \\ &\varphi\left(e^{-1}\right)\left(\varphi(a)\varphi\left(a_{e,f}^\dagger\right)\right)\varphi(e) \\ &\quad = \varphi\left(e^{-1} a a_{e,f}^\dagger e\right) = \varphi\left(\left(a a_{e,f}^\dagger\right)^*\right) \\ &\quad = \left(\varphi\left(a a_{e,f}^\dagger\right)\right)^* = \left(\varphi(a)\varphi\left(a_{e,f}^\dagger\right)\right)^*, \end{aligned} \tag{28}$$

and similarly

$$\varphi\left(f^{-1}\right)\varphi\left(a_{e,f}^\dagger\right)\varphi(a)\varphi(f) = \left(\varphi\left(a_{e,f}^\dagger\right)\varphi(a)\right)^*. \tag{29}$$

Therefore $\varphi(a_{e,f}^\dagger)$ is weighted MP-inverse of $\varphi(a)$ with weights $\varphi(e)$, $\varphi(f)$. Now by the uniqueness of weighted MP-inverse we get

$$\varphi\left(a_{e,f}^\dagger\right) = (\varphi(a))_{\varphi(e),\varphi(f)}^\dagger. \tag{30}$$

□

Proposition 20. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras and $a \in \mathfrak{A}^\dagger$.*

(i) *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a multiplicative $*$-function, then $\varphi(\mathfrak{D}_{e,f}(b)) \subset \mathfrak{D}_{\varphi(e),\varphi(f)}(\varphi(b))$.*

(ii) *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a bijection multiplicative $*$-function, then $\varphi(\mathfrak{D}_{e,f}(b)) = \mathfrak{D}_{\varphi(e),\varphi(f)}(\varphi(b))$.*

Proof. (i) Suppose that $\varphi(a) \in \varphi(\mathfrak{D}_{e,f}(b))$. Then $a \in \mathfrak{D}_{e,f}(b)$ and so $b^{-1}a^\dagger b$ is weighted MP-inverse of $b^{-1}ab$ with weights e, f. By applying Proposition 19 we get $\varphi(a) \in \mathfrak{D}_{\varphi(e),\varphi(f)}(\varphi(b))$.

(ii) Since φ is bijection multiplicative $*$-function, φ^{-1} is also multiplicative $*$-function. From here and part (i) we obtain the desired assertion. □

Corollary 21. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a multiplicative $*$-function, then $\varphi(\mathfrak{D}(b)) \subset \mathfrak{D}(\varphi(b))$. Moreover, if φ is bijection, then $\varphi(\mathfrak{D}(b)) = \mathfrak{D}(\varphi(b))$.*

Corollary 22. *Assume that $b \in \mathfrak{A}^{-1}$. Then,*

(i) *if $u \in \mathfrak{A}$ is a unitary, then $\mathfrak{D}_{e,f}(u^*bu) = u^*\mathfrak{D}_{u^*eu,u^*fu}(b)u$;*

(ii) *if $v \in \mathfrak{A}^{-1}$, then $\mathfrak{D}_{e,f}(v^{-1}bv) = v^{-1}\mathfrak{D}_{v^{-1}ev,v^{-1}fv}(b)v$.*

Proof. Since the maps $a \mapsto u^*au$ and $a \mapsto u^{-1}au$ are bijection multiplicative $*$-functions, the results follow from Proposition 20. □

Corollary 23. *If $u \in \mathfrak{R}$ is a unitary, then $\mathfrak{D}(u^*bu) = u^*\mathfrak{D}(b)u$, and if $v \in \mathfrak{R}^{-1}$, then $\mathfrak{D}(v^{-1}bv) = v^{-1}\mathfrak{D}(b)v$.*

Remark 24. The notion of dual covariance is not studied to Drazin inverses because it is easy to see that, for every Drazin invertible element a^D and for every $b \in \mathfrak{R}^{-1}$, we have

$$\left(bab^{-1}\right)^D = ba^Db^{-1}. \tag{31}$$

In fact, for any $b \in \mathfrak{R}^{-1}$ we have $\{a \in \mathfrak{R}^D : (bab^{-1})^D = ba^Db^{-1}\} = \mathfrak{R}^D$ where $\mathfrak{R}^D$ is the set of all Drazin invertible elements of $\mathfrak{R}$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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